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A MULTI-SYMPLECTIC SCHEME FOR RLW EQUATION *)

Ya-juan Sun Meng-zhao Qin
(*ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

The Hamiltonian and multi-symplectic formulations for RLW equation are considered in this paper. A new twelve-point difference scheme which is equivalent to multi-symplectic Preissmann integrator is derived based on the multi-symplectic formulation of RLW equation. And the numerical experiments on solitary waves are also given. Comparing the numerical results for RLW equation with those for KdV equation, the inelastic behavior of RLW equation is shown.

Mathematics subject classification: 65N15.
Key words: Multi-Symplectic Scheme, RLW Equation.

1. Introduction

We consider the regularized long-wave (RLW) equation [1]

$$u_t + (V'(u))_x - \sigma u_{xxt} = 0, \tag{1}$$

where $V(u) = \frac{1}{6}u^3$. The equation was first put forward by Peregrine (1966) [2] to describe the development of long wave behaviour. B. Benjamin, J. L. Bona and J. J. Mahoney (1972) [1] studied in detail the existence, uniqueness and stability of solutions of equation (1) and considered it as a more suitable posed model for long wave. (1) was called as BBM equation by P. J. Olver (1979) [6], who studied the conservation laws of equation (1) and proved that it possesses only three independent conservation laws. At the same year, the comparative numerical experiments between RLW equation and KdV equation were given by M. E. Alexander and J. L. Morris [5], the results showed that the interaction of two-solitary wave is inelastic during evolution process. It was known very well that KdV equation as a completely integrable model has infinite family of conservation laws, so the interaction between solitons is elastic. Therefore, it was clear that RLW equation and KdV equation describe different physical phenomena.

The purpose of this paper is to present a multi-symplectic formulation for RLW equation, then by using multi-symplectic Preissmann scheme to derive a new twelve-point scheme. At last we give the comparative numerical experiments on solitary waves for RLW equation and KdV equation by using the new twelve-point scheme given in this paper and the twelve-point scheme given by Pingfu Zhao and Mengzhao Qin [11]. These numerical results obtained in this paper further confirm the inelastic behaviour of the RLW equation and show that the multi-symplectic twelve-point scheme given in [11] treats the KdV equation more successfully than the Galerkin method got in [5]. In addition, we simulate the interaction of three solitary waves.

An outline of this paper is as follows. In section 2, we present the Hamiltonian formulation and the multi-symplectic formulation for RLW equation. In section 3, we derive a new twelve -point difference scheme based on multi-symplectic Preissmann scheme. Numerical experiments are presented in section 4.

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2. Symplectic and Multi-symplectic Formulations of RLW Equation

In this section, we discuss symplectic and multi-symplectic formulations for RLW equation. First, (1) can be written as the classical Hamiltonian formulation

$$u_t = -(1-\sigma D^2)^{-1} D \frac{\delta H}{\delta u},$$

where the Hamilton operator is $-(1-\sigma D^2)^{-1}D$, $D = \frac{\partial}{\partial x}$, and the Hamiltonian function is $H = \int \frac{1}{6} u^3 dx$.

Let $u = \varphi_x$, (1) can be rewritten as the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\delta L}{\delta \varphi_t} = \frac{\delta L}{\delta \varphi}$$

with Lagrangian functional

$$L(\varphi, \varphi_t) = \int (\frac{1}{2}\varphi_t\varphi_x + \frac{1}{2}\sigma\varphi_{xt}\varphi_{xx} + \frac{1}{6}\varphi_x^3) dx.$$

Here, the canonical 1-form on infinite dimensional functional space TQ with coordinates (φ, φ_t) can be considered as

$$\theta = \frac{\delta L}{\delta \varphi_t} d\varphi = \frac{1}{2}(1-\sigma D^2)\varphi_x d\varphi = \frac{1}{2}(1-\sigma D^2) u D^{-1} du.$$

Now, the canonical two-form on TQ is

$$\Omega = -d\theta = -\frac{1}{2}(1-\sigma D^2) du \wedge D^{-1} du = \frac{1}{2}(\sigma du \wedge du_x - du \wedge D^{-1} du).$$

Second, we can describe (1) by using multi-symplectic language. According to the multi-symplectic concept introduced by Bridges [6], the RLW equation can be reformulated as the following

$$M z_t + K z_x = \bigtriangledown_z S(z), \quad z \in R^5, \quad (x,t) \in R^2, \tag{2}$$

with state variable $z = (\varphi, u, v, w, p)^T$ and the Hamiltonian function

$$H = up - \frac{1}{6}u^3 + \frac{1}{2}\sigma v w,$$

where

$$M = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2}\sigma & 0 & 0 \\ 0 & \frac{1}{2}\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\frac{1}{2}\sigma & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sigma & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$u = \varphi_x, \ v = u_x, \ w = u_t, \ p = \frac{1}{2}\varphi_t + \frac{1}{2}\varphi_x^2 - \sigma\varphi_{xxt}. \tag{3}$$

Taking exterior derivative on the two side of (2), we have

$$M dz_t + K dz_x = \bigtriangledown_z^2 S(z) dz.$$

Then take the wedge product with dz on the two side of the equality above, a multi-symplectic conservation law can be derived

$$\frac{\partial}{\partial t}(du \wedge d\varphi + \sigma dv \wedge du) + \frac{\partial}{\partial x}(2dp \wedge d\varphi + \sigma dw \wedge du) = 0. \tag{4}$$

Let a fiber bundle $\pi_{XY}: Y \longrightarrow X$ be the covariant configuration bundle, in which the coordinates of the base space X is denoted by (x,t), the coordinates of the bundle space $Y = X \times R$ is denoted by (x,t,y). Denote $J^2(Y)$ as the 2^{th} jet bundle of Y, if $\varphi : X \longrightarrow Y$ is a section of Y, then the coordinates of the third jet prolongation $j^3(\varphi)$ of the section φ can be given by $(x,t,\varphi,\varphi_\mu,\varphi_{\mu_1\mu_2},\varphi_{\mu_1\mu_2\mu_3})$, where $\varphi_\mu = \partial_\mu\varphi$ is the first partial derivative, $\varphi_{\mu_1\mu_2} = \partial_{\mu_1}\partial_{\mu_2}\varphi$ is the second partial derivative, $\varphi_{\mu_1\mu_2\mu_3} = \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\varphi$ is the third partial derivative. According to Marsden's Lagrangian theory [7], (2) are equivalent to an Euler-Lagrange equation with the Lagrangian density on $J^2(Y)$

$$\mathcal{L}(x,t,\varphi,\varphi_t,\varphi_x,\varphi_{xt},\varphi_{xx}) = (\frac{1}{2}\varphi_t\varphi_x + \frac{1}{2}\sigma\varphi_{xt}\varphi_{xx} + \frac{1}{6}\varphi_x^3)dx \wedge dt. \tag{5}$$

The Cartan form on $J^3(Y)$ can be taken as

$$\begin{aligned}\theta_{\mathcal{L}} = &-\tfrac{1}{2}\varphi_x d\varphi \wedge dx + (\tfrac{1}{2}\varphi_t + \tfrac{1}{2}\varphi_x^2 - \sigma\varphi_{xxt})d\varphi \wedge dt + \tfrac{1}{2}\sigma\varphi_{xt}d\varphi_x \wedge dt \\ &-\tfrac{1}{2}\sigma\varphi_{xx}d\varphi_x \wedge dx - (\tfrac{1}{2}\varphi_x\varphi_t + \tfrac{1}{3}\varphi_x^3 - \sigma\varphi_{xxt}\varphi_x + \tfrac{1}{2}\sigma\varphi_{xx}\varphi_{xt})dx \wedge dt,\end{aligned}$$

then the corresponding multi-symplectic form is $\Omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$.

In Lagrangian sense, (1) has multi-symplectic form formula

$$\int_U j^3(\varphi)^*(j^3(V)\rfloor j^3(W)\rfloor\Omega_{\mathcal{L}}) = 0, \tag{6}$$

where U is a smooth manifold with smooth closed boundary. Taking $V, W \in \pi_{X,Y}$-vertical vector $V = \varphi_t\frac{\partial}{\partial\varphi}$, $W = \varphi_x\frac{\partial}{\partial\varphi}$, then

$$\begin{aligned}j^3(V) &= (\varphi_t,\varphi_{tt},\varphi_{tx},\varphi_{ttt},\varphi_{txt},\varphi_{txx},\varphi_{tttt},\varphi_{tttx},\varphi_{ttxx},\varphi_{txxx}),\\ j^3(W) &= (\varphi_x,\varphi_{xt},\varphi_{xx},\varphi_{xtt},\varphi_{xxt},\varphi_{xxx},\varphi_{xttt},\varphi_{xttx},\varphi_{xtxx},\varphi_{xxxx}),\end{aligned}$$

and computing (6), the Marsden's multi-symplectic form formula for RLW equation is

$$\begin{aligned}j^3(\varphi)^*(j^3(V)\rfloor j^3(W)\rfloor\Omega_{\mathcal{L}}) = &-\frac{1}{2}(\varphi_{tx}\varphi_x - \varphi_{xx}\varphi_t + \sigma\varphi_{txx}\varphi_{xx} - \sigma\varphi_{xxx}\varphi_{tx})dx + \frac{1}{2}(\varphi_{tt}\varphi_x - \varphi_{xt}\varphi_t)dt\\ &+ \frac{1}{2}(2\sigma\varphi_{ttxx}\varphi_x - 2\sigma\varphi_{xxxt}\varphi_t + \sigma\varphi_{ttx}\varphi_{xx} - \sigma\varphi_{xxt}\varphi_{tx})dt\\ &+ (\varphi_x\varphi_{tx}\varphi_x - \varphi_x\varphi_{xx}\varphi_t)dt.\end{aligned}$$

Using Stokes theorem, we have

$$\begin{aligned}0 = &\int_U D_t(\frac{1}{2}(\varphi_{tx}\varphi_x - \varphi_{xx}\varphi_t + \sigma\varphi_{txx}\varphi_{xx} - \sigma\varphi_{xxx}\varphi_{tx})) + D_x(\frac{1}{2}(\varphi_{tt}\varphi_x - \varphi_{xt}\varphi_t))dx \wedge dt\\ &+ \int_U D_x(\frac{1}{2}(2\sigma\varphi_{ttxx}\varphi_x - 2\sigma\varphi_{xxxt}\varphi_t + \sigma\varphi_{ttx}\varphi_{xx} - \sigma\varphi_{xxt}\varphi_{tx} + 2\varphi_x\varphi_{tx}\varphi_x - 2\varphi_x\varphi_{xx}\varphi_t))dx \wedge dt.\end{aligned}$$

Since U is arbitrary, thus a conservation law

$$\begin{aligned}0 = &D_t(\tfrac{1}{2}(\varphi_{tx}\varphi_x - \varphi_{xx}\varphi_t + \sigma\varphi_{txx}\varphi_{xx} - \sigma\varphi_{xxx}\varphi_{tx})) + D_x(\tfrac{1}{2}(\varphi_{tt}\varphi_x - \varphi_{xt}\varphi_t))dx \wedge dt\\ &+D_x(\tfrac{1}{2}(2\sigma\varphi_{ttxx}\varphi_x - 2\sigma\varphi_{xxxt}\varphi_t + \sigma\varphi_{ttx}\varphi_{xx} - \sigma\varphi_{xxt}\varphi_{tx} + 2\varphi_x\varphi_{tx}\varphi_x - 2\varphi_x\varphi_{xx}\varphi_t))dx \wedge dt\end{aligned} \tag{7}$$

can be obtained. By using the transformation (3), we notice that (7) is the Bridges's multi-symplectic conservation law (4) for RLW equation, i.e., conservation law (7) is equivalent to conservation law (4).

Finally, we show that the multi-symplectic structure on $J^3(Y)$ can be induced to a symplectic structure on a infinite-dimensional manifold. Let Q be the infinite-dimensional manifold with coordinates φ and $i: x \longmapsto (x,t)$ be the inclusion map of Σ into X, $J^3(Y)_\Sigma$ is the collection of restrictions of holonomic sections of $J^3(Y)$ to Σ, then we may define the canonical 1-form on $J^3(Y)_\Sigma$

$$\theta_\Sigma(j^3(\varphi)\circ i)V = \int i^* j^3(\varphi)^*(V\rfloor\theta_{\mathcal{L}}), \tag{8}$$

where $V \in T_{j^3(\varphi)\circ i}(J^3(Y)_\Sigma)$. Taking $V \in \pi_{X,J^3(Y)}$-vertical vector,

$$V = (V_\varphi, V_{\varphi_x}, V_{\varphi_t}, V_{\varphi_{xx}}, V_{\varphi_{xt}}, V_{\varphi_{tt}}, V_{\varphi_{ttt}}, V_{\varphi_{ttx}}, V_{\varphi_{txx}}, V_{\varphi_{xxx}}),$$

then we have

$$j^3(\varphi)^*(V\rfloor\theta_{\mathcal{L}}) = -\frac{1}{2}(\varphi_x - \sigma\varphi_{xxx})V_\varphi dx + V_\varphi(\frac{1}{2}\varphi_t + \frac{1}{2}\varphi_x^2 - \sigma\varphi_{xxt})dt + \frac{1}{2}\sigma\varphi_{xt}V_{\varphi_x}dt,$$

$$i^*j^3(\varphi)^*(V\rfloor\theta_{\mathcal{L}}) = \frac{1}{2}\varphi_x V_\varphi dx - \frac{1}{2}\sigma\varphi_{xxx}V_\varphi dx,$$

$$\theta_\Sigma(j^3(\varphi)\circ i)V = \int \frac{1}{2}(1-\sigma D^2)uV_\varphi dx. \tag{9}$$

The equality (9) implies

$$\theta_\Sigma = \int \frac{1}{2}(1-\sigma D^2)uD^{-1}du dx,$$

it follows

$$\Omega_\Sigma = -d\theta_\Sigma = \int \frac{1}{2}(\sigma du\wedge du_x - du\wedge D^{-1}du)dx.$$

3. Multi-symplectic Scheme Based on Multi-symplectic Preissmann Scheme

Multi-symplecticity as a geometric property of the PDEs is noticed recently in [9,10,11], and a natural way to discrete the PDEs is to preserve the property. Based on this idea, Bridges and Reich [8-9] introduced the concept of multi-symplectic integrators, i.e., numerical methods which preserve a discrete version of multi-symplectic conservation law. As a example, we discretize (2) by using Preissmann scheme and obtain

$$\frac{u_{j+\frac{1}{2}}^{i+1} - u_{j+\frac{1}{2}}^{i}}{2\triangle t} + \frac{p_{j+1}^{i+\frac{1}{2}} - p_j^{i+\frac{1}{2}}}{\triangle x} = 0, \tag{10}$$

$$\frac{\varphi_{j+\frac{1}{2}}^{i+1} - \varphi_{j+\frac{1}{2}}^{i}}{2\triangle t} - \sigma\frac{v_{j+\frac{1}{2}}^{i+1} - v_{j+\frac{1}{2}}^{i}}{2\triangle t} - \sigma\frac{w_{j+1}^{i+\frac{1}{2}} - w_j^{i+\frac{1}{2}}}{2\triangle x} = p_{j+\frac{1}{2}}^{i+\frac{1}{2}} - \frac{1}{2}(u_{j+\frac{1}{2}}^{i+\frac{1}{2}})^2, \tag{11}$$

$$\frac{u_{j+\frac{1}{2}}^{i+1} - u_{j+\frac{1}{2}}^{i}}{2\triangle t} = \frac{w_{j+\frac{1}{2}}^{i+\frac{1}{2}}}{2}, \tag{12}$$

$$\frac{\varphi_{j+1}^{i+\frac{1}{2}} - \varphi_j^{i+\frac{1}{2}}}{\triangle x} = u_{j+\frac{1}{2}}^{i+\frac{1}{2}}, \tag{13}$$

where $p_j^i \approx p(j\triangle x, i\triangle t)$, $p_{j+\frac{1}{2}}^i = \frac{1}{2}(p_j^i + p_{j+1}^i)$, $p_{j+\frac{1}{2}}^{i+\frac{1}{2}} = \frac{1}{4}(p_j^i + p_{j+1}^i + p_j^{i+1} + p_{j+1}^{i+1})$, $\triangle x$ and $\triangle t$ are the space step length and time step length respectively.

It is clear, Integrator (9)-(12) preserves the discrete multi-symplectic conservation law

$$\begin{aligned} &\frac{du_{j+\frac{1}{2}}^{i+1}\wedge d\varphi_{j+\frac{1}{2}}^{i+1} - du_{j+\frac{1}{2}}^{i}\wedge d\varphi_{j+\frac{1}{2}}^{i}}{2\triangle t} + \frac{\sigma dv_{j+\frac{1}{2}}^{i+1}\wedge du_{j+\frac{1}{2}}^{i+1} - \sigma dv_{j+\frac{1}{2}}^{i}\wedge du_{j+\frac{1}{2}}^{i}}{2\triangle t} \\ &+ \frac{dp_{j+1}^{i+\frac{1}{2}}\wedge d\varphi_{j+1}^{i+\frac{1}{2}} - dp_j^{i+\frac{1}{2}}\wedge d\varphi_j^{i+\frac{1}{2}}}{\triangle x} + \frac{\sigma dw_{j+1}^{i+\frac{1}{2}}\wedge du_{j+1}^{i+\frac{1}{2}} - \sigma dw_j^{i+\frac{1}{2}}\wedge du_j^{i+\frac{1}{2}}}{2\triangle x} = 0. \end{aligned} \tag{14}$$

In practice, we usually need to know the values of u only, which inspires us to eliminate φ, v, w, p, and obtain a twelve-point difference scheme for u. Define difference operator $\partial_t u_j^i$, $\partial_{xxt} u_j^i$, $\bar{u}_j^i$

$$\begin{aligned}\partial_t u_j^i &= u_j^{i+1} - u_j^{i-1},\\ \partial_{xxt} u_j^i &= u_{j+1}^{i+1} - u_j^{i+1} - u_{j+1}^{i-1} + u_j^{i-1} - u_{j-1}^{i+1} + u_{j-2}^{i+1} + u_{j-1}^{i-1} - u_{j-2}^{i-1},\\ \bar{u}_j^i &= \frac{1}{4}(u_j^i + u_j^{i+1} + u_{j+1}^{i+1} + u_{j+1}^i),\end{aligned}$$

then a twelve-point scheme

$$\begin{aligned}&\tfrac{1}{4\triangle t}(\partial_t u_{j+1}^i + 3\partial_t u_j^i + 3\partial_t u_{j-1}^i + \partial_t u_{j-2}^i) - \tfrac{\sigma}{\triangle x^2 \triangle t}\partial_{xxt} u_j^i\\ &= \tfrac{-1}{\triangle x}(V'(\bar{u}_j^{i-1}) - V'(\bar{u}_{j-2}^{i-1}) + V'(\bar{u}_j^i) - V'(\bar{u}_{j-2}^i))\end{aligned} \tag{15}$$

can be obtained. Discretizing the Lagrangian (5) by using

$$\begin{aligned}\varphi_x &\approx \frac{\varphi_{j+1}^i + \varphi_{j+1}^{i+1} - \varphi_{j-1}^i - \varphi_{j-1}^{i+1}}{4\triangle x},\\ \varphi_{xx} &\approx \frac{\varphi_{j+1}^i - 2\varphi_j^i + \varphi_{j-1}^i + \varphi_{j+1}^{i+1} - 2\varphi_j^{i+1} + \varphi_{j-1}^{i+1}}{2\triangle x},\\ \varphi_{xt} &\approx \frac{\varphi_{j+1}^{i+1} - \varphi_{j+1}^i - \varphi_j^{i+1} + \varphi_j^i}{\triangle x \triangle t},\\ \varphi_t &\approx \frac{\varphi_{j+1}^{i+1} - \varphi_{j+1}^i + \varphi_j^{i+1} - \varphi_j^i}{2\triangle t},\end{aligned}$$

we get the following scheme from the discrete variational principle

$$\begin{aligned}&\tfrac{1}{16\triangle x\triangle t}(\varphi_{j+2}^{i+1} - \varphi_{j+2}^{i-1} + 2\varphi_{j+1}^{i+1} - 2\varphi_{j-1}^{i+1} - 2\varphi_{j+1}^{i-1} + 2\varphi_{j-1}^{i-1} - \varphi_{j-2}^{i+1} + \varphi_{j-2}^{i-1})\\ &-\tfrac{\sigma}{4\triangle x^3\triangle t}(\varphi_{j+2}^{i+1} - 2\varphi_{j+1}^{i+1} + 2\varphi_{j-1}^{i+1} - \varphi_{j-2}^{i+1} - \varphi_{j+2}^{i-1} + 2\varphi_{j+1}^{i-1} - 2\varphi_{j-1}^{i-1} + \varphi_{j-2}^{i-1})\\ &+\tfrac{1}{4\triangle x}(V'(\tfrac{\varphi_{j+2}^{i+1} - \varphi_j^{i+1} + \varphi_{j+2}^i - \varphi_j^i}{4\triangle x}) + V'(\tfrac{\varphi_{j+2}^i - \varphi_j^i + \varphi_{j+2}^{i-1} - \varphi_j^{i-1}}{4\triangle x})\\ &-V'(\tfrac{\varphi_j^i - \varphi_{j-2}^i + \varphi_j^{i-1} - \varphi_{j-2}^{i-1}}{4\triangle x}) - V'(\tfrac{\varphi_j^{i+1} - \varphi_{j-2}^{i+1} + \varphi_j^i - \varphi_{j-2}^i}{4\triangle x})) = 0.\end{aligned} \tag{16}$$

In the equality above, if let $\frac{\varphi_{j+1}^i - \varphi_j^i}{\triangle x} = u_j^i$, we have that (16) is equivalent to (15).

4. Numerical Experiments

In this section, we present these plots of the one-, two- and three-solitary wave solutions for RLW equation, and compare the numerical results with the ones given in [11] for KdV equation.

The following initial boundary conditions are used:

One-solitary wave

$$u(x,0) = 3c\,sech^2(kx+d), \quad u(0,t) = u(2,t), \tag{17}$$

where $k = \frac{1}{2}(\frac{c}{m})^{\frac{1}{2}} = \frac{1}{2}(\frac{1}{\sigma})^{\frac{1}{2}}$, $d = -k$, $c = 0.3$, $m = 4.84(10^{-4})$.

Two-solitary wave

$$\begin{aligned}u(x,0) = 3c_1 sech^2(k_1 x + d_1) + 3c_2 sech^2(k_2 x + d_2),\\ u(-2+\tfrac{4}{3}, t) = u(2+\tfrac{4}{3}, t),\end{aligned} \tag{18}$$

where $k_1 = \frac{1}{2}(\frac{c_1}{m})^{\frac{1}{2}}$, $k_2 = \frac{1}{2}(\frac{c_2}{m})^{\frac{1}{2}}$, $d_1 = d_2 = -5$, $c_1 = 0.3$, $c_2 = 0.1$.

Three-solitary wave

$$\begin{aligned}u(x,0) = 3c_1 sech^2(k_1 x + d_1) + 3c_2 sech^2(k_2 x + d_2) + 3c_3 sech^2(k_3 x + d_3),\\ u(-2+\tfrac{2}{3}, t) = u(2+\tfrac{2}{3}, t),\end{aligned} \tag{19}$$

where $k_1 = \frac{1}{2}(\frac{c_1}{m})^{\frac{1}{2}}$, $k_2 = \frac{1}{2}(\frac{c_2}{m})^{\frac{1}{2}}$, $k_3 = \frac{1}{2}(\frac{c_3}{m})^{\frac{1}{2}}$, $d_1 = d_2 = d_3 = -5$, $c_1 = 0.6$, $c_2 = 0.3$, $c_3 = 0.15$.

First, we consider the propagation of a solitary wave, and use (17) to do this. The computation is done for $x \in [0, 2]$, with the time step $\triangle t = 0.05$ and the space step $\triangle x = 2/201$, see Fig.1.

Next, we consider the interaction of two colliding solitary waves by using (18). The computation is done for $x \in [-2 + 4/3, 2 + 4/3]$, with the time step $\triangle t = 0.01$ and the space step $\triangle x = 4/301$. Comparing Fig.2, Fig.3 with Fig.4, Fig.5, we notice an emergence of a third small amplitude solitary wave which is similar to that showed in [5] for RLW equation.

Finally, we also test our integrators on a boundary state of three solitary waves with the initial boundary condition (19). The computation is done for $x \in [-2 + 2/3, 2 + 2/3]$, with the time step $\triangle t = 0.002$ and the space step $\triangle x = 4/301$. Comparing Fig.6, Fig.7 with Fig.8, Fig.9, we observe the emergence of a small amplitude wave in negative direction for RLW equation.

In conclusion, these numerical results given in this section further confirm the inelastic behavior of RLW equation, and show that RLW equation and KdV equation describe the different physical behaviour.

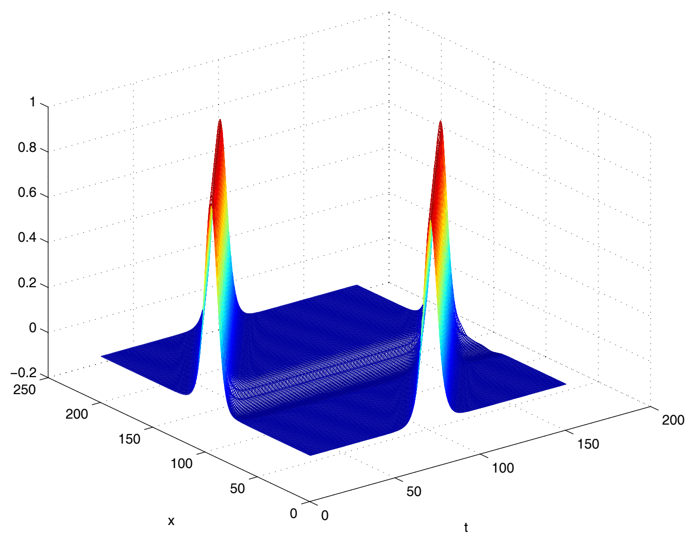
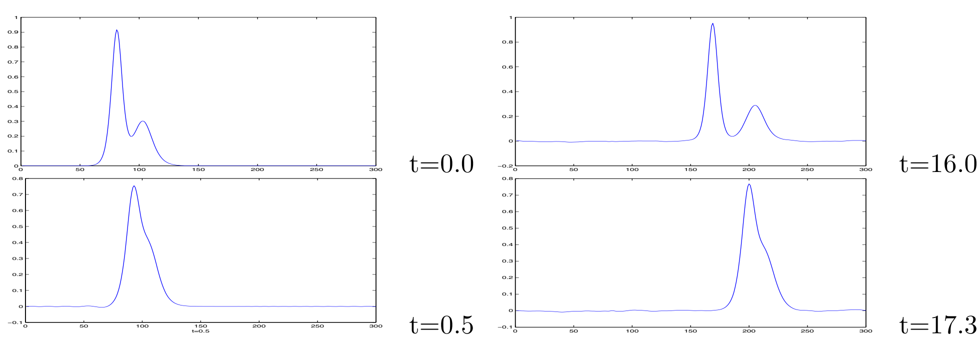


Figure 1. A three dimensional version of the evolutions of one-solitary wave for RLW equation



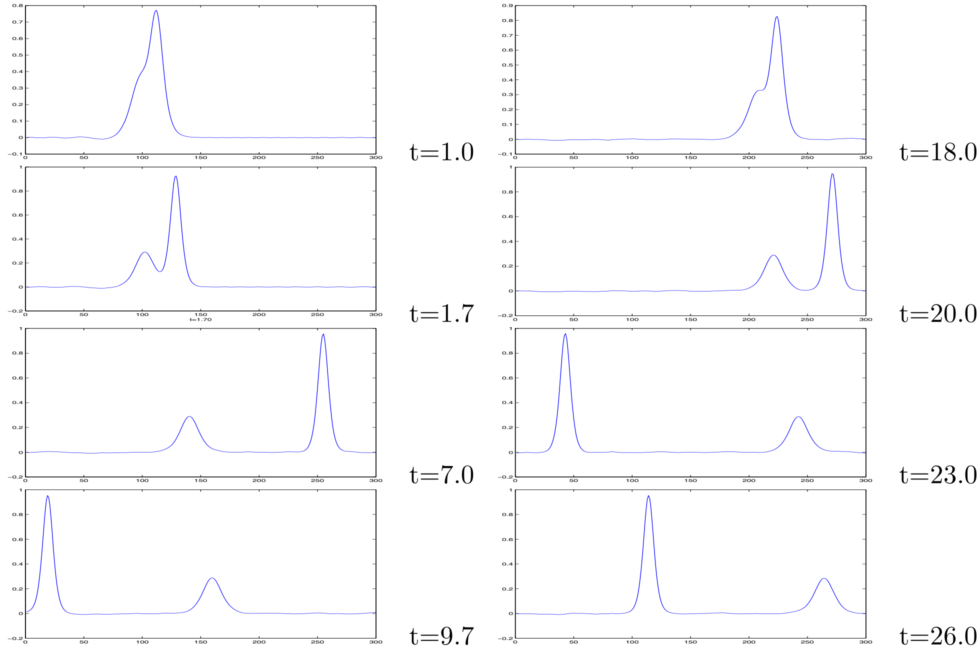


Figure 2. The temporal developments of two-solitary wave for KdV equation

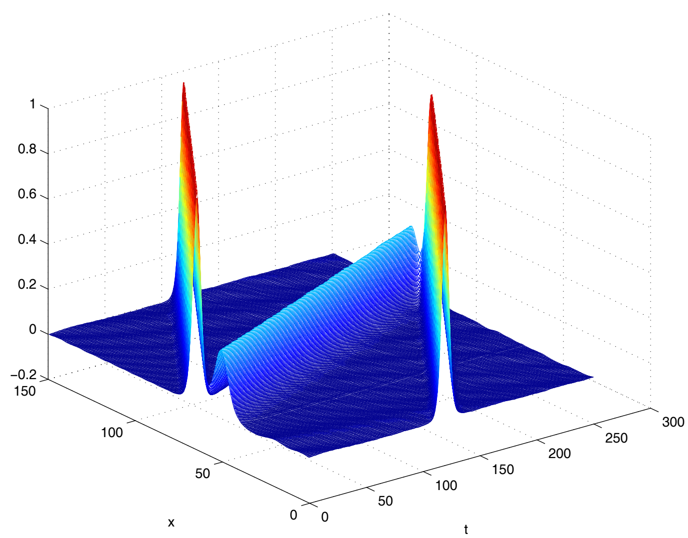
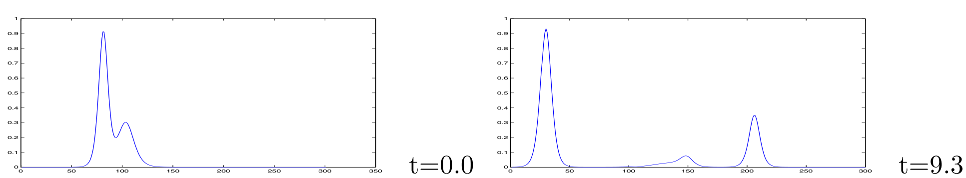


Figure 3. A three dimensional version of the evolutions of two-solitary wave for KdV equation



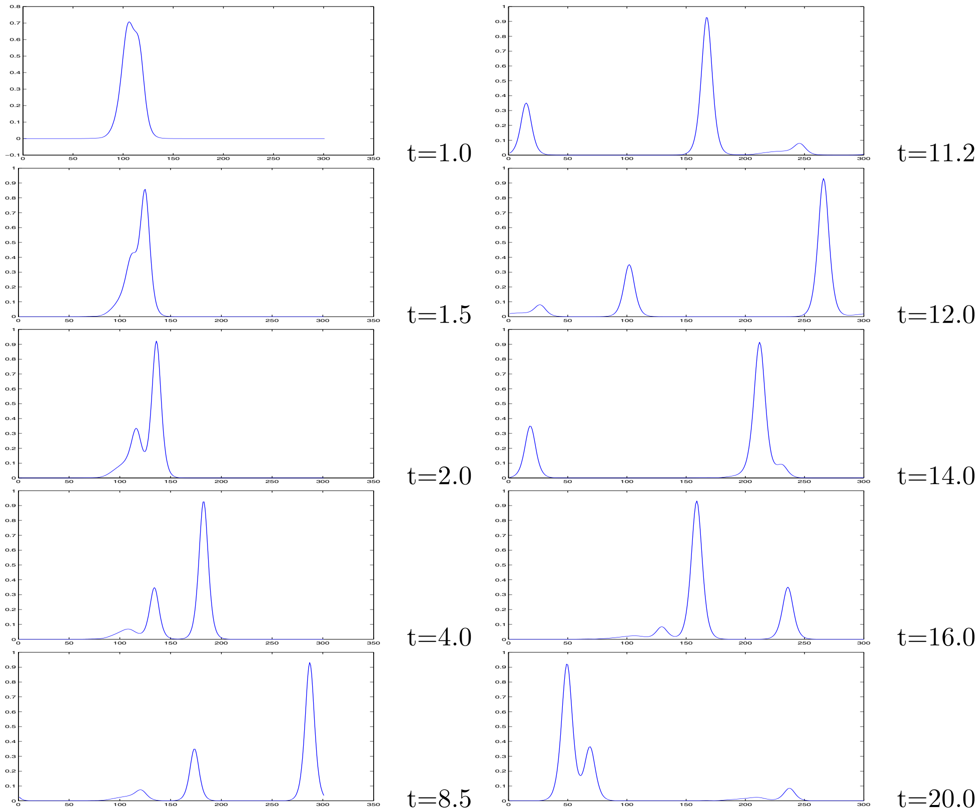


Figure 4. The temporal developments of two-solitary wave for RLW equation

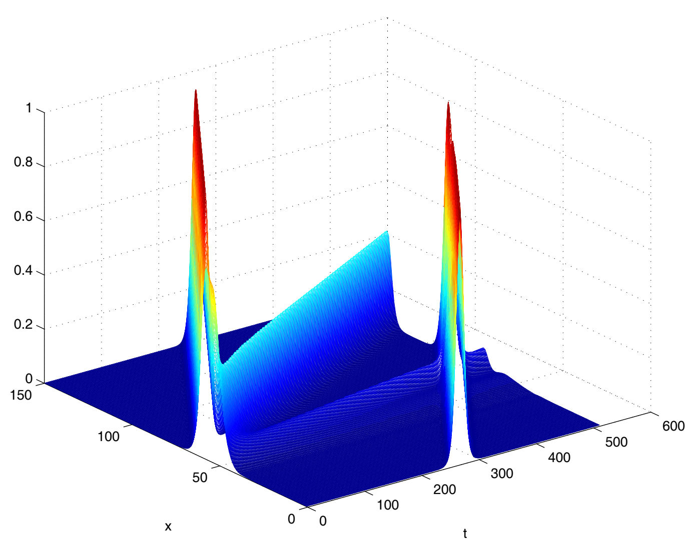


Figure 5. A three dimensional version of the evolutions of two-solitary wave for RLW equation

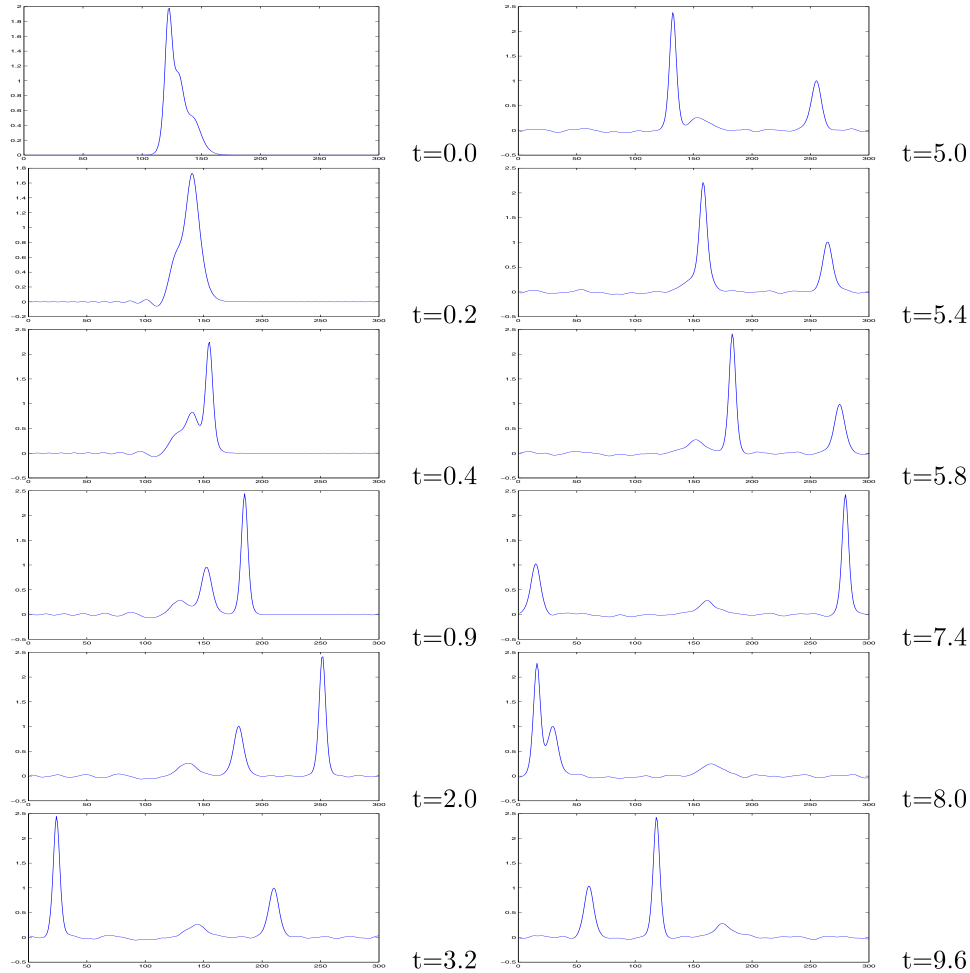


Figure 6. The temporal developments of three solitary wave for KdV equation

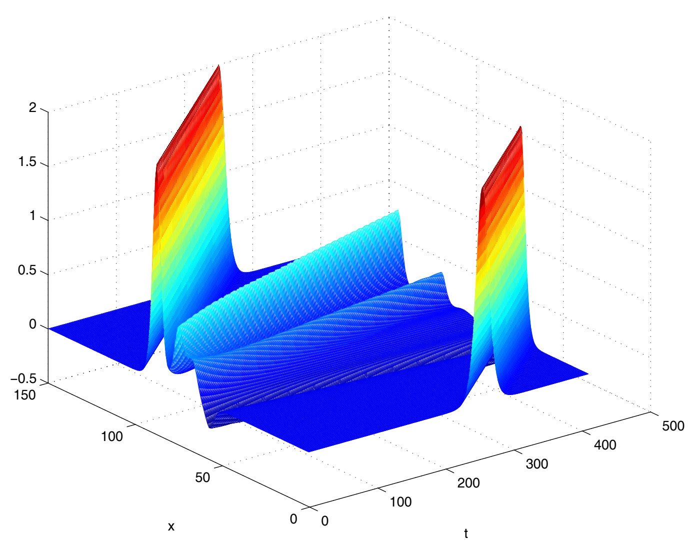


Figure 7. A three dimensional version of the evolutions of three-solitary wave for KdV equation

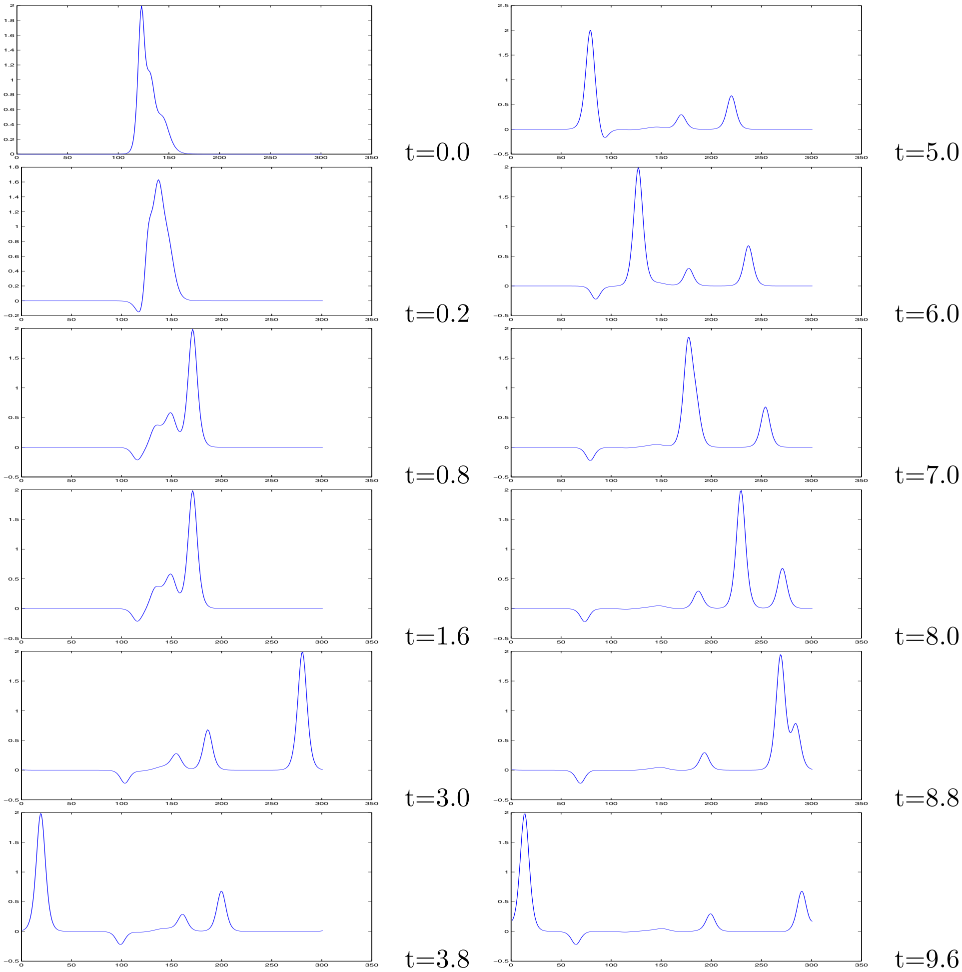


Figure 8. The temporal developments of three-solitary wave for RLW equation

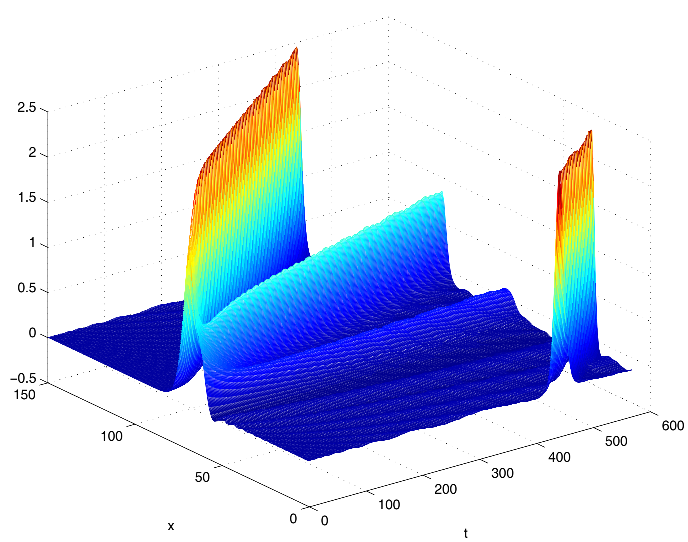


Figure 9. A three dimensional version of the evolutions of three-solitary wave for RLW equation

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