Journal of Computational Mathematics, Vol.24, No.3, 2006, 435–443.

IMAGE SEGMENTATION BY PIECEWISE CONSTANT MUMFORD-SHAH MODEL WITHOUT ESTIMATING THE CONSTANTS *

Xue-cheng Tai
(*CIPR,Department of Mathematics, University of Bergen, Norway*)

Chang-hui Yao
(*Department of Mathematics, University of Bergen, Norway*)

Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this work, we try to use the so-called Piecewise Constant Level Set Method (PCLSM) for the Mumford-Shah segmentation model. For image segmentation, the Mumford-Shah model needs to find the regions and the constant values inside the regions for the segmentation. In order to use PCLSM for this purpose, we need to solve a minimization problem using the level set function and the constant values as minimization variables. In this work, we test on a model such that we only need to minimize with respect to the level set function, i.e., we do not need to minimize with respect to the constant values. Gradient descent method and Newton method are used to solve the Euler-Lagrange equation for the minimization problem. Numerical experiments are given to show the efficiency and advantages of the new model and algorithms.

Mathematics subject classification: 35G25, 65K10.
Key words: PCLSM, Image Segmentation, Mumford-Shah model.

1. Introduction

Level set methods, originally introduced by Osher and Sethian [13], have been developed to be one of the most successful tools for the computation of evolving geometries, which appear in many practical applications. They use zero level set of some functions to trace interfaces separated a domain Ω into subdomains. For a recent survey on the level set methods see [15, 10, 14].

In [7, 6, 9] some variants of the level set methods of [13], so-called "piecewise constant level set method (PCLSM)", were proposed. The method can be used for different applications. In [12, 17], applications to inverse problems involving elliptic and reservoir equations are shown. In [7, 6, 9, 18], the ideas have been used for image segmentation problem.

Image segmentation is one of the foundational tasks of computer vision. Its goal is to partition a given image into regions which contain distinct objects. One of the most common forms of segmentation are based on assumption that distinct objects in an image have different approximately constant(or slowly varying) colors (or roughnesses in the case of monochrome imagery). In this paper, we use the Mumford-Shah model [11] for image segmentation with PCLSM. In [2, 8, 15], the Osher-Sethian level set idea was used to solve the Mumford-Shah model. No matter whether we use the Osher-Sethian method or PCLSM, we need to minimize with respect to the level set functions and the constant values. In this work, we shall propose a model such that only the level set function is the minimization variable. The advantage of this

* Received March 1, 2006.

model is that we just have one variable to minimize. The resulted algorithms are more stable and have a slightly faster convergence rate. This model eliminates the question about when and how often to update the constant values.

The organization of our paper is as follows. In Section 2, firstly, we begin with a quick examination of what constitutes a piecewise constant level set method and the classical Mumford-Shah model. The origin of the new idea is developed as well. In Section 3, gradient descent method and Newton method of [18] are used for the new model. At the end of this section, some remarks are given about how to obtain an initial guess for Newton method. In Section 4, numerical experiments are shown to demonstrate the efficiency of the new model and the algorithms. For every example, the initial values of level set function and the values of some parameters are shown. The differences between one-variable model and two-variable model are compared in this section, too. In Section 5, we make some conclusions on this new model.

2. One-variable Mumford-Shah Model with PCLSM

Firstly, We shall recall PCLSM of [7]. The essential idea of PCLSM of [7] is to use a piecewise constant level set function to identify the subdomains. Assume that we need to partition the domain Ω into subdomains Ω_i, $i = 1, 2, \ldots, n$, and the number of subdomains is a priori known. In order to identify the subdomains, we try to identify a piecewise constant level set function ϕ such that

$$\phi = i \quad \text{in } \Omega_i, \quad i = 1, 2, \ldots, n. \tag{1}$$

Thus, for any given partition $\{\Omega_i\}_{i=1}^n$ of the domain Ω, it corresponds to a unique PCLS function ϕ which takes the values $1, 2, \cdots, n$. Associated with such a level set function ϕ, the characteristic functions of the subdomains are given as

$$\psi_i = \frac{1}{\alpha_i} \prod_{j=1, j\neq i}^{n} (\phi - j), \quad \alpha_i = \prod_{k=1, k\neq i}^{n} (i - k). \tag{2}$$

If ϕ is given as in (1), then we have $\psi_i(x) = 1$ for $x \in \Omega_i$, and $\psi_i(x) = 0$ elsewhere. We can use the characteristic functions to extract geometrical information for the subdomains and the interfaces between the subdomains. For example,

$$\text{Length}(\partial\Omega_i) = \int_\Omega |\nabla \psi_i| dx, \quad \text{Area}(\Omega_i) = \int_\Omega \psi_i dx. \tag{3}$$

Define

$$K(\phi) = (\phi - 1)(\phi - 2)\cdots(\phi - n) = \prod_{i=1}^{n} (\phi - i). \tag{4}$$

At every point in Ω, the level set function ϕ satisfies

$$K(\phi) = 0. \tag{5}$$

This level set idea has been used for Mumford-Shah image segmentation in [7]. For a given digital image $u_0 : \Omega \mapsto R$, which may be corrupted by noise and blurred, the piecewise constant Mumford-Shah segmentation model is to find subdomains Ω_i and constant values c_i by minimizing:

$$\sum_i \int_{\Omega_i} |c_i - u_0|^2 dx + \beta \text{ Length } (\Gamma). \tag{6}$$

The curve Γ is the one that separates the domain Ω into subdomains Ω_i such that $\Omega = \bigcup_i \Omega_i \cup \Gamma$ and $\Gamma = \bigcup_i \partial\Omega_i$. In [2], the traditional level set idea of [13] was used to represent the curve Γ and to solve the problem (6). In [7, 16, 18], the PCLSM were used for identifying the regions and the constant values. The minimization problem there is:

$$\min_{\substack{\mathbf{c},\phi \\ K(\phi)=0}} \Big\{ F(\mathbf{c}, \phi) = \frac{1}{2} \int_\Omega |u(\mathbf{c}, \phi) - u_0|^2 dx + \beta \int_\Omega |\nabla \phi| dx \Big\},$$