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ON MATRIX AND DETERMINANT IDENTITIES FOR COMPOSITE FUNCTIONS*

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Abstract

We present some matrix and determinant identities for the divided differences of the composite functions, which generalize the divided difference form of Faà di Bruno's formula. Some recent published identities can be regarded as special cases of our results.

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1. Introduction

The Bell polynomials which are studied extensively by Bell [2] have played a very important role in combinatorial analysis. Many sequences such as the Stirling numbers and the Lah numbers are special values of the Bell polynomials. Recently, Abbas and Bouroubi [1] proposed two new methods for determining new identities for the Bell polynomials, and Yang [20] generalized one of the methods. By studying the matrices related to the Bell polynomials, Wang and Wang [16] gave a unified approach to various lower triangular matrices such as the Stirling matrices of the first kind and of the second kind [5,6], the Lah matrix [14] and so on. The Faà di Bruno formula [7,8] on higher derivatives of composite functions has wide applications in many branches of mathematics, notably in numerical analysis [9,15,17]. The Bell polynomial is one of the representation tools of the Faà di Bruno's formula. Chu [3] used the properties of the Bell polynomials and the Faà di Bruno formula to obtain several classical determinant identities for composite functions which generalized Mina's identities and their extensions due to Kedlaya [11] and Wilf [19].

Divided differences as the coefficients in a Newton form arise in numerical analysis, which also have applications in the study of spline interpolation and polynomial interpolation. Given a function h, for distinct points $x_0, x_1, \cdots, x_n$, the divided differences of h are defined recursively as

$$
\begin{aligned}
&h[x_0] = h(x_0),\\
&h[x_0, x_1, \cdots, x_n] = \frac{h[x_0, x_1, \cdots, x_{n-1}] - h[x_1, x_2, \cdots, x_n]}{x_0 - x_n}, \qquad n \geq 1.
\end{aligned}
$$

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By extending the definition for $h[x_0, x_1, \cdots, x_n]$ in the case of distinct arguments, we have a similar formula for $x_0 \le x_1 \le \cdots \le x_n$ as follows

$$h[x_0, x_1, \cdots, x_n] = \begin{cases} \dfrac{h[x_0, x_1, \cdots, x_{n-1}] - h[x_1, x_2, \cdots, x_n]}{x_0 - x_n}, & \text{if } x_n \neq x_0, \\ \dfrac{h^{(n)}(x_0)}{n!}, & \text{if } x_n = x_0. \end{cases}$$

For more basic properties of divided differences, one can refer to a recent reference [4]. In [18], Wang and Wang proposed a divided difference form of Faà di Bruno's formula to obtain an explicit divided difference formula for a composite function.They also provided a new proof of Faà di Bruno's formula. It is also noted that Floater and Lyche [10] independently obtained similar results.

Usually, derivatives can be understood as the limit of divided differences. That is to say, divided differences may be taken as discrete derivatives. This gives a motivation to present some matrix identities and determinant identities for the divided differences of the composite functions. This paper is organized as follows. In Section 2, some notation will be introduced. Section 3 is devoted to the matrix identities for composite functions with divided differences. Using the divided difference form of the Faà di Bruno's formula, Section 4 presents some determinant identities for composite functions.

2. Notation

In this section we will introduce some notation. Let the composite function $h(t) = f(g(t))$. Then the divided difference form of Faà di Bruno's formula [10,18] is described as follows.

Proposition 2.1. *For $n \ge 1$, if f and g are sufficiently smooth functions, then*

$$h[t_0, t_1, \cdots, t_n] = \sum_{m=1}^{n} f[g(t_0), g(t_1), \cdots, g(t_m)] \times \sum_{m=\nu_0 \le \nu_1 \le \cdots \le \nu_m = n} \prod_{i=1}^{m} g[t_{i-1}, t_{\nu_{i-1}}, t_{\nu_{i-1}+1}, \cdots, t_{\nu_i}].$$

In particular, Faà di Bruno's formula holds when $t_0 = t_1 = \cdots = t_n = t$, namely,

$$h^{(n)}(t) = \sum_{m=1}^{n} f^{(m)}(g(t)) B_{n,m}\Big(g'(t), g''(t), \cdots, g^{(n)}(t)\Big),$$

where $B_{n,m}$ is the exponential partial Bell polynomial defined as

$$B_{n,m}(x_1, x_2, \cdots, x_n) = \sum_{\substack{c_1+2c_2+\cdots+nc_n=n \\ c_1+c_2+\cdots+c_n=m}} \frac{n!}{c_1!(1!)^{c_1} c_2!(2!)^{c_2} \cdots c_n!(n!)^{c_n}} x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}.$$

Further, we define

$$e_{n,m}(\phi, \{t_i\}_{i=0}^{n}) = \begin{cases} \displaystyle\sum_{m=\nu_0 \le \nu_1 \le \cdots \le \nu_m = n} \prod_{i=1}^{m} \phi[t_{i-1}, t_{\nu_{i-1}}, t_{\nu_{i-1}+1}, \cdots, t_{\nu_i}], & 1 \le m \le n, \\ 0, & m > n \text{ or } 0 = m < n, \\ 1, & m = n = 0. \end{cases} \tag{2.1}$$

Write

$$e_{n,m}(\phi,t)=\sum_{\substack{k_1+k_2+\cdots k_m=n\\ k_1,k_2,\cdots,k_m\geq 1}}\prod_{i=1}^{m}\frac{\phi^{(k_i)}(t)}{k_i!},\qquad 1\leq m\leq n,$$

and for $n\geq 1$ put $e_{n,0}(\phi,t)=0$, $e_{0,0}(\phi,t)=1$ by convention. Moreover, denote

$$B_{n,m}(\phi,t)=B_{n,m}(\phi'(t),\cdots,\phi^{(n)}(t)).$$

If $t_0=t_1=\cdots=t_n=t$, then $e_{n,m}(\phi,\{t_i\}_{i=0}^{n})$ reduces to $e_{n,m}(\phi,t)$ and we have

$$e_{n,m}(\phi,t)=\frac{m!}{n!}B_{n,m}(\phi,t). \tag{2.2}$$

Hence, from Eq. (2.1), Proposition 2.1 can be rewritten as

$$h[t_0,t_1,\cdots,t_n]=\sum_{m=1}^{n}f[g(t_0),g(t_1),\cdots,g(t_m)]e_{n,m}(g,\{t_i\}_{i=0}^{n}). \tag{2.3}$$

In addition, let the lower triangular matrices be of the form:

$$E_n(\phi,\{t_i\}_{i=0}^{n})=\begin{pmatrix} e_{1,1}(\phi,\{t_i\}_{i=0}^{1}) & 0 & \cdots & 0\\ e_{2,1}(\phi,\{t_i\}_{i=0}^{2}) & e_{2,2}(\phi,\{t_i\}_{i=0}^{2}) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ e_{n,1}(\phi,\{t_i\}_{i=0}^{n}) & e_{n,2}(\phi,\{t_i\}_{i=0}^{n}) & \cdots & e_{n,n}(\phi,\{t_i\}_{i=0}^{n}) \end{pmatrix},$$

$$E(\phi,\{t_i\}_{i=0}^{\infty})=\begin{pmatrix} e_{1,1}(\phi,\{t_i\}_{i=0}^{1}) & 0 & \cdots & 0\\ e_{2,1}(\phi,\{t_i\}_{i=0}^{2}) & e_{2,2}(\phi,\{t_i\}_{i=0}^{2}) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

$$E_n(\phi,t)=\begin{pmatrix} e_{1,1}(\phi,t) & 0 & \cdots & 0\\ e_{2,1}(\phi,t) & e_{2,2}(\phi,t) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ e_{n,1}(\phi,t) & e_{n,2}(\phi,t) & \cdots & e_{n,n}(\phi,t) \end{pmatrix},$$

$$E(\phi,t)=\begin{pmatrix} e_{1,1}(\phi,t) & 0 & \cdots & 0\\ e_{2,1}(\phi,t) & e_{2,2}(\phi,t) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

$$B_n(\phi,t)=\begin{pmatrix} B_{1,1}(\phi,t) & 0 & \cdots & 0\\ B_{2,1}(\phi,t) & B_{2,2}(\phi,t) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ B_{n,1}(\phi,t) & B_{n,2}(\phi,t) & \cdots & B_{n,n}(\phi,t) \end{pmatrix},$$

$$B(\phi,t)=\begin{pmatrix} B_{1,1}(\phi,t) & 0 & \cdots & 0\\ B_{2,1}(\phi,t) & B_{2,2}(\phi,t) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

and write the determinant

$$\det_{1\leq i,j\leq n}(a_{ij})=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

3. Matrix Identities for Composite Functions

This section deals with some interesting matrix identities for composite functions. It is well known that the Leibniz formula for higher derivatives is basic and important in calculus. A divided difference form of this formula is given [13] is stated below.

Lemma 3.1. *Let $\chi(x) = \varphi(x)\psi(x)$. If φ and ψ are sufficiently smooth functions, then for arbitrary nodes $x_0, x_1, \cdots, x_n$, we have*

$$\chi[x_0, x_1, \cdots, x_n] = \sum_{\nu=0}^{n} \varphi[x_0, x_1, \cdots, x_\nu]\psi[x_\nu, x_{\nu+1}, \cdots, x_n].$$

The above is called the Steffensen formula which is a generalization of the Leibniz formula. If $x_0 = x_1 = \cdots = x_n = x$, then the Leibniz formula holds, namely,

$$\chi^{(n)}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} \varphi^{(\nu)}(x)\psi^{(n-\nu)}(x).$$

Hence, using the Steffensen formula, we have the following theorem.

Theorem 3.1. *Let $h(t) = f(g(t))$. For arbitrary nodes $t_0, t_1, \cdots, t_n$, if f and g are sufficiently smooth functions, then we have*

$$E_n(h, \{t_i\}_{i=0}^n) = E_n(g, \{t_i\}_{i=0}^n)E_n(f, \{g(t_i)\}_{i=0}^n). \tag{3.1}$$

In the special case $t_0 = t_1 = \cdots = t_n = t$, we have

$$E_n(h, t) = E_n(g, t)E_n(f, g(t)),$$
$$B_n(h, t) = B_n(g, t)B_n(f, g(t)).$$

Proof. Assume initially that the t_i are distinct. Let $\omega_0(t) = 1$, $\omega_k(t) = \prod_{i=0}^{k-1}(t - t_i)$ for $k \geq 1$. The Newton interpolation of the function $P_{h,r}(t) = \prod_{i=0}^{r-1}(h(t) - h(t_i))$, $r \geq 1$, at the nodes $t_0, t_1, \cdots, t_n$ can be written as

$$N_n(P_{h,r}, \{t_i\}_{i=0}^n; t) = \sum_{k=0}^{n} P_{h,r}[t_0, \cdots, t_k]\omega_k(t).$$

Using the Steffensen formula, we have

$$P_{h,r}[t_0, \cdots, t_k] = \sum_{j=0}^{k} \omega_r[t_0, \cdots, t_j]H[t_j, \cdots, t_k],$$

where $H(t) = \prod_{i=0}^{r-1} h[t, t_i]$. Since

$$\omega_r[t_0, \cdots, t_j] = \begin{cases} 0, & j \neq r, \\ 1, & j = r, \end{cases}$$

it follows that

$$P_{h,r}[t_0, \cdots, t_k] = \begin{cases} 0, & k < r, \\ H[t_r, \cdots, t_k] = e_{k,r}(h, \{t_i\}_{i=0}^k), & k \geq r. \end{cases}$$

Thus, the Newton interpolation of $P_{h,r}(t)$ at the nodes $t_0, \cdots, t_n$ can be rewritten as

$$N_n(P_{h,r}, \{t_i\}_{i=0}^n; t) = \sum_{k=r}^{n} e_{k,r}(h, \{t_i\}_{i=0}^k)\omega_k(t).$$

Let $x = g(t), x_i = g(t_i), i = 0, 1, \cdots, n$. Then

$$P_{h,r}(t) = \prod_{i=0}^{r-1}(f(x) - f(x_i)).$$

Similarly, the Newton interpolation of $P_{h,r}(t)$ at the nodes $x_0, \cdots, x_n$ is

$$N_n\Big(P_{h,r}, \{x_i\}_{i=0}^n; x\Big) = \sum_{\nu=r}^{n} e_{\nu,r}\Big(f, \{x_i\}_{i=0}^{\nu}\Big)\omega_\nu(x) = \sum_{\nu=r}^{n} e_{\nu,r}\Big(f, \{g(t_i)\}_{i=0}^{\nu}\Big)P_{g,\nu}(t), \tag{3.2}$$

where $P_{g,\nu}(t) = \prod_{i=0}^{\nu-1}(g(t) - g(t_i))$. Replacing $P_{g,\nu}(t)$ in (3.2) by its Newton's polynomial $\sum_{k=\nu}^{n} e_{k,\nu}(g, \{t_i\}_{i=0}^k)\omega_k(t)$ yields

$$\begin{aligned} P_n(t) &= \sum_{\nu=r}^{n} e_{\nu,r}(f, \{g(t_i)\}_{i=0}^{\nu}) \sum_{k=\nu}^{n} e_{k,\nu}(g, \{t_i\}_{i=0}^k)\omega_k(t) \\ &= \sum_{k=r}^{n} \omega_k(t) \sum_{\nu=r}^{k} e_{k,\nu}(g, \{t_i\}_{i=0}^k) e_{\nu,r}(f, \{g(t_i)\}_{i=0}^{\nu}), \end{aligned}$$

which is a polynomial of degree less than $n + 1$. It is not difficult to check that

$$P_n(t_i) = N_n(P_{h,r}, \{t_i\}_{i=0}^n; t_i), \quad i = 0, 1, \cdots, n.$$

Therefore, by the uniqueness of the interpolation polynomials, one readily has

$$P_n(t) = N_n(P_{h,r}, \{t_i\}_{i=0}^n; t).$$

Hence, equating the coefficients of $N_n(P_{h,r}, \{t_i\}_{i=0}^n; t)$ and $P_n(t)$ yields

$$e_{k,r}(h, \{t_i\}_{i=0}^k) = \sum_{\nu=r}^{k} e_{k,\nu}(g, \{t_i\}_{i=0}^k) e_{\nu,r}(f, \{g(t_i)\}_{i=0}^{\nu}). \tag{3.3}$$

We can let the t_i coalesce in (3.3) provided g is smooth enough. This completes the proof. □

Theorem 3.1 can be rewritten as

$$E_n(h, \{t_i\}_{i=0}^n)^T = E_n(f, \{g(t_i)\}_{i=0}^n)^T E_n(g, \{t_i\}_{i=0}^n)^T.$$

The above form seems better than (3.1) since $h = f \circ g$. From Eq. (3.3), we also obtain another matrix identity.

Theorem 3.2. *Let $h(t) = f(g(t))$. For arbitrary nodes $t_0, t_1, \cdots$, if f and g are sufficiently smooth functions, then we have*

$$E(h, \{t_i\}_{i=0}^\infty) = E(g, \{t_i\}_{i=0}^\infty) E(f, \{g(t_i)\}_{i=0}^\infty). \tag{3.4}$$

In the special case $t_0 = t_1 = \cdots = t$, we have

$$\begin{aligned} E(h, t) &= E(g, t)E(f, g(t)), \\ B(h, t) &= B(g, t)B(f, g(t)). \end{aligned}$$

It is worth noting that, from the first columns of matrices $E_n(h, \{t_i\}_{i=0}^n)$ and $E(h, \{t_i\}_{i=0}^\infty)$, we can recover the divided difference form of the Faà di Bruno's formula because $e_{n,1}(f, \{t_i\}_{i=0}^n) = f[t_0, t_1, \cdots, t_n]$. More generally, assume that $\alpha \geq 0$ is an integer. We let $f^{<0>}(t) = t$ and denote by $f^{<\alpha>}$ the $\alpha - 1$ times compositions for the function f. For example, $f^{<1>} = f$, $f^{<2>} = f \circ f$. Then we have

$$E_n\left(f^{<\alpha>}, \{t_i\}_{i=0}^n\right) = \prod_{k=0}^{\alpha-1} E_n\left(f, \left\{f^{<k>}(t_i)\right\}_{i=0}^n\right),$$
$$E\left(f^{<\alpha>}, \{t_i\}_{i=0}^\infty\right) = \prod_{k=0}^{\alpha-1} E\left(f, \left\{f^{<k>}(t_i)\right\}_{i=0}^\infty\right).$$

Thus, we only need to compute the elements in the first columns of $E_n(f^{<\alpha>}, \{t_i\}_{i=0}^n)$ or $E(f^{<\alpha>}, \{t_i\}_{i=0}^\infty)$, and can get the explicit expressions for arbitrary order divided differences of $f^{<\alpha>}$. For instance, if $\alpha = 3$, then we have

$$\begin{aligned} & f^{<3>}[t_0, t_1, \cdots, t_n] \\ = & \sum_{m=1}^n f\left[f^{<2>}(t_0), f^{<2>}(t_1), \cdots, f^{<2>}(t_m)\right] \sum_{k=m}^n e_{n,k}\left(f, \{t_i\}_{i=0}^n\right) e_{k,m}\left(f, \{f(t_i)\}_{i=0}^k\right). \end{aligned}$$

Further, if we denote by $f^{<-1>}$ the inverse function of f, then it follows that

$$I_n = E_n\left(f, \{t_i\}_{i=0}^n\right) E_n\left(f^{<-1>}, \{f(t_i)\}_{i=0}^n\right),$$
$$I_n = E_n\left(f^{<-1>}, \{t_i\}_{i=0}^n\right) E_n\left(f, \left\{f^{<-1>}(t_i)\right\}_{i=0}^n\right),$$

where I_n is the identity matrix of order n. Similarly, we have

$$I = E\left(f, \{t_i\}_{i=0}^\infty\right) E\left(f^{<-1>}, \{f(t_i)\}_{i=0}^\infty\right),$$
$$I = E\left(f^{<-1>}, \{t_i\}_{i=0}^\infty\right) E\left(f, \left\{f^{<-1>}(t_i)\right\}_{i=0}^\infty\right),$$

where I is the infinite identity matrix.

4. Determinant Identities for Composite Functions

As early as in 1900s, Mina [12] proposed a beautiful determinant identity as follows

$$\det_{0 \leq i,j \leq n}\left[\frac{d^j}{dx^j}\left(f(x)\right)^i\right] = \left(f'(x)\right)^{\frac{n(n+1)}{2}} \prod_{k=0}^n k!.$$

Recently, several extensions of this classical result, involving formal power series and the Vandermonde determinant, have appeared in the literature [3,11,19]. In this section, by mean of the divided difference form of the Faà di Bruno's formula, we give some new determinant identities via divided differences, which include recent results as special cases.

Theorem 4.1. *Let $h_i(t) = f_i(g(t))$, $i = 0, 1, \cdots, n$. If $f_i(t)$ and $g(t)$ are smooth enough, then we have*

$$\det_{0 \leq i,j \leq n}\Big[h_i[t_0, t_1, \cdots, t_j]\Big] = \det_{0 \leq i,j \leq n}\Big[f_i[g(t_0), g(t_1), \cdots, g(t_j)]\Big] \prod_{0 \leq i < j \leq n} g[t_i, t_j],$$

$$\det_{1\le i,j\le n}\left[h_i[t_0,t_1,\cdots,t_j]\right]=\det_{1\le i,j\le n}\left[f_i[g(t_0),g(t_1),\cdots,g(t_j)]\right]\prod_{0\le i<j\le n}g[t_i,t_j].$$

Proof. We only prove the first identity because the proof of the second identity is similar to that of the first one. By definition, we have

$$e_{j,m}(g,\{t_k\}_{k=0}^j)=\sum_{m=\nu_0\le\nu_1\le\cdots\le\nu_m=j}\prod_{k=1}^m g[t_{k-1},t_{\nu_{k-1}},t_{\nu_{k-1}+1},\cdots,t_{\nu_k}]=0,$$

for $0=m<j$ or $m>j$. By Proposition 2.1 one readily verifies that

$$\begin{aligned}&h_i[t_0,t_1,\cdots,t_j]\\ =&\sum_{m=0}^n f_i[g(t_0),g(t_1),\cdots,g(t_m)]\sum_{m=\nu_0\le\nu_1\le\cdots\le\nu_m=j}\prod_{k=1}^m g[t_{k-1},t_{\nu_{k-1}},t_{\nu_{k-1}+1},\cdots,t_{\nu_k}]\\ =&\sum_{m=0}^n f_i[g(t_0),g(t_1),\cdots,g(t_m)]e_{j,m}(g,\{t_k\}_{k=0}^j),\end{aligned}$$

where $0\le i\le n$, $1\le j\le n$. Hence

$$\det_{0\le i,j\le n}\left[h_i[t_0,t_1,\cdots,t_j]\right]=\det_{0\le i,j\le n}\left[f_i[g(t_0),g(t_1),\cdots,g(t_j)]\right]\det_{0\le i,j\le n}\left[e_{i,j}(g,\{t_k\}_{k=0}^i)\right].$$

Noting that $\det_{0\le i,j\le n}[e_{i,j}(g,\{t_k\}_{k=0}^i)]$ is upper triangular,

$$e_{0,0}(g,\{t_k\}_{k=0}^0)=1,\quad e_{j,j}(g,\{t_k\}_{k=0}^j)=\prod_{k=1}^j g[t_{k-1},t_j]\quad\text{for }\ 1\le j\le n,$$

we obtain

$$\det_{0\le i,j\le n}\left[h_i[t_0,t_1,\cdots,t_j]\right]=\det_{0\le i,j\le n}\left[f_i[g(t_0),g(t_1),\cdots,g(t_j)]\right]\prod_{0\le i<j\le n}g[t_i,t_j].$$

Therefore, the proof is complete. □

Letting $t_0=t_1=\cdots=t_n=t$ in Theorem 4.1, we get the following two identities due to Chu [3] obtained by using the Faà di Bruno's formula.

Corollary 4.1. *If $f_i(t)$, $i=0,1,\cdots,n$, and $g(t)$ are smooth enough, then*

$$\det_{0\le i,j\le n}\left[\frac{d^j}{dt^j}h_i(t)\right]=\det_{0\le i,j\le n}\left[f_i^{(j)}(g(t))\right](g'(t))^{\frac{n(n+1)}{2}},\tag{4.1}$$

$$\det_{1\le i,j\le n}\left[\frac{d^j}{dt^j}h_i(t)\right]=\det_{1\le i,j\le n}\left[f_i^{(j)}(g(t))\right](g'(t))^{\frac{n(n+1)}{2}}.\tag{4.2}$$

Theorem 4.2. *If $g(t)$ is smooth enough and $g_i(t)=(g(t))^i$, $i=0,1,\cdots,n$, then*

$$\det_{0\le i,j\le n}\left[g_i[t_0,t_1,\cdots,t_j]\right]=\prod_{0\le i<j\le n}g[t_i,t_j],\tag{4.3}$$

$$\det_{1\le i,j\le n}\left[g_i[t_0,t_1,\cdots,t_j]\right]=\prod_{0\le i<j\le n}g[t_i,t_j].\tag{4.4}$$

In particular, if $t_0 = t_1 = \cdots = t_n = t$*, then*

$$\det_{0\le i,j\le n}\left[\frac{d^j}{dt^j}g_i(t)\right] = \det_{1\le i,j\le n}\left[\frac{d^j}{dt^j}g_i(t)\right] = (g'(t))^{\frac{n(n+1)}{2}}\prod_{k=1}^{n}k!.$$

Proof. We take $f_i(x) = x^i, i = 0,1,\cdots,n$. Consider the divided difference of $f_i(x)$ at the nodes $g(t_0), g(t_1),\cdots,g(t_j)$, $j = 0,1,\cdots n$, respectively. By the fundamental properties of divided differences we find that, for $j > i$, $f_i[g(t_0), g(t_1),\cdots,g(t_j)]$ is equal to 0 and $f_i[g(t_0), g(t_1),\cdots,g(t_j)]$ is equal to 1 when $j = i$. Thus, $\det_{0\le i,j\le n}[f_i[g(t_0), g(t_1),\cdots,g(t_j)]]$ is lower triangular and all of the terms on the main diagonal are equal to 1. Since g_i can be viewed as the composition of f_i and g, it follows from Theorem 4.1 that

$$\det_{0\le i,j\le n}\left[g_i[t_0,t_1,\cdots,t_j]\right] = \prod_{0\le i<j\le n} g[t_i,t_j].$$

Clearly, $g_0[t_0] = 1$ and $g_0[t_0,t_1,\cdots,t_j] = 0,\ j = 0,1,\cdots,n$. Hence, the identity (4.4) is also obtained and the proof is complete. □

In fact, when $t_0,t_1,\cdots,t_n$ are distinct, $\prod_{0\le i<j\le n} g[t_i,t_j]$ can be regarded as a quotient of two Vandermonde determinants which are $\det_{0\le i,j\le n}[(g(t_i))^j]$ and $\det_{0\le i,j\le n}[t_i^j]$. In the rest of this section we will present weighted determinant identities.

Theorem 4.3. *Let* $h_i(t) = f_i(g(t))$ *and* $s_{ij}(t) = h_i(t)\overline{\omega}_j(t)$, $i,j = 0,1,\cdots,n$*. For* $0 \le i \le n$*, if* $f_i,\overline{\omega}_i$ *and* g *are smooth enough, then*

$$\begin{aligned}&\det_{0\le i,j\le n}\left[s_{ij}[t_0,t_1,\cdots,t_j]\right]\\ &= \det_{0\le i,j\le n}\left[f_i[g(t_0),g(t_1),\cdots,g(t_j)]\right]\prod_{0\le i<j\le n} g[t_i,t_j]\prod_{k=0}^{n}\overline{\omega}_k(t_k).\end{aligned} \tag{4.5}$$

In particular, for $t_0 = t_1 = \cdots = t_n = t$*, we have*

$$\det_{0\le i,j\le n}\left[\frac{d^j}{dt^j}s_{ij}(t)\right] = \det_{0\le i,j\le n}\left[f_i^{(j)}(g(t))\right](g'(t))^{\frac{n(n+1)}{2}}\prod_{k=0}^{n}\overline{\omega}_k(t).$$

Proof. By the Steffensen formula, it follows that

$$s_{ij}[t_0,t_1,\cdots,t_j] = \sum_{k=0}^{j} h_i[t_0,\cdots,t_k]\cdot\overline{\omega}_j[t_k,\cdots,t_j].$$

Hence,

$$\det_{0\le i,j\le n}\left[s_{ij}[t_0,t_1,\cdots,t_j]\right] = \det_{0\le i,j\le n}\left[h_i[t_0,t_1,\cdots,t_j]\right]\det_{0\le i,j\le n}[a_{i,j}],$$

where $\det_{0\le i,j\le n}[a_{i,j}]$ is upper triangular, and $a_{i,i} = \overline{\omega}_i(t_i)$, $i = 0,1,\cdots,n$. Then

$$\det_{0\le i,j\le n}[a_{i,j}] = \prod_{k=0}^{n}\overline{\omega}_k(t_k). \tag{4.6}$$

From Theorem 4.1, there holds

$$\det_{0\le i,j\le n}\Big[h_i[t_0,t_1,\cdots,t_j]\Big]=\det_{0\le i,j\le n}\Big[f_i[g(t_0),g(t_1),\cdots,g(t_j)]\Big]\prod_{0\le i<j\le n}g[t_i,t_j]. \tag{4.7}$$

In view of (4.6) and (4.7), we obtain (4.5). This completes the proof. □

Now we establish a similar identity with weaker weight functions as follows.

Theorem 4.4. *Let $h_i(t):=f_i(g(t))$, and $s_{ij}(t)=h_i(t)\overline{\omega}_j(t)$, $i,j=1,2,\cdots,n$. If the weight function $\overline{\omega}_i$ is a polynomial of degree less than i, f_i and g are smooth enough for $1\le i\le n$, then*

$$\begin{aligned}&\det_{1\le i,j\le n}\Big[s_{ij}[t_0,t_1,\cdots,t_j]\Big]\\&=\det_{1\le i,j\le n}\Big[f_i[g(t_0),g(t_1),\cdots,g(t_j)]\Big]\prod_{0\le i<j\le n}g[t_i,t_j]\prod_{k=1}^{n}\overline{\omega}_k(t_k).\end{aligned}$$

In particular, for $t_0=t_1=\cdots=t_n=t$, we have

$$\det_{1\le i,j\le n}\left[\frac{d^j}{dt^j}s_{ij}(t)\right]=\det_{1\le i,j\le n}\Big[f_i^{(j)}(g(t))\Big](g'(t))^{\frac{n(n+1)}{2}}\prod_{k=1}^{n}\overline{\omega}_k(t).$$

Proof. The key of the proof is that, for $i<j$,

$$\overline{\omega}_i[t_0,t_1,\cdots,t_j]=0.$$

The rest of the proof is similar to that of Theorem 4.3. □

Finally, we close this section by stating two more theorems without giving proofs.

Theorem 4.5. *For $i,j=0,1,\cdots,n$, let $g_i(t)=(g(t))^i$ and $s_{ij}(t)=g_i(t)\overline{\omega}_j(t)$. If g and $\overline{\omega}_i$ $(i=0,1,\cdots,n)$ are smooth enough, then*

$$\det_{0\le i,j\le n}\Big[s_{ij}[t_0,t_1,\cdots,t_j]\Big]=\prod_{0\le i<j\le n}g[t_i,t_j]\prod_{k=0}^{n}\overline{\omega}_k(t_k).$$

Especially for $t_0=t_1=\cdots=t_n=t$, we have

$$\det_{0\le i,j\le n}\left[\frac{d^j}{dt^j}s_{ij}(t)\right]=(g'(t))^{\frac{n(n+1)}{2}}\prod_{k=0}^{n}k!\overline{\omega}_k(t).$$

Theorem 4.6. *For $i,j=1,\cdots,n$, let $g_i(t)=(g(t))^i$ and $s_{ij}(t)=g_i(t)\overline{\omega}_j(t)$. If g is smooth enough and $\overline{\omega}_i$ is a polynomial of degree less than i for each $1\le i\le n$, then*

$$\det_{1\le i,j\le n}\Big[s_{ij}[t_0,t_1,\cdots,t_j]\Big]=\prod_{0\le i<j\le n}g[t_i,t_j]\prod_{k=1}^{n}\overline{\omega}_k(t_k).$$

Especially for $t_0=t_1=\cdots=t_n=t$, we have

$$\det_{1\le i,j\le n}\left[\frac{d^j}{dt^j}s_{ij}(t)\right]=(g'(t))^{\frac{n(n+1)}{2}}\prod_{k=1}^{n}k!\overline{\omega}_k(t).$$

More determinant identities on derivatives of composite functions and formal power series iterations can be found in [3].

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