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USING THE SKEW-SYMMETRIC ITERATIVE METHODS FOR SOLUTION OF AN INDEFINITE NONSYMMETRIC LINEAR SYSTEMS*

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Abstract

The concept of the field of value to localize the spectrum of the iteration matrices of the skew-symmetric iterative methods is further exploited. Obtained formulas are derived to relate the fields of values of the original matrix and the iteration matrix. This allows us to determine theoretically that indefinite nonsymmetric linear systems can be solved by this class of iterative methods.

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1. Introduction

Consider a system of linear equations

$$Au = f, \tag{1.1}$$

where A is an $N \times N$ matrix, $u = (u_1, ..., u_N)^T$ is the vector of solution, $f = (f_1, ..., f_N)^T$ is the vector of the right-hand side. The system of linear equations (1.1) can be solved by the following iterative method [9]:

$$B(\omega)\frac{y_{k+1} - y_k}{\tau} + Ay_k = f, \tag{1.2}$$

where $A, B(\omega)$ are nonsingular matrices of size $N \times N$ in a finite-dimensional Euclidean space. In the following $B(\omega)$ will be shortly denoted by B, whenever there is no possible confusion. We rewrite (1.2) as

$$y_{k+1} = Gy_k + \tilde{f}, \quad \tilde{f} = \tau B^{-1} f,$$

where

$$G = I - \tau B^{-1} A \tag{1.3}$$

is the corresponding iteration matrix.

Let λ_j be an eigenvalue of the matrix $F = B^{-1}A$, μ_j be an eigenvalue of G, and $Sp(G)$ be the set of all eigenvalues (the spectrum) of G in (1.3). Investigation of convergence of iterative methods can be used in two ways:

(i) Spectral estimate, where we research spectrum of G, [12] and require that

$$|\mu_j| < 1, \quad j = 1, ..., n; \tag{1.4}$$

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(ii) Energy-norm estimate, where we check norm of G, [9] and demand that

$$\|G\|_D < 1, \tag{1.5}$$

with D being Hermitian positive definite matrix. The first one provides necessary and sufficient conditions and the second one only sufficient conditions.

Using (1.4) we have

$$\mu_j = 1 - \tau\lambda_j, \quad \tau > 0,$$

and, according to [12], a necessary and sufficient condition for the convergence of the iterative method is

$$|1 - \tau\lambda_j| < 1. \tag{1.6}$$

Let

$$\Re(\lambda) = \min_j Re(\lambda_j), \quad \rho^2 = \max_j |\lambda_j|^2.$$

Then by taking into account the introduced notation, from (1.6) we have

$$\tau < \frac{2\Re(\lambda)}{\rho^2}. \tag{1.7}$$

See [7] for details.

Definition 1.1. *Matrix F is named positive stable if $Re(\lambda_j(F)) > 0, \forall j = 1, \ldots, n$.*

Theorem 1.1. *If the matrix $F = B^{-1}A$ is positive stable, then the iterative method* (1.2) *converges if the condition* (1.7) *holds.*

To investigate convergence of iterative methods we have to know eigen-property about matrix F, a product of the matrices A and B, which determinates the iterative method. It was shown above that if F is positive stable, then the method converges. For example, if the matrix A is positive real and the matrix B is symmetric positive definite, then from Lyapunov theorem [3] we know that F is a positive stable matrix. So, we obtain convergence for a wide class of iterative methods (Jacobi, SSOR, CG and so on).

Energy-norm [9] and spectral [12] sufficient conditions were previously obtained for convergence of skew-symmetric iterative method in [6] and [2], when the initial matrix A is positive real. If the matrix A losses this property, the question about sufficient conditions of convergence remains open. Unfortunately, there is no discussion about the convergence in the sense of spectral or energy-norm. We propose to use field-of-values to solve this problem.

The field-of-values [3] of the matrix A is defined as

$$H(A) = \{x^*Ax : x \in \mathcal{C}^n, x^*x = 1\}.$$

It is known [4] that

$$Sp(A) \subseteq H(A).$$

We will begin with short description of skew-symmetric iterative methods (SSM), then we will show that the field-of-values works for investigating the convergence of SSM, using formulas from [7], which connect fields-of-values of the matrices A, B, their symmetric part A_0 and skew-symmetric part A_1, and eigenvalues of the iteration matrix G. These formulas will be used to derive sufficient conditions for convergence about matrix A when it is not positive real.