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A PRIORI ERROR ESTIMATES FOR LEAST-SQUARES MIXED FINITE ELEMENT APPROXIMATION OF ELLIPTIC OPTIMAL CONTROL PROBLEMS*

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Abstract

In this paper, a constrained distributed optimal control problem governed by a first-order elliptic system is considered. Least-squares mixed finite element methods, which are not subject to the Ladyzhenkaya-Babuska-Brezzi consistency condition, are used for solving the elliptic system with two unknown state variables. By adopting the Lagrange multiplier approach, continuous and discrete optimality systems including a primal state equation, an adjoint state equation, and a variational inequality for the optimal control are derived, respectively. Both the discrete state equation and discrete adjoint state equation yield a symmetric and positive definite linear algebraic system. Thus, the popular solvers such as preconditioned conjugate gradient (PCG) and algebraic multi-grid (AMG) can be used for rapid solution. Optimal a priori error estimates are obtained, respectively, for the control function in $L^2(\Omega)$-norm, for the original state and adjoint state in $H^1(\Omega)$-norm, and for the flux state and adjoint flux state in $H(\mathrm{div};\Omega)$-norm. Finally, we use one numerical example to validate the theoretical findings.

Mathematics subject classification: 49K20, 49M25, 65N15, 65N30.
Key words: Optimal control, Least-squares mixed finite element methods, First-order elliptic system, A priori error estimates.

1. Introduction

Optimal control problems governed by partial differential equations (PDEs) are playing an increasingly important role in modern scientific and engineering applications. Basically, the goal of optimal control is achieving some desired objective. Nowadays, a variety of finite element methods are widely used in solving such optimal control problems. Systematic introductions of the finite element method for PDEs and optimal control problems can be found in, e.g., [9, 18, 19, 23, 25].

In this paper, we shall consider the distributed optimal control problem for elliptic equations via least-squares mixed finite element methods. Let us consider the following second-order elliptic boundary value problem

$$\begin{cases} -\mathrm{div}(\mathcal{A}\nabla y) = f + u, & \text{in } \Omega, \\ y = 0, & \text{on } \Gamma_D, \\ -\mathcal{A}\nabla y \cdot \boldsymbol{n} = 0, & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\Gamma = \Gamma_D \cup \Gamma_N$, such that Γ_D is nonempty and $\boldsymbol{n}$ is the outward unit normal to Γ_N.

Introducing the flux $\boldsymbol{\sigma} = -\mathcal{A}\nabla y$, we derive the equivalent first-order elliptic system

$$\begin{cases} \mathrm{div}\boldsymbol{\sigma} = f + u, & \text{in } \Omega, \\ \boldsymbol{\sigma} + \mathcal{A}\nabla y = 0, & \text{in } \Omega, \\ y = 0, & \text{on } \Gamma_D, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = 0, & \text{on } \Gamma_N. \end{cases} \tag{1.2}$$

In this work, we consider the following type quadratic cost functional

$$\mathcal{J}(y, \boldsymbol{\sigma}, u) = \frac{1}{2}\Big(\int_\Omega (y - y_d)^2 + \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\sigma}_d)^2 + \gamma \int_\Omega u^2 \Big), \tag{1.3}$$

over the admissible control set U_{ad}

$$U_{ad} = \{u \in L^2(\Omega) : \xi_1 \le u \le \xi_2,\ a.e. \text{ in } \Omega\}, \tag{1.4}$$

where the bounds ξ_1, $\xi_2 \in \mathbb{R}$ fulfill $\xi_1 < \xi_2$.

The optimal control problem we considered is to seek optimal state variables y^* and $\boldsymbol{\sigma}^*$, and optimal control u^* in the admissible set U_{ad}, such that the functional (1.3) is minimized subject to problem (1.2), where y_d and $\boldsymbol{\sigma}_d$ are two given desired states. The positive penalty parameter γ can be used to change the relative importance of the terms appearing in the definition of the functional. A precise formulation of this problem including a functional analytic setting is given in the next section.

Recently, classical mixed finite element methods have been proved effectively for solving various optimal control problems, see, e.g., [7, 8, 12, 26]. In summary, they have an advantage of approximating the unknown scaler variable and its diffusive flux simultaneously. Besides, these methods can approximate the unknown variable and its flux to a same order of accuracy. However, It is well-known that these methods usually result in typical saddle-point type linear algebraic systems that are symmetric but indefinite. Although significant progress has been made in the development of methods for such algebraic systems, their numerical solution is still challenging and computationally demanding. Furthermore, the spaces used for the approximation of the different unknowns y and $\boldsymbol{\sigma}$ must satisfy strict Ladyzhenkaya-Babuska-Brezzi consistency condition.

To circumvent those difficulties appeared in using classical mixed finite element methods, least-squares mixed finite element method, based on transforming a second-order PDE into a first-order system, was introduced by Pehlivanov et al. [20], where a least-squares residual minimization is introduced for the mixed system in unknown variable y and unknown velocity flux $\boldsymbol{\sigma}$. Subsequently, there has been an extensive research of first-order type least-squares mixed finite element methods for various problems and different definition of minimization functionals, see, e.g., [4–6, 10, 11, 15, 21, 22]. It is well known that least-squares mixed finite element methods have two typical advantages: First, they are not subjected to the Ladyzhenkaya-Babuska-Brezzi consistency condition, so the choice of finite element spaces becomes flexible. Second, these methods result in a symmetric and positive definite system, which can be solved using PCG or AMG solvers quickly.

Least-squares methods to optimal control problems governed by a first-order *div-curl* elliptic system was first studied in [13], where a least-squares minimization functional, including both