Journal of Computational Mathematics
Vol.33, No.2, 2015, 158–178.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1211-m4267

MIXED DISCONTINUOUS GALERKIN TIME-STEPPING METHOD FOR LINEAR PARABOLIC OPTIMAL CONTROL PROBLEMS*

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Abstract

In this paper, we discuss the mixed discontinuous Galerkin (DG) finite element approximation to linear parabolic optimal control problems. For the state variables and the co-state variables, the discontinuous finite element method is used for the time discretization and the Raviart-Thomas mixed finite element method is used for the space discretization. We do not discretize the space of admissible control but implicitly utilize the relation between co-state and control for the discretization of the control. We derive a priori error estimates for the lowest order mixed DG finite element approximation. Moveover, for the element of arbitrary order in space and time, we derive a posteriori $L^2(0,T;L^2(\Omega))$ error estimates for the scalar functions, assuming that only the underlying mesh is static. Finally, we present an example to confirm the theoretical result on a priori error estimates.

Mathematics subject classification: 35K10, 65N30.
Key words: A priori error estimates, A posteriori error estimates, Mixed finite element, Discontinuous Galerkin method, Parabolic control problems.

1. Introduction

It is well known that finite element approximation of the optimal control problems has been an important and hot topic in engineering design, and has been extensively studied, see, e.g., [13, 14, 19, 23, 30]. For the optimal control problems governed by elliptic or parabolic state equations, a priori error estimates of finite element approximations were studied in, e.g., [1, 12, 18, 22, 24, 25, 29]. There also exist lots of works concentrating on the adaptivity of various optimal control problems, see, e.g., [12, 22–25].

In many control problems, the objective functional contains the gradient of the state variables. For example, in the flow control problem, the gradient stands for Dracy velocity and it is an important physics variable, or, in the temperature control problem, large temperature gradients during cooling or heating may lead to its destruction. Thus, the accuracy of the gradient is important in the numerical approximation of the state equations. In the finite element community, mixed finite element methods should be used for discretization of the state

* Received September 23, 2012 / Revised version received October 9, 2014 / Accepted November 17, 2014 / Published online March 13, 2015 /

equations in such cases, since both the scalar variable and its flux variable can be approximated in the same accuracy by using mixed finite element methods. In computational optimal control, mixed finite element methods are not as widely used as in engineering simulations. Recently, Chen [4] studied a priori error estimates and superconvergence of RT0 mixed finite element methods for elliptic optimal control problems, and used the RT projection operator and the superconvergence properties of mixed finite element methods for elliptic problems to derive the superconvergence properties of the control, the state and the co-state. However, the convergence order is $h^{3/2}$ since the analysis was restricted by the low regularity of the control. Furthermore, using the postprocessing technique, the author derived the superconvergence of all variables. In [3], Chen used the postprocessing projection operator, which was defined by Meyer and Rösch [26], to prove a quadratic superconvergence result of the control with mixed finite element methods, while Chen and Liu [7] considered a posteriori error estimates for linear elliptic optimal control problems by RT mixed finite element methods. In [5, 6], the authors considered error estimates and superconvergence of RT mixed finite element methods for optimal control problems governed by semilinear elliptic equations. In those paper, L^∞-error estimates and H^{-1}-error estimates are derived, respectively. For a priori error error estimates and a posteriori error estimates of mixed finite element methods for parabolic optimal control problems, see, e.g., [8, 33].

In recent years, the discontinuous Galerkin (DG) discretization has been proved useful in computing time-dependent convection and diffusion equations; see, e.g., [10,11] for the DG time-stepping method where only time discretization is discontinuous. It will be simply referred as to the DG method in this paper, although we are aware that there exist several DG discretization schemes in the literature. Furthermore this method has been found useful in computing optimal control of parabolic equations; see, e.g., [27, 28]. However, there is a lack of error analysis for the DG approximation combined with mixed finite element approximation for linear parabolic optimal control problems.

We consider the following linear-quadratic optimal control problems:

$$\min_{u\in K\subset X}\left\{\frac{1}{2}\int_0^T\left(\|\boldsymbol{p}-\boldsymbol{p}_d\|_{0,\Omega}^2+\|y-y_d\|_{0,\Omega}^2+\|u\|_{0,\Omega}^2\right)dt\right\} \tag{1.1}$$

subject to the state equation

$$y_t+\mathrm{div}\boldsymbol{p}=f+u,\qquad x\in\Omega,\ t\in I, \tag{1.2}$$

$$\boldsymbol{p}=-A\nabla y,\qquad x\in\Omega,\ t\in I, \tag{1.3}$$

$$y=0,\qquad x\in\partial\Omega,\ t\in I, \tag{1.4}$$

$$y(0)=y_0(x),\qquad x\in\Omega, \tag{1.5}$$

where Ω is a convex polygon domain in $\mathbb{R}^2$ and $I=[0,T]$. We assume that $f,\ y_d\in L^2(I;L^2(\Omega))$, $\boldsymbol{p}_d\in L^2(I;(L^2(\Omega))^2)$ and $y_0\in H_0^1(\Omega)$. Moreover, we assume that the coefficient matrix $A(x)=(a_{ij}(x))_{2\times 2}\in W^{1,\infty}(\bar{\Omega};\mathbb{R}^{2\times 2})$ is a symmetric positive definite matrix. $K\subset X=L^2(I;L^2(\Omega))$ is a set defined by

$$K=\left\{u\in X:\ \int_0^T\int_\Omega u\ dxdt\geq 0\right\}.$$

The purpose of this work is to investigate the mixed DG finite element methods for linear parabolic optimal control problems. Firstly, we use the piecewise constant functions to discretize the time variable t and use the lowest order Raviart-Thomas mixed finite element to