Journal of Computational Mathematics
Vol.34, No.2, 2016, 159–173.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1510-m2014-0085

OPTIMAL SOLVER FOR MORLEY ELEMENT DISCRETIZATION OF BIHARMONIC EQUATION ON SHAPE-REGULAR GRIDS*

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Abstract

This paper presents an optimal solver for the Morley element problem for the boundary-value problem of the biharmonic equation by decomposing it into several subproblems and solving these subproblems optimally. The optimality of the proposed method is mathematically proved for general shape-regular grids.

Mathematics subject classification: 65F08, 65N30, 65N99
Key words: Biharmonic equation, Morley element, Optimal solver, Precondition, Exact sequence.

1. Introduction

The boundary-value problems of the biharmonic equation are frequently encountered in solid and fluid mechanics and material sciences. Various finite element methods have been developed for discretizing the boundary-value problems; these methods lead to ill-conditioned linear systems with condition numbers of the order $\mathcal{O}(h^{-4})$, which are difficult to solve. The Morley element [23] is among the simplest [32], and it has been proven numerically and theoretically to be well-suited to an adaptively generated mesh [18, 30]. In this paper, we study the numerical solution of the Morley element system for the boundary-value problem of the biharmonic equation discretized on shape-regular grids.

Multilevel methods are among the most efficient techniques for solving linear systems. They can be used in the design of direct iterative methods and in the design of preconditioners for other kinds of iterative schemes. In particular, geometric multigrid methods, which are based on a nested sequence of multilevel grids, have been extensively studied for the Morley element problem ([8,25,28,43] and references therein). The efficiency of these methods, however, depends crucially on the multilevel structures of the underlying grids. Because such structures are not naturally available in most unstructured grids, multigrid methods of this type are generally quite difficult to use in practice. For the linear systems generated by other finite elements, similar conditions prevail [5, 11, 16, 33, 34, 38, 39]. Methods that do not rely on the underlying grid including algebraic multigrid methods have also been studied in the literature [6, 7, 22, 27, 29].

* Received October 8, 2014 / Accepted October 29, 2015 /
Published online March 6, 2016 /

When applied to the fourth-order finite element problem, however, these methods are not very efficient. Further, there is still no theory to support methods of this type. Recently, a mathematically provable optimal solver for fourth order problems discretised on grids without hierarchical structure was presented in [41], where optimal preconditioners are designed for various finite element systems with unified formulation. In this paper, instead of designing an iterative method or a preconditioner directly in a unified way, we solve the Morley element problem specifically by decomposing it into three subproblems and solving these subproblems with provably optimal solvers.

At the continuous level, the boundary-value problem of the biharmonic equation can be decomposed into two Poisson problems and one Stokes problem. Similar to the connection between the biharmonic problem and the Stokes problem [13], this decomposition depends on the exact relation between $H_0^2(\Omega)$ and $H_0^1(\Omega)^2$ whereby the curl space of $H_0^2(\Omega)$ is just the kernel of div in $(H^1(\Omega))^2$. A discrete decomposition associated with some exact relation between discrete spaces is expected when the continuous spaces are discretised respectively. Fortunately, a similar exact relation can be found between the Morley element space and the vector Crouzeix-Raviart element space (see Lemma 2.2 and [12]). Consequently, the Morley element problem for the biharmonic problem is decomposed into Morley element problems for the Poisson equation and Crouzeix-Raviart element problem for the Stokes problem. These subproblems are decoupled from each other. Decompositions of this type were first studied by Arnold and Falk [2] for a modified Morley element problem (see historical remark in Section 2). Huang [21] studied the decomposition of the original Morley element problem and designed a fast solver for the subproblems based on the AMG algorithm, and so did Huang et. al. [20]. In the present paper, we construct solvers for the subproblems in the framework of the fast auxiliary space preconditioning (FASP) method with the aid of a provably optimal Poisson solver, and an provably optimal solver for the original Morley element problem is thus obtained.

We note that a discrete Poisson system generated by Morley element is solved in this algorithm. It is known that the Morley element does not provide a consistent approximation for $H^1(\Omega)$. However, the Morley element scheme for Poisson equation can find its application this way. Moreover, the approximation and stability of the Morley element scheme for the Poisson equation are sufficient for it to be optimally preconditioned by other convergent schemes, particularly the linear element scheme. Thus a provably optimal solver for the Morley element problem for the Poisson equation is established. This same strategy works for other finite element problems for the Poisson equation as well. The Crouzeix-Raviart element problem for the Stokes problem can be provably optimally solved with the aid of the linear element problem for the Poisson equation on general shape-regular grids. For Stokes problem on multilevel grids, the optimal-order multigrid solver has been designed ([9]). The solver reported in the present paper can still be expected to be practically useful, as the Crouzeix-Raviart element problem for the Stokes problem is numerically and theoretically suited to the adaptively generated grids [19].

The rest of the paper is organized as follows. In Section 2, we describe the model problems and show their decomposition into subproblems. In this paper, we focus ourselves on the Dirichlet boundary value problem of the biharmonic equation, which is referred by "Dirichlet biharmonic problem" in the sequel. In Section 3, our optimal solvers for the subproblems are introduced. Both theoretical analysis and numerical experiments are given. Finally, some concluding remarks are given in Section 4.