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TWO-STEP SCHEME FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, a stochastic linear two-step scheme has been presented to approximate backward stochastic differential equations (BSDEs). A necessary and sufficient condition is given to judge the $\mathbb{L}_2$-stability of our numerical schemes. This stochastic linear two-step method possesses a family of 3-order convergence schemes in the sense of strong stability. The coefficients in the numerical methods are inferred based on the constraints of strong stability and n-order accuracy ($n \in \mathbb{N}^+$). Numerical experiments illustrate that the scheme is an efficient probabilistic numerical method.

Mathematics subject classification: 60H35, 60H10, 65C05.
Key words: Backward stochastic differential equation, Stochastic linear two-step scheme, Local truncation error, Stability and convergence.

1. Introduction

Probabilistic numerical methods to solve BSDEs could rely on time discretization of the stochastic process and approximations of the conditional expectations. This paper is not concerned with the analysis of methods which calculate the conditional expectations but with the discretization procedure. As far as we all know, Zhang [22] developed a modified explicit Euler scheme to solve BSDEs and demonstrated that the convergence order of this modified explicit Euler scheme is $\frac{1}{2}$. Bouchard and Touzi [2] presented an implicit Euler scheme based on [22]. In [7], Gobet and Labart generalized Zhang's result such that the convergence order of error estimates of the scheme reached 1 in the sense of strong $\mathbb{L}_p (p \geq 1)$. Even for the linear regression multi-step forward dynamic programming (MDP) scheme [10], its better convergence order is also the same as that of [7]. Thus, if time discretization of BSDEs relies on the Euler scheme (explicit or implicit), the order of discretization errors locates in $[\frac{1}{2}, 1]$. Obviously, we cannot obtain a high order time discretization of BSDEs via the Euler scheme. To overcome this problem, the high order schemes [4, 5, 12] are developed for BSDEs. In the analysis of Section 4.4 of [10], its authors also maintained that it is significant to develop high-order discretization error schemes for BSDEs.

Naturally, there is an interesting question, namely what is the highest order of discretization error of the multi-step method? Owing to the fact that the estimation of the highest order in the framework of the multi-step method is complex, we try to demonstrate the highest order in the framework of stochastic linear two-step method of BSDEs in this paper. And this research is conducive to investigating the highest order in the case of the stochastic linear multi-step method of BSDEs.

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Now, we adopt the stochastic linear two-step scheme to compute the solution of the BSDE in the form as below, for $i = N-2, N-3, \cdots, 0$

$$\begin{cases} Y_i^\pi = \mathbb{E}_i\big[\alpha_1 Y_{i+1}^\pi + \alpha_2 Y_{i+2}^\pi + h(\beta_0 f_i + \beta_1 f_{i+1} + \beta_2 f_{i+2})\big], \\ Z_i^\pi = \mathbb{E}_i\big[\frac{2}{h} Y_{i+1}^\pi (W_{i+1} - W_i)^\top - \frac{1}{2h} Y_{i+2}^\pi (W_{i+2} - W_i)^\top\big], \end{cases} \tag{1.1}$$

where $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_{t_i}]; f_i = f(t_i, X_i^\pi, Y_i^\pi, Z_i^\pi)$; $\alpha_1, \alpha_2, \beta_0, \beta_1$ and β_2 are real numbers that will be given in the following section; h is the time step.

In what follows, the previous papers with respect to the multi-step schemes of BSDEs are provided as the number of steps becomes 2.

A kind of high order multi-step schemes for BSDEs are presented in [23]. If the number of steps is 2, the scheme can be written as

$$\begin{cases} Y_i^\pi = \mathbb{E}_i[\frac{4}{3} Y_{i+1}^\pi - \frac{1}{3} Y_{i+2}^\pi] + \frac{2h}{3} f_i, \\ Z_i^\pi = \mathbb{E}_i\big[\frac{2}{h} Y_{i+1}^\pi (W_{i+1} - W_i)^\top - \frac{1}{2h} Y_{i+2}^\pi (W_{i+2} - W_i)^\top\big]. \end{cases} \tag{1.2}$$

The convergence order of scheme (1.2) can attain 2 (see [21]). Our scheme (1.1) has better estimates than those of (1.2) although they have the same numerical scheme with respect to Z. It is natural because the scheme (1.1) has more accurate estimates about $\mathbb{E}_i[\int_{t_i}^{t_{i+2}} f(s, X_s, Y_s, Z_s)dt]$.

In [10], authors developed the multi-step forward dynamic programming equation to implement the approximation of solutions of BSDEs. If the number of steps is 2, this multi-step is given by, for all $i \in \{N-1, \cdots, 0\}$

$$\begin{cases} Y_i^\pi = \mathbb{E}_i\big[Y_{i+2}^\pi + f(t_i, X_i^\pi, Y_{i+1}^\pi, Z_i^\pi)h + f(t_{i+1}, X_{i+1}^\pi, Y_{i+2}^\pi, Z_{i+1}^\pi)h\big], \\ hZ_i^\pi = \mathbb{E}_i\Big[\big(Y_{i+2}^\pi + f(t_{i+1}, X_{i+1}^\pi, Y_{i+2}^\pi, Z_{i+1}^\pi)h\big)\Delta W_i^\top\Big]. \end{cases} \tag{1.3}$$

It is straightforward that the scheme in [15] and the scheme (1.3) are coincided by using the tower property of conditional expectations. Hence, the scheme (1.3) is not a high order scheme. And the highest convergence order of the scheme (1.3) can attain 1.

In [4], the linear multi-step schemes are provided for BSDEs. As the number of steps is 2, this linear multi-step scheme is given by, for all $i \in \{N-2, \cdots, 0\}$,

$$\begin{cases} Y_i^\pi = \mathbb{E}_i\left[\sum_{j=1}^2 a_j Y_{i+j}^\pi + h\sum_{j=0}^2 b_j f\left(Y_{i+j}^\pi, Z_{i+j}^\pi\right)\right], \\ Z_i^\pi = \mathbb{E}_i\left[\sum_{j=1}^2 \alpha_j H_{i,j}^Y Y_{i+j}^\pi + h\sum_{j=1}^2 \beta_j H_{i,j}^f f\left(Y_{i+j}^\pi, Z_{i+j}^\pi\right)\right], \end{cases} \tag{1.4}$$

where $a_j, b_j, \alpha_j, \beta_j$ are real numbers; $H_{i,j}^\psi = \left(\frac{1}{jh}\int_{t_i}^{t_{i+j}} \psi^l(\frac{u-t_i}{jh})dW_u^l\right)_{1\le l\le d}; (\psi^l)_{1\le l\le d} \in \mathcal{B}_{[0,1]}^m$; $\mathcal{B}_{[0,1]}^m$ denotes the set of bounded measurable function. And its convergence order is 2.

In this paper, our major contributions are: (i) the relationship between $\mathbb{L}_2$-stability of this scheme and the Dahlquist's root condition is rigorously presented, namely necessary and sufficient conditions. (ii) the scheme (1.1) has a class of 3-order convergence schemes in the sense of strong stability. This conclusion is presented as $\beta_0 \neq 0$ and we maintain that the condition $2h\beta_0 L_f < 1$ should be held to ensure the existence of iterations. Meanwhile, if $\beta_0 = 0$, we deduce that the scheme (1.1) has a kind of 2-order convergence schemes in the sense of strong stability. (iii) the relationship between α_1, α_2 and $\beta_0, \beta_1, \beta_2$ is also ascertained under the constraints of strong stability and convergence.

The paper is organized as follows. In Section 2, we present some fundamental definitions and properties that are used in the following sections. And we show the stochastic linear two-step scheme for BSDEs. The solvability of Y is firstly established under $\beta_0 \neq 0$. Secondly,