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# THE NYSTRÖM METHOD FOR ELASTIC WAVE SCATTERING BY UNBOUNDED ROUGH SURFACES*

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**Abstract**

We consider a numerical algorithm for the two-dimensional time-harmonic elastic wave scattering by unbounded rough surfaces with Dirichlet boundary condition. A Nyström method is proposed for the scattering problem based on the integral equation method. Convergence of the Nyström method is established with convergence rate depending on the smoothness of the rough surfaces. In doing so, a crucial role is played by analyzing the singularities of the kernels of the relevant boundary integral operators. Numerical experiments are presented to demonstrate the effectiveness of the method.

*Mathematics subject classification:* 35P25, 45P05.
*Key words:* Elastic wave scattering, Unbounded rough surface, Nyström method.

## 1. Introduction

We consider the two-dimensional time-harmonic elastic scattering problem for unbounded rough surfaces with Dirichlet boundary condition. This kind of problem has attracted much attention since it has a wide range of applications in diverse scientific areas such as seismology and non-destructive testing.

Given the incident wave and the rough surface, the direct scattering problem is to determine the distribution of the scattered wave and develop an efficient algorithm to simulate the scattered wave. The well-posedness of the acoustic scattering problems by unbounded rough surfaces have been extensively studied via either a variational approach [5, 8–10] or the integral equation method [11, 12, 26, 27, 29] based on the classical Fredholm theory [15, 22] or the generalized Fredholm theory [13]. When applying the integral equation method, the direct scattering

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problem can be transformed to an equivalent second-kind integral equation with logarithmic singular periodic kernels. One of the most efficient method for the numerical solution to this kind of integral equation is the Nyström method, which was first applied to the acoustic scattering by sound-soft obstacles [14], and has been extended to acoustic scattering by sound-hard obstacles [19]. Compared with the second-kind integral equation defined on a finite interval, a non-trivial extension of the Nyström method is the work [25], which has extended it to deal with the second-kind integral equation defined on the real line. Thus, it enables us to obtain the numerical solution of the acoustic scattering problems by unbounded rough surfaces, see [20,25] for the sound-soft case and [21] for the penetrable case.

Although a large number of results have been obtained for the acoustic case, there are few available results for elastic scattering by unbounded rough surfaces, especially for the computational aspect. The unique solvability of the elastic scattering by rough surfaces with Dirichlet boundary conditions has been established in [2], while the existence result was given in [3] using the boundary integral equation method (see also [1] for a comprehensive discussion). The authors in [18] studied the well-posedness of the elastic scattering problem by unbounded rough surfaces via the variational approach. Some numerical algorithms are proposed to solve elastic scattering problems while the scatterers are bounded obstacles, such as boundary element method [6, 7], spectral algorithm [17, 23] and Nyström method [16, 28]. However, to the best of our knowledge, few numerical algorithms are presented for elastic wave scattering by unbounded rough surfaces.

The purpose of this paper is to develop the Nyström method for two-dimensional time-harmonic elastic scattering problem for unbounded rough surfaces with Dirichlet boundary condition. Our method is based on the integral equation formulations given in [1, 3], which can be reduced to a class of integral equations on the real line. A crucial role of our method is played by a thorough analysis on the singularities of the kernels in the relevant integral equations, which involves the Green tensor for Navier equation in the half-space. By splitting off the logarithmic singularity in the related kernels and using the asymptotic behaviour of the Bessel functions, we obtain the convergence of the Nyström method with convergence rate depending on the smoothness of the rough surfaces. Several numerical examples are presented to verify our theoretical results and show the effectiveness of our method.

The paper is organized as follows. In Section 2, we give a brief introduction to a mathematical model of the scattering problem and present the existed well-posedness result using the integral equation method. Section 3 is devoted to analyzing the singularities for the relevant kernels included in the integral expression of the solution. In Section 4, we establish the convergence of the Nyström method. Numerical experiments are given to show the effectiveness of the proposed method in Section 5. Finally, we give a conclusion in Section 6.

## 2. The Well-Posedness of the Scattering Problem

In this section, we present the existed results for the well-posedness of the two-dimensional elastic wave scattering problem by unbounded rough surfaces. First, we introduce some notations and function spaces used throughout this paper. We will use bold lowercase letters to denote all vectors and vector fields, and use bold capital letters to denote matrices or matrix functions. For a vector $\mathbf{a} = (\mathrm{a}_1, \mathrm{a}_2)^\top$, we define $\mathbf{a}^\perp = (\mathrm{a}_2, -\mathrm{a}_1)^\top$ which is orthogonal to $\mathbf{a}$, and unless otherwise stated, we use $|\mathbf{a}|$ to denote its Euclidean norm. For any $b \in \mathbb{R}$, we define the