Journal of Computational Mathematics
Vol.42, No.6, 2024, 1554–1578.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2303-m2022-0201

BÉZIER SPLINES INTERPOLATION ON STIEFEL AND GRASSMANN MANIFOLDS*

Ines Adouani[1)]
Department of Mathematics, University of Sousse, Tunisia
Email: ines.adouani.issatso@gmail.com
Chafik Samir
LIMOS-CNRS, University of Clermont Auvergne, France
Email: chafik.samir@uca.fr

Abstract

We propose a new method for smoothly interpolating a given set of data points on Grassmann and Stiefel manifolds using a generalization of the De Casteljau algorithm. To that end, we reduce interpolation problem to the classical Euclidean setting, allowing us to directly leverage the extensive toolbox of spline interpolation. The interpolated curve enjoy a number of nice properties: The solution exists and is optimal in many common situations. For applications, the structures with respect to chosen Riemannian metrics are detailed resulting in additional computational advantages.

Mathematics subject classification: 49K15, 65D10, 49K30.
Key words: Optimization, Bézier spline, Curve fitting, Grassmann manifolds, Stiefel manifolds, Canonical metrics.

1. Introduction

Data points on Riemannian manifolds are fundamental objects in many fields including, subspace filtering, machine learning, Signal-image-video processing and medical imaging [6, 7, 22]. To cite but few examples, tracking, face and action recognition and statistical shape analysis [1, 2, 35, 39, 42]. In many real-world applications, Stiefel manifold and Grassmann manifold are most commonly preferred as representation on Riemannian manifolds. A common limitation in many of these applications has been the geometric structure on underlying manifolds, e.g. Grassmann and Stiefel manifolds [1]. As increasingly real-world applications have to deal with non-vector data, a great number of algorithms for manifold embedding and manifold learning have been introduced. Recently, many efforts have been made to develop important geometric and statistical tools: Riemannian exponential map and its inverse, means, distributions, geodesic [6, 9, 25].

Motivated by the success of these approaches, we are interested in the problem of fitting smooth curves to a finite set of data points on a special class of Riemannian manifold $\mathcal{M}$: The Grassmann manifold of all p-dimensional subspace of $\mathbb{R}^n$ and the Stiefel manifold of p-orthonormal vectors in $\mathbb{R}^n$. More precisely, given a finite set of points $X_0, X_1, \cdots, X_N$ in $\mathcal{M}$ and a distinct and ordered instants of time ($0 = t_0 < t_1 < \cdots < t_N = 1$), we seek a spline $\alpha : [0, 1] \to \mathcal{M}$ that best fits the given data $X_0, X_1, \cdots, X_N$ and is sufficiently smooth. More importantly, we focus on the search space of smooth regression splines where data points verify

* Received September 5, 2022 / Revised version received December 14, 2022 / Accepted March 27, 2023 / Published online November 22, 2023 /
1) Corresponding author

orthogonality constraints and t_i are distinct and ordered time instants. We show that this problem occurs in many real situations and leads to specific optimization methods, usually called optimization on Riemannian manifolds [1, 14]. In fact, one way to tackle the two conflicting goals of being sufficiently smooth while passing sufficiently close to the data points at the given t_i, is to express the spline α as the minimizer of the following functional:

$$\begin{aligned} E : \mathcal{W} &\to \mathbb{R}^+ \\ \alpha &\mapsto \frac{\lambda}{2}\int_{t_0}^{t_N} \left\langle \frac{D^2\alpha(t)}{Dt^2}, \frac{D^2\alpha(t)}{Dt^2} \right\rangle_{\mathcal{M}} dt + \frac{1}{2N}\sum_{i=0}^{N} d^2_{\mathcal{M}}\big(\alpha(t_i), X_i\big), \end{aligned} \tag{1.1}$$

where $\mathcal{W}$ denotes the underlying space of admissible C^2 splines in $\mathcal{M}$, $\langle .\,, . \rangle_{\mathcal{M}}$ and $d_{\mathcal{M}}$ denote the Riemannian metric and distance, respectively. We are thus facing an optimization problem in infinite dimension. It is well-known that when the Riemannian manifold $\mathcal{M}$ reduces to the Euclidean space $\mathbb{R}^n$, solutions of the optimization problem (1.1) under the interpolation constraint

$$\sum_{i=0}^{N} d^2_{\mathcal{M}}\big(\alpha(t_i), X_i\big) = 0$$

are cubic splines.

In this paper, we propose a geometric algorithm that generates a solution of problem (1.1), which is expressed as interpolating Bézier splines with a certain degree of smoothness. We adopt the Grassmann manifold as an example of symmetric spaces. Therefore, using the definition of geodesic curves and taking into account the rich and nice structure of these spaces, we present a novel approach to generate a C^2 Bézier spline that interpolates a given data set of points at specified knot times. As for the Stiefel manifold, in short, the task is that of regression on a homogeneous space [1] in the purpose to estimate/predict missing data from few available observations. By observations we mean any data points that can be obtained from temporal acquisitions. For example, medical images at different time instants are usually used to analyze the evolution of a disease. In this context and due to logistic and time constraints, it is very common to store few discrete moments only. Then at each time instant, we have a data point that is represented as an element of a manifold [25]. So there is a need to estimate missing data points on such manifold at non observed time instants. Several discrete-time models on smooth manifolds and Lie groups have been studied in the literature [34]. Here, we consider a continuous-time model only of class C^1 and will address the problem of regularized non-linear regression from finite observations [12]. In the main and in both cases, we start from the energy minimization formulation of linear least-squares in the Euclidean space $\mathbb{R}^n$ and generalize this concept to these manifolds. The proposed method is geometrically simple, extensible and easy to implement. In fact, we illustrate the relevance of the proposed method with various experiments.

In a general fitting problem, a vast number of methods for fitting smooth curves to a given finite set of points on Riemannian manifolds have appeared recently. For instance, motivated by the work of Noakes *et al.* general Riemannian manifolds, the authors in [38] derive a fourth-order differential equation associated to this variational problem involving the covariant derivative and the curvature tensor and prove that the optimal curves are approximating cubic splines. On the other hand, Samir *et al.* [36] choose to solve the optimization problem (1.1) by means of a steepest-descent algorithm on an adequate set of curves where the steepest-descent direction for the energy function E is defined with respect to the first-order and second-order Palais