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CONVERGENCE AND STABILITY OF THE SPLIT-STEP THETA METHOD FOR A CLASS OF STOCHASTIC VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY NOISE*

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Abstract

In this paper, we investigate the theoretical and numerical analysis of the stochastic Volterra integro-differential equations (SVIDEs) driven by Lévy noise. The existence, uniqueness, boundedness and mean square exponential stability of the analytic solutions for SVIDEs driven by Lévy noise are considered. The split-step theta method of SVIDEs driven by Lévy noise is proposed. The boundedness of the numerical solution and strong convergence are proved. Moreover, its mean square exponential stability is obtained. Some numerical examples are given to support the theoretical results.

Mathematics subject classification: 65C30.
Key words: Stochastic Volterra integro-differential equations, Existence and uniqueness, Stability, Split-step theta method, Convergence.

1. Introduction

In this paper, we concern the following SVIDE driven by Lévy noise:

$$\begin{aligned} Y(t) = \varphi(t) &+ \int_0^t f\left(Y(z), \int_0^z \kappa(z-s)Y(s)\,ds\right) dz \\ &+ \int_0^t g\left(Y(z), \int_0^z \kappa(z-s)Y(s)\,ds\right) dw(z) \\ &+ \int_0^t \int_{\mathbf{Z}} \gamma\left(Y(z), \int_0^z \kappa(z-s)Y(s)\,ds, \xi\right) \tilde{N}(dz, d\xi) \end{aligned} \tag{1.1}$$

for $t \in [0,\infty)$, where $\varphi : [0,\infty) \to \mathbb{R}$ and $\|\varphi\|_\infty^2 = \max_{t\in[0,\infty)} |\varphi(t)|^2 < \infty$. Here $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R}\times\mathbb{R}\times\mathbf{Z} \to \mathbb{R}$ are measurable functions. The kernel κ: $[0,\infty) \to \mathbb{R}$ is continuous.

As we known, SVIDE (1.1) can be regarded as an extension of stochastic differential equations (SDEs) (see [14] and the references cited therein) and special types of stochastic Volterra integral equations (SVIEs) (see, e.g. [2, 7] and the references therein). SVIDEs and SDEs are used to mathematically formulate many problems in different kinds of fields. The theoretical analysis of SVIDEs has gained abundant attention in recent decades (see [13, 16] and the references therein).

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In general, explicit solutions of SDEs and SVIDEs are rarely available and we have to resort to numerical methods to gain their approximate solutions. A large number of studies on numerical methods for SDEs have emerged (see, e.g. [5, 6, 15, 17, 24–26]), however, there are only a few numerical results about SVIDEs and SVIEs (see, e.g. [10, 12, 19, 22] and the references therein). Although numerical solutions of SVIEs have attracted more and more attention recently, but the research in this area is still limited. In particular, Liang *et al.* [11] obtained super-convergence of the Euler-Maruyama method for SVIEs. In 2020, we studied theoretical and numerical analysis of the Euler-Maruyama method for the generalized SVIDEs under global Lipschitz condition [23] and a class of SVIDEs with non-globally Lipschitz continuous coefficients [21].

It is necessary to incorporate event driven uncertainty such as market crashes, central bank announcements, changes in credit ratings, defaults, etc. which can have sudden and significant effects on the movements of stock price into a model, and this can be expressed by jumps. The evolution of economics, finance and many other random quantities are often modeled by SVIDEs driven by Lévy noise, which offer the most flexible, numerically accessible mathematical frameworks ([4] and the references therein). Some progress has been made in the recent decades [1, 3, 8, 18, 20].

To the best of our knowledge, due to some new difficulties caused by the stochastic integral (see [9]) and Lévy noise, these are the first results in the literature for such generalized SVIEs driven by Lévy noise.

This paper is organized as follows. We consider the existence, uniqueness, boundedness and mean square exponential stability of the analytic solution of SVIDE (1.1) in Section 2. The split-step theta (SST) method of SVIDE (1.1) is presented and its boundedness, convergence and mean square exponential stability are established in Section 3. Finally, we give some numerical examples in Section 4 to illustrate the theoretical results of SVIDE (1.1).

2. Theoretical Analysis of SVIDE Driven by Lévy Noise

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denote a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ be the expectation corresponding to $\mathbb{P}$. Let $\mathbf{Z} \subseteq \mathbf{R}_+ - \{\mathbf{0}\}$ be the range space of the impulsive jumps. A one-dimensional Brownian motion defined on the probability space is denoted by $w(t)$ and $N(dt, d\xi)$ is a Poisson random measure defined on σ-finite measure space $(\mathbf{Z}, \mathcal{L}, \nu)$ with intensity measure $\nu \not\equiv 0$ for the case when $\nu \equiv 0$. Set

$$\tilde{N}(dt, d\xi) := N(dt, d\xi) - \nu(d\xi)dt,$$

where

$$\nu(d\xi) = \lambda\phi(\xi)d\xi, \quad \phi(\xi) = \frac{1}{\sqrt{2\pi\xi}} \exp\left(-\frac{(\ln\xi)^2}{2}\right), \quad 0 < \xi < \infty.$$

Moreover, we assume that $w(t)$ is independent of $\tilde{N}(t, \cdot)$. The family of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes $\{x(t)\}_{t\in[0,T]}$ such that $\mathbb{E}|x(t)|^p < \infty$ $(p \geq 1)$ is denoted by $\mathcal{L}^p([0,T]; \mathbb{R})$. We denoted by $\mathcal{M}^2([0,T]; \mathbb{R})$ the family of processes $\{x(t)\}_{t\in[0,T]}$ in $\mathcal{L}^2([0,T]; \mathbb{R})$ such that

$$\mathbb{E}\left(\int_0^T |x(t)|^2\, dt\right) < \infty.$$