Journal of Computational Mathematics
Vol.43, No.1, 2025, 1–17.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2307-m2022-0264

A STABILIZER FREE WEAK GALERKIN FINITE ELEMENT METHOD FOR BRINKMAN EQUATIONS*

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Abstract

We develop a stabilizer free weak Galerkin (SFWG) finite element method for Brinkman equations. The main idea is to use high order polynomials to compute the discrete weak gradient and then the stabilizing term is removed from the numerical formulation. The SFWG scheme is very simple and easy to implement on polygonal meshes. We prove the well-posedness of the scheme and derive optimal order error estimates in energy and L^2 norm. The error results are independent of the permeability tensor, hence the SFWG method is stable and accurate for both the Stokes and Darcy dominated problems. Finally, we present some numerical experiments to verify the efficiency and stability of the SFWG method.

Mathematics subject classification: 65N30, 65N15.
Key words: Brinkman equations, Weak Galerkin method, Stabilizer free, Discrete weak differential operators.

1. Introduction

In this paper, we consider the following Brinkman model: Seek unknown fluid velocity $\mathbf{u}$ and pressure p satisfying

$$-\mu\Delta\mathbf{u} + \mu\kappa^{-1}\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega \in \mathbb{R}^d$ is a polygonal ($d=2$) or polyhedral domain ($d=3$), μ is the fluid viscosity coefficient and κ denotes the permeability tensor of the porous medium, $\mathbf{f}$ represents the momentum source term, and the boundary value $\mathbf{g}$ satisfies the compatibility condition $\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n} = 0$.

For simplicity, we consider the Brinkman equations with boundary condition $\mathbf{g} = \mathbf{0}$ and take the viscosity coefficient μ to be 1. Assume that the permeability κ is piecewise constant and there exist two constants $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1\xi^t\xi \le \xi^t\kappa^{-1}\xi \le \lambda_2\xi^t\xi, \quad \forall\, \xi \in \mathbb{R}^d,$$

where ξ is a column vector and ξ^t is the transpose of ξ. We consider that λ_1 is the unit size and λ_2 may be the case of large size.

* Received November 25, 2022 / Revised version received April 6, 2023 / Accepted July 26, 2023 / Published online December 11, 2023 /
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The Brinkman equations (1.1)-(1.3) can be seen as a modified version of Darcy's law obtained by adding viscous forces to the Navier-Stokes equations [5]. This model has been applied in many fields, such as power engineering, petroleum industry, geology, geophysics, and so on [4, 9, 10, 20]. Mathematically speaking, the Brinkman equations have different properties due to the varying permeability tensor κ. When κ is very large, the Brinkman equations are similar to Stokes equations. Conversely, when κ is small and close to zero, the equations are similar to Darcy equations. Therefore, the numerical method designed for Brinkman equations should be efficient and stable for both the Stokes and Darcy equations. To achieve this goal, one natural attempt is to directly apply the existing stable Stokes elements (e.g. Mini-element, $P_2 - P_0$ element, nonconforming Crouzeix-Raviart element) or the stable Darcy element (e.g. Raviart-Thomas element) to the Brinkman equations. However, numerical experiments in [22] show that when applying stable Darcy element the convergence would deteriorate when κ is relatively large and vice versa. To overcome this difficulty, many recent studies have attempted to develop suitable modified elements for Brinkman equations. For instance, Burman *et al.* [6] add stabilizing terms penalizing the jumps on the normal component of the velocity field. Juntunen *et al.* [15] generalize the classical Mini-element, and obtain a stable finite element method for varying permeability. An $H(\text{div})$-conforming element is applied to a geometric multi-grid method [16] based on the DG method. In recent years, some new numerical approaches have been developed for Brinkman equations, for example, virtual element methods [7], hybridizable discontinuous Galerkin method [18, 19], mixed discontinuous Galerkin method [28], weak Galerkin methods [14, 24, 36], and so on.

The weak Galerkin (WG) finite element method is first proposed by Wang and Ye [29] for the second-order elliptic equations. They introduced the weak differential operators to approximate the classical differential operators in the variational form. A unified study on WG methods with other discontinuous Galerkin methods for solving partial differential equations has been presented in [11, 12]. The discrete weak gradient is computed by the RT_k or BDM_k elements, which limits the finite element partition to triangular meshes. In order to extend the partition to polygonal meshes, a stabilizing term is added to the WG scheme in [30]. This stabilized WG finite element method has been applied to various equations, see [13, 21, 23, 25–27, 31, 32]. However, such a stabilizing term also increases the difficulty of theoretical analysis and the complexity of algorithm implementation. Therefore, efforts have been made to remove the stabilizing term from the numerical scheme. A popular and efficient strategy is to raise the degree of the polynomial that approximates the weak gradient [33]. The specific degree of polynomial depends on the number of edges of polygonal meshes. Such a stabilizer free WG method has been applied to Stokes equations [8], parabolic equations [2,37], wave equations [17], biharmonic equations [34], and so on.

The purpose of this paper is to establish a stabilizer free weak Galerkin (SFWG) method for Brinkman equations. Adopting high order piecewise polynomial space to approximate the weak gradient of velocity, we establish a simple numerical scheme on general polygonal meshes without any stabilizing term. Furthermore, we prove the well-posedness of the numerical scheme and derive the optimal order error estimates. The corresponding energy and L^2 error estimates are independent of the permeability κ, so the SFWG method is suitable for both the Stokes and Darcy dominated problems. Besides, in programming, the calculation of the stiffness matrix is simpler and more intuitive since there is no stabilizing term.

The outline of the paper is summarized as follows. In Section 2, we introduce some basic notations and the weak formulation of Brinkman model. Section 3 is devoted to constructing