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IMAGE SPACE BRANCH-REDUCTION-BOUND ALGORITHM FOR GLOBALLY SOLVING THE SUM OF AFFINE RATIOS PROBLEM*

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Abstract

This article presents an image space branch-reduction-bound algorithm for globally solving the sum of affine ratios problem. The algorithm works by solving its equivalent problem, and by using convex hull and concave hull approximation of bilinear function, we can construct the affine relaxation problem of the equivalent problem, which can be used to compute the lower bounds during the branch-and-bound search. By subsequently refining the initial image space rectangle and solving a series of affine relaxation problems, the proposed algorithm is convergent to the global optima of the primal problem. For improving the convergence speed, an image space region reducing method is adopted for compressing the investigated image space rectangle. In addition, the global convergence of the algorithm is proved, and its computational complexity is analyzed. Finally, comparing with some existing methods, numerical results indicate that the algorithm has better computational performance.

Mathematics subject classification: 90C26, 90C32.
Key words: Sum of affine ratios, Global optimization, Affine relaxation problem, Branch-reduction-bound, Computational complexity.

1. Introduction

Consider the following sum of affine ratios problem defined by

$$\begin{aligned} \min G(x) = \sum_{i=1}^{p} \frac{c_i^T x + f_i}{d_i^T x + g_i} \\ \text{s.t.} \quad x \in D = \{x \in R^n \mid Ax \le b\}, \end{aligned} \tag{1.1}$$

where $p \ge 2$, A is an $m \times n$ order real matrix, b is an m dimensional column vector, D is a nonempty bounded polyhedron set, $c_i^T x + f_i$ and $d_i^T x + g_i$ are all affine functions defined over D, and for any $x \in D$, and for each ratio, the denominator $d_i^T x + g_i \ne 0$.

From 1980s, the problem (1.1) has attracted a growing attention of many practitioners and researchers. From the perspective of applications, the general form and special form of the problem (1.1) have a wide range of applications in information theory, optical processing of in-

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formation, macroeconomic planning, cluster analysis, loading problem, optimum transportation plan, game theory problem, optimal paths in graph theory, computer vision, location problem, and so on, see [10–13,17,22,34,35]. From a theoretical point of view, since the objective function of the problem (1.1) is neither quasiconvex nor quasiconcave, that is to say, it is a nonconvex global optimization problem, which usually possesses many local optimal solutions that are not globally optimal. Therefore, there are some important theoretical and computational difficulties in solving the problem (1.1).

Especially in the past 30 years, many algorithms have been proposed and developed for globally solving the problem (1.1). For example, Konno *et al.* [18] and Cambini *et al.* [2] propose separately a parametric simplex algorithm and a parametric linear programming algorithm for solving the problem (1.1) with only two affine ratios terms, and with that the numerator and denominator of each affine ratio are positive over the feasible region. By solving the corresponding equivalent concave minimization problem, Konno and Yamashita [19] propose an outer approximation algorithm for solving the problem (1.1) with the assumptions that all numerators must be nonnegative and all denominators must be positive over the feasible region. By iteratively searching the image space of affine ratios, Falk and Palocsay [4] first propose an image space analysis method for solving the sum of affine ratios problem. Based on the monotonic optimization theory, and by solving a parametric linear programming problem at each iteration, Phuong and Tuy [24] propose a unified monotonic optimization algorithm for globally solving the generalized affine fractional programming problem which includes the sum of affine ratios problem.

In addition, a large number of branch-and-bound algorithms have been also proposed for solving the problem (1.1). For example, Quesada and Grossman [25], and Konno and Fukaishi [16] propose two rectangular branch-and-bound algorithms for solving the problem (1.1) with the assumptions that each numerator must be nonnegative and each denominator must be positive over the feasible region. By using trapezoidal partition and concave envelope bound, Kuno [20, 21] propose two trapezoidal branch-and-bound algorithms for globally solving the problem (1.1) with the assumptions that all numerators and denominators of affine ratios must be positive over the feasible region. By utilizing simplicial partition and Lagrangian duality bound technique, Benson [1] presents a simplicial branch-and-bound duality bound method for globally solving the problem (1.1). By replacing each denominator with the upper bound of its interval, to construct the affine relaxation of the original problem, Ji and Zhang [7] propose a branch-and-bound algorithm for solving the problem (1.1) with that all numerators and denominators of ratios must be positive over the feasible region. Recently, by utilizing the new two-level affine relaxation technique to construct the linear relaxation problem, Jiao *et al.* [14] present a rectangle branch-and-bound algorithm for globally solving the generalized affine multiplicative programming problem which includes the sum of affine ratios problem; by using region division and reduction techniques, Shen *et al.* [27, 29, 31] propose three different polynomial-time approximation algorithms for special cases of the problem (1.1) and the generalized affine fractional programming problem, respectively; by using the well-known concave envelope and convex underestimation method to derive the relaxation problem, Shen *et al.* [28] present a simplicial branch-and-bound algorithm for globally solving the sum of convex-convex ratios problem. For an excellent review of fractional programming algorithms, we can refer to Schaible and Shi [26] and Stancu-Minasian [36].

In this paper, we will present a practical image space branch-and-bound algorithm for globally solving the problem (1.1), where the branch-and-bound search takes place in the image