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A DECOUPLED, LINEARLY IMPLICIT AND UNCONDITIONALLY ENERGY STABLE SCHEME FOR THE COUPLED CAHN-HILLIARD SYSTEMS*

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Abstract

We present a decoupled, linearly implicit numerical scheme with energy stability and mass conservation for solving the coupled Cahn-Hilliard system. The time-discretization is done by leap-frog method with the scalar auxiliary variable (SAV) approach. It only needs to solve three linear equations at each time step, where each unknown variable can be solved independently. It is shown that the semi-discrete scheme has second-order accuracy in the temporal direction. Such convergence results are proved by a rigorous analysis of the boundedness of the numerical solution and the error estimates at different time-level. Numerical examples are presented to further confirm the validity of the methods.

Mathematics subject classification: 65M12, 65M22, 65M70.
Key words: Coupled Cahn-Hilliard system, Leap-frog method, Scalar auxiliary variable, Error estimate.

1. Introduction

In this paper, we consider the coupled Cahn-Hilliard system

$$\frac{\partial u}{\partial t} = M_u \Delta\big(-\epsilon_u^2 \Delta u + f(u,v)\big), \tag{1.1a}$$

$$\frac{\partial v}{\partial t} = M_v \big[\Delta\big(-\epsilon_v^2 \Delta v + g(u,v)\big) - \sigma(v-\bar{v})\big] \tag{1.1b}$$

in $\Omega \times (0,T]$ with periodic boundary conditions and following initial conditions:

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad v(\mathbf{x},0) = v_0(\mathbf{x}), \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$, M_u and M_v are the mobility constants that control the speed of u and v move. ϵ_u and ϵ_v represent the interfacial width between macrophases (described by u) and microphases (described by v). σ is related to the connectivity

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between the two components of the copolymer. $\bar{v} = \int_\Omega v d\mathbf{x}/|\Omega|$ is the mass ratio between two polymers. $f(u,v) = u^3 - u + \alpha v + \beta v^2$ and $g(u,v) = v^3 - v + \alpha u + 2\beta uv$ are the first variational derivative of the double welled potential

$$W(u,v) = \frac{1}{4}(u^2-1)^2 + \frac{1}{4}(v^2-1)^2 + \alpha uv + \beta uv^2,$$

where α and β are two coupling parameters. The coupled Cahn-Hilliard system was proposed to study the phase transition of the mixture of a homopolymer and a copolymer [2, 3]. And it has been widely used in the study of physics and materials.

Theoretical analysis for the coupled Cahn-Hilliard system has been well done recently. Here we refer interested readers to [7, 8, 21] for the existence, uniqueness and regularity of solutions, weak formulation and the well-posedness of the system, and global existence and decay estimates to the system. It is noticed that system (1.1) has two distinguishing features. For one thing, taking the inner product of (1.1a) and (1.1b) with 1, we have

$$\frac{d}{dt}\int_\Omega u d\mathbf{x} = \frac{d}{dt}\int_\Omega v d\mathbf{x} = 0.$$

This indicates that the phase variable's mass are conserved. For another thing, taking the inner product of (1.1a) and (1.1b) with $-(-\Delta)^{-1}u_t$ and $-(-\Delta)^{-1}v_t$, respectively, we can obtain the energy dissipation law

$$\frac{d}{dt}E(u,v) = \int_\Omega \left[\frac{\delta E(u,v)}{\delta u}u_t + \frac{\delta E(u,v)}{\delta v}v_t\right] d\mathbf{x} = -\frac{1}{M_u}\|u_t\|_{-1}^2 - \frac{1}{M_v}\|v_t\|_{-1}^2 \le 0 \tag{1.3}$$

with the energy defined by

$$E(u,v) = \int_\Omega \frac{\epsilon_u^2}{2}|\nabla u|^2 + \frac{\epsilon_v^2}{2}|\nabla v|^2 + W(u,v)d\mathbf{x} + \frac{\sigma}{2}\|(v-\bar{v})\|_{-1}^2, \tag{1.4}$$

where $\|\cdot\|_{-1}$ is the norm defined in H_{per}^{-1}.

In the past several decades, many efforts have been done to develop energy-stable schemes for the coupled Cahn-Hilliard system. Typical ways include the convex splitting approach [9, 10], the stabilized approach [17, 22, 29], the Lagrange multiplier approach [4, 5, 12], the invariant energy quadratization approach [32, 33, 36], the scalar auxiliary variable (SAV) approach [1, 11, 14, 18, 25, 26, 28], the relaxation approach [13, 15, 16] and so on [6, 23, 27, 34, 35, 37]. Generally speaking, a class of couple system was obtained by using the usual time-discretization with the mentioned approaches. For example, Yang and Kim [31] proposed a coupled scheme by using the Lagrange multiplier approach and the second-order backward difference formula (BDF2) method. Li *et al.* [24] presented a linearized implicit and coupled scheme by using the Crank-Nicolson-type method and a nonlinearly stabilized splitting approach. Li and Mei [20] developed a family of coupled schemes by using the BDF2 method and the SAV approach. The coupled schemes required some additional decoupled or iterative methods to get the numerical approximations.

In recent years, there are some decoupled schemes for solving the coupled Cahn-Hilliard system. In [19], a linearly implicit scheme was constructed by using an extension of the typical invariant energy quadratization approach with the Crank-Nicolson method, BDF1 method, and BDF2 method, respectively. In [30], a linearly implicit scheme was constructed by using BDF2 method and an efficient variant of the SAV approach and the energy relaxation technique. However, these references have no convergence results of the decoupled schemes.