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ADAPTIVE STOCHASTIC MESHFREE METHODS FOR OPTIMAL CONTROL PROBLEM GOVERNED BY RANDOM ELLIPTIC EQUATIONS*

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Abstract

In this paper, a radial basis function method combined with the stochastic Galerkin method is considered to approximate elliptic optimal control problem with random coefficients. This radial basis function method is a meshfree approach for solving high dimensional random problem. Firstly, the optimality system of the model problem is derived and represented as a set of deterministic equations in high-dimensional parameter space by finite-dimensional noise assumption. Secondly, the approximation scheme is established by using finite element method for the physical space, and compactly supported radial basis functions for the parameter space. The radial basis functions lead to the sparsity of the stiff matrix with lower condition number. A residual type a posteriori error estimates with lower and upper bounds are derived for the state, co-state and control variables. An adaptive algorithm is developed to deal with the physical and parameter space, respectively. Numerical examples are presented to illustrate the theoretical results.

Mathematics subject classification: 65N15, 65N30, 65N50, 65C20.
Key words: Radial basis function method, Meshfree method, Random elliptic equation, A posteriori error estimates, Stochastic Galerkin (SG) method, Optimal control problem.

1. Introduction

Optimal control problems governed by partial different equations (PDEs) have been the major research topics in applied mathematics and control theory. Numerical methods for deterministic optimal control problems governed by PDEs have been well developed and investigated for several decades, see, e.g. [28, 39, 40, 50, 53, 54, 57]. It is well known that uncertainty, such as the uncertainty of coefficients, boundary conditions or parameters of PDEs, arises in many complex real-world problems which possess lots of physical and engineering interests.

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These problems can be described by different kinds of stochastic partial differential equations (SPDEs), e.g. [4–8, 19]. Hence, it is very important to take into account the uncertainty in optimal control problem governed by SPDEs.

The uniqueness of optimal solution for optimal control problem governed by SPDEs was given in [12] which viewed it as a stochastic saddle point problem. The numerical approximations for optimal control problem governed by SPDEs have been the concerned issues just in recent years. Some efficient numerical methods were discussed, such as [3, 13, 26, 29, 30, 33, 35, 36, 43, 48, 51, 55]. Among them, Alexandrian *et al.* [3] dealt with the optimal control of systems governed by PDE with uncertain parameter fields by using quadratic approximations; Geierbash *et al.* [26], Martin *et al.* [43] presented stochastic gradient method for optimal control with random parameters; Hou *et al.* [30] applied the finite element method to approximate the stochastic optimal control problem; Ulbrich *et al.* [33] gave an approximate robust formulation that employed quadratic model of the involved functions to the design problems with uncertain parameters; Kouri *et al.* [35, 36] considered the trust-region algorithm for PDE-constrained optimization under uncertainty; Rosseel *et al.* [48] used stochastic finite element to the optimal control problem constrained by PDE with uncertain controls. However, in many physical systems, small changes in uncertain parameters can lead to large jumps of the state variables. Nobile *et al.* [44] considered an elliptic PDE describing a Darcy flow in a porous media whose diffusion coefficient is always modeled as a lognormal random field. The covariance structure of this random field is parameterized by a scalar value ν that governs the smoothness of each random field realization by taking different values. In the micro electro mechanical system (MEMS) design [2], since the non-linear nature of the electrostatic driving force, material properties, or geometric parameters would lead to a well known phenomena of pull-in. In order to numerically compute the random solution well for these practical problems, we need to accurately resolve the discontinuities in the random region.

It is well known that adaptive finite element methods are very important to deal with deterministic discontinuities in optimal control problems, see, e.g. [10, 32, 40, 54]. Adaptive finite element approximation uses a posteriori error indicators to guide the mesh refinement or adjustment procedure. Kohls *et al.* [32] presented a unifying framework for the a posteriori error analysis of control constrained linear-quadratic optimal control problems. It only refines the area where the error indicator is larger so that higher density of nodes is distributed over the area where the solution is difficult to be approximated. In practice, there are four major types of adaptive finite element methods, namely, the h-type (mesh refinement), the p-type (order enrichment), the r-type (mesh redistribution), and the h-p type, which is the combination of the h-type and p-type. The research of stochastic adaptive computational methods is most for SPDEs. In [42], an adaptive sparse grid collocation strategy using piecewise multi-linear hierarchical basis functions was developed for the solution of stochastic differential equations. Hierarchical surplus was used as an error indicator to automatically detect the discontinuity region in the parameter space and adaptively refine the collocation points in this region. In [59, 60], an adaptive multi-element generalized polynomial chaos (ME-gPC) method was developed for elliptic PDEs with random data. Local and global a posteriori error estimators were generated by constructing a reduced space using much smaller number of terms in the enhanced polynomial chaos space to capture the approximation errors of the parameter space. The methods in [42, 59, 60] are of h-type adaptive for both the physical space and the parameter space. An adaptive stochastic Galerkin FEM with hierarchical tensor representations for elliptic PDEs with random data was considered in [24] as a continuation of the papers [20–23]. This method