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# A FIRST-ORDER, SEMI-IMPLICIT, AND UNCONDITIONALLY ENERGY-STABLE SCHEME FOR AN INCOMPRESSIBLE FERROHYDRODYNAMICS FLOW*

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**Abstract**

In this paper, we propose and analyze a first-order, semi-implicit, and unconditionally energy-stable scheme for an incompressible ferrohydrodynamics flow. We consider the constitutive equation describing the behavior of magnetic fluid provided by Shliomis, which consists of the Navier-Stokes equation, the magnetization equation, and the magnetostatics equation. By using an existing regularization method, we derive some prior estimates for the solutions. We then bring up a rigorous error analysis of the temporal semi-discretization scheme based on these prior estimates. Through a series of experiments, we verify the convergence and energy stability of the proposed scheme and simulate the behavior of ferrohydrodynamics flow in the background of practical applications.

*Mathematics subject classification:* 65N30, 65N12, 65M60.
*Key words:* Unconditionally energy-stable scheme, Ferrohydrodynamics, Magnetic fluid, Prior estimates, Error analysis.

## 1. Introduction

Ferrohydrodynamics (FHD) studies the hydrodynamics affected by magnetic polarization force. The main type of magnetic fluid, also called ferrofluid, produced by the behavior of FHD is colloidal ferromagnetic fluid. This colloid is made of nanoscale (3-15 nm), ferromagnetic, and single-domain particles suspended in organic solvents or water, including slowly precipitated suspensions. Due to Brownian motion, these ferromagnetic particles stay suspended. Such particles are coated with a molecular layer of dispersant to prevent them from adhering to each other during stirring. Colloidal ferrofluid can maintain its magnetization and liquid fluidity in a magnetic field, which attracts much attention to its distinctiveness. Thus such a ferrofluid can be controlled by an external magnetic field, making them widely used in electronic devices, medicine, industrial technology, and so on [7, 26, 27, 29, 30]. For instance, they are used for separation of industrial scrap metals [34], zero-leakage shaft sealing [6], impactless printing [20], drag reduction [21], ferrofluid pumping [11, 39, 40], etc. Other potentially valuable applications include high-power transformer cooling [8], hyperthermia [17], magnetic drug targeting [1], etc.

The modeling of FHD can be traced back to the 1960s [22], as a study field similar to magnetohydrodynamics (MHD) [32]. At the beginning it is to compare the modeling ideas of FHD with the relatively well-known MHD. Although they are both studying continuum

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fluid mechanics, they are actually very different. In MHD, the body force acting on the fluid is the Lorentz force generated by the current, while in FHD the Kelvin force is caused by magnetization dominates. Likewise, MHD emphasizes the interaction between electromagnetic fields and conductive fluids, while ferrofluids are typically dielectric and absence of free charges. In general, the constitutive equation of MHD combines the Navier-Stokes equation, the Maxwell equation, and Ohm's law [13]. When it comes to FHD modeling, there are currently two widely accepted models: Rosensweig model [29–31] and Shliomis model [35,36]. The Rosensweig model is composed of the Navier-Stokes equation, the angular momentum equation, the magnetization equation, and the magnetostatics equation. The Shliomis model covers the intrinsic spins of ferromagnetic particles through the extra magnetic torque term. So the angular momentum equation vanishes in the governing system.

Currently, the research works on mathematical analysis of FHD are not very rich, although the problem has attracted much attention in recent years. Amirat *et al.* [2, 5] presented some rigorous proof of the existence of global-in-time weak solutions for the regularized FHD model (the magnetization equation was handled by a regularization method), and local-in-time strong solutions for the FHD model [3, 4]. Nochetto *et al.* [25] established the global existence of weak solutions for the Rosensweig model by the DiPerna-Lions theory. Based on the existence of solutions for the FHD problem, our key interest is to discretize it using existing numerical methods and establish numerical analysis. Nochetto *et al.* [24] proposed and analyzed an energy-stable fully-implicit scheme for the Rosensweig model, which was also the first work to provide a stable numerical algorithm for the FHD equations. Wu and Xie [38] proposed and analyzed a fully discrete finite element method for the Shliomis model. In addition, it is worth mentioning that there exists some progress in modeling the diffusion interface model of two-phase ferrofluids and their corresponding numerical algorithms [23, 41].

In this paper, our goal is to design a stable numerical algorithm for the Shliomis model. This is a challenging task. The main difficulties come from:

(i) How to deal with a large number of nonlinear terms in the constitutive equation.

(ii) Ensuring the energy stability of the discrete scheme.

(iii) Establishing a complete error analysis.

We consider providing a first-order, semi-implicit, and unconditionally energy-stable scheme. To obtain error analysis, we supplement some results on the regularity of strong solutions with some reasonable assumptions. This paper only analyzes the error of the temporal semi-discretization instead of the full discretization. If spatial discretization is proposed, such as the finite difference method or finite element method, the error analysis can be theoretically proven. In fact, we have implemented a full discretization scheme (5.1)-(5.5) on the basis of simulating some real applications, resulting in some interesting phenomena of ferrofluid flow and magnetization.

The paper is organized as follows. In Section 2, we introduce the constitute equation of the Shliomis model: preliminaries are introduced in Section 2.1 and some prior estimates are proven in Section 2.2. In Section 3, we propose a first-order, semi-implicit, and unconditionally energy-stable scheme for the Shliomis equations and prove its discrete stability. In Section 4, we carry out the error analysis of the proposed scheme. Finally in Section 5, after computing a smooth solution problem to illustrate the stability and convergence of the proposed scheme, some numerical experiments related to practical applications are presented to emulate the behavior of ferrofluids.