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A WEAK GALERKIN MIXED FINITE ELEMENT METHOD FOR LINEAR ELASTICITY WITHOUT ENFORCED SYMMETRY*

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Abstract

A weak Galerkin mixed finite element method is studied for linear elasticity problems without the requirement of symmetry. The key of numerical methods in mixed formulation is the symmetric constraint of numerical stress. In this paper, we introduce the discrete symmetric weak divergence to ensure the symmetry of numerical stress. The corresponding stabilizer is presented to guarantee the weak continuity. This method does not need extra unknowns. The optimal error estimates in discrete H^1 and L^2 norms are established. The numerical examples in 2D and 3D are presented to demonstrate the efficiency and locking-free property.

Mathematics subject classification: 65M60, 74B10, 74S05.
Key words: Linear elasticity, Discrete symmetric weak divergence, Mixed finite element method, Weak Galerkin finite element method.

1. Introduction

Elasticity problems are the important branches of solid mechanics, which discuss the deformation of elastomer under the external force. Elasticity is used in construction, machinery, chemical industry and other fields. In practical engineering, stress is crucial to study dilation, von Mises stress and so on. Hence, adopting the mixed formulation to solve the elasticity problem is efficient. In this paper, we consider the following linear elasticity problem:

$$-\nabla \cdot \sigma = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\Lambda^{-1}\sigma = \varepsilon(\boldsymbol{u}) \quad \text{in } \Omega, \tag{1.1b}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } \Gamma_D, \tag{1.1c}$$

$$\sigma\mathbf{n} = \boldsymbol{t}_N \quad \text{on } \Gamma_N, \tag{1.1d}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is an open and bounded domain with the Lipschitz continuous boundary $\partial\Omega = \Gamma_D \cup \Gamma_N, \boldsymbol{u}$ is the displacement field, $\sigma(\boldsymbol{u}) = [\sigma_{ij}]_{i,j=1,\ldots,d}$ is the symmetric stress tensor, $\boldsymbol{f}$ is the body force and $\mathbf{n}$ is the unit outward normal of $\partial\Omega$. The notation $\nabla \cdot \sigma(\boldsymbol{u})$ denotes that the divergence operator is applied to each row of $\sigma(\boldsymbol{u})$. For homogeneous and isotropic materials with linear elasticity, there have

$$\sigma = \Lambda\varepsilon(\boldsymbol{u}) = 2\mu\varepsilon(\boldsymbol{u}) + \lambda\nabla \cdot \boldsymbol{u}\,\mathrm{I},$$

$$\Lambda^{-1}\sigma = \frac{1}{2\mu}\sigma - \frac{\lambda}{2\mu(2\mu + d\lambda)}\mathrm{tr}(\sigma)\mathrm{I},$$

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where $\varepsilon(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})/2$ is the strain tensor, $\mathrm{tr}(\sigma)$ is the trace of σ, λ and μ are the Lamé constants such that $0 < \alpha_1 \leq \lambda \leq \alpha_2 < +\infty$ and $0 < \beta_1 \leq \mu \leq \beta_2 < +\infty$. The symbol I is the identity matrix. Here the superscript T denotes the transposition of matrix. Denote the Young's modulus and the Poisson ratio by E and ν, where $E > 0$ and $0 < \nu < 0.5$. For linear plane strain, the Lamé constants are

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{1.2}$$

The key of numerical methods in mixed formulation is the symmetric constraint of stress, which can be classified into two main categories. One is to strongly enforce the symmetry in the finite element space (cf. [1, 2, 4]). The other is to weakly enforce the symmetry. For example, in [3, 11, 19], the different mixed finite element methods were introduced, where the symmetry of the stress tensor was imposed by introducing a Lagrange multiplier. The simple and efficient Hellinger-Reissner type mixed finite element was presented by Viebahn *et al.* [12]. The symmetry of stress was weakly enforced without the necessary of additional degrees of freedom. The theoretical analysis of the inf-sup condition was not straight forward, which was verified by numerical examples. The reduced symmetry elements were developed in [6]. There were two ways to reduce symmetry, that is, based on the Stokes problems in two dimensions and based on the nice property of several interpolation operators in three dimensions. Superconvergent HDG method with weakly symmetric stresses in [8] was analysed, in which the bubble functions were used to derive superconvergent method. The methods in [6,8] based on the following form:

$$-\nabla \cdot \sigma = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.3a}$$

$$\Lambda^{-1}\sigma - \nabla \boldsymbol{u} + \rho = 0 \quad \text{in } \Omega, \tag{1.3b}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } \partial\Omega, \tag{1.3c}$$

where $\rho = (\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^{\mathrm{T}})/2$ is the rotation.

Weak Galerkin finite element method (WG-FEM) was first introduced by Wang and Ye [14] for second order elliptic problems. The main idea is to replace the classical differential operators by the discrete weak differential operators. The piecewise polynomial functions are adopted to approximate the exact solution on each element and each edge. The advantages of WG-FEM are flexibility on the construction of finite element space and mesh generation. There have some WG-FEMs with the mixed formulation to solve linear elasticity problems. Chen and Xie [7] presented the WG-FEM with strong symmetric stresses, which was equivalent to the corresponding HDG algorithm. Wang and Zhang [17] introduced the WG-FEM with optimal polynomial approximation spaces, satisfying symmetry of the weak Galerkin finite element space. Wang *et al.* [16] developed the hybridized WG-FEM, which used the Lagrange multiplier on the boundary of elements to reduce computational costs. Wang *et al.* [18] presented the modified WG-FEM for linear elasticity problems. This method replaced the boundary stress tensor by the average of the interior stress tensor to reduce the degrees of freedom of the linear system. However, the above WG-FEMs required the strongly symmetric stress.

In this paper, the weak Galerkin mixed finite element method without enforced symmetry is studied. The piecewise discontinuous polynomial spaces are adopted to approximate the displacement and stress. The discrete symmetric weak divergence and stabilizer are first introduced to guarantee the symmetry of numerical stress and the weak continuity, respectively. The equivalent algorithm is given to simplify the computation. Our numerical examples are conducted from the following aspects: Optimal convergence rates, locking-free property, and