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NUMERICAL ERGODICITY AND UNIFORM ESTIMATE OF MONOTONE SPDES DRIVEN BY MULTIPLICATIVE NOISE*

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Abstract

We analyze the long-time behavior of numerical schemes for a class of monotone stochastic partial differential equations (SPDEs) driven by multiplicative noise. By deriving several time-independent a priori estimates for the numerical solutions, combined with the ergodic theory of Markov processes, we establish the exponential ergodicity of these schemes with a unique invariant measure, respectively. Applying these results to the stochastic Allen-Cahn equation indicates that these schemes always have at least one invariant measure, respectively, and converge strongly to the exact solution with sharp time-independent rates. We also show that these numerical invariant measures are exponentially ergodic and thus give an affirmative answer to a question proposed in [J. Cui *et al.*, Stochastic Process. Appl., 134 (2021)], provided that the interface thickness is not too small.

Mathematics subject classification: 60H35, 60H15, 65L60.
Key words: Monotone stochastic partial differential equation, Stochastic Allen-Cahn equation, Numerical invariant measure, Numerical ergodicity, Time-independent strong error estimate.

1. Introduction

Recently, a lot of researchers investigated numerical analysis of the following SPDEs under the homogeneous Dirichlet boundary condition in a finite time horizon:

$$\begin{aligned}
&\mathrm{d}X(t,\xi) = \big(\Delta X(t,\xi) + f\big(X(t,\xi)\big)\big)\mathrm{d}t + g\big(X(t,\xi)\big)\mathrm{d}W(t,\xi),\\
&X(t,\xi) = 0, \qquad (t,\xi) \in \mathbb{R}_+ \times \partial\mathscr{O},\\
&X(0,\xi) = X_0(\xi), \quad \xi \in \mathscr{O},
\end{aligned} \tag{1.1}$$

where the physical domain $\mathscr{O} \subset \mathbb{R}^d$ ($d = 1,2,3$) is a bounded open set with smooth boundary. Here, f is assumed to be of monotone type with polynomial growth, g satisfies the usual Lipschitz condition in an infinite-dimensional setting, and W is an infinite-dimensional Wiener process, see Section 2.1. Eq. (1.1) includes the following semiclassical stochastic Allen-Cahn equation, arising from phase transition in materials science by stochastic perturbation, as a special case

$$\mathrm{d}X = \Delta X \mathrm{d}t + \alpha^{-2}(X - X^3)\mathrm{d}t + \mathrm{d}W, \quad X(0) = X_0, \tag{1.2}$$

where $\alpha > 0$ is the interface thickness, see, e.g. [2,3,8,15,19,30–33] and references therein.

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The numerical behavior in the infinite time horizon, especially the numerical ergodicity of SPDEs, including Eq. (1.1) (and Eq. (1.2)), is a natural and intriguing question. As a significant asymptotic behavior, the ergodicity characterizes the case of temporal average coinciding with spatial average, which has vital applications in quantum mechanics, fluid dynamics, financial mathematics, and many other fields [17, 20]. The spatial average, i.e. the mean of a given function for the stationary law, called the invariant measure of diffusion, is also known as the ergodic limit, which is desirable to compute in many practical applications. Then, one has to investigate a stochastic system over long time intervals, which is one of the main difficulties from the computational perspective. Generally, the explicit expression of the invariant measure for a nonlinear infinite-dimensional stochastic system is rarely available. Therefore, it is usually impossible to precisely compute the ergodic limit for a nonlinear SPDE driven by multiplicative noise; exceptional examples are gradient Langevin systems (driven by additive noise), see, e.g. [5, 23]. For this reason, it motivated and fascinated a lot of investigations in the recent decade for constructing numerical algorithms that can inherit the ergodicity of the original system and approximate the ergodic limit efficiently.

In finite-dimensional case, much progress has been made in the design and analysis of numerical approximations of the desired ergodic limits, see, e.g. [26,34,36] and references therein for numerical ergodicity of stochastic ordinary differential equations (SODEs). On the contrary, the construction and analysis of numerical ergodic limits for SPDEs are still in their early stages. Bréhier *et al.* [6,9,10] firstly studied Galerkin-based linear implicit Euler scheme and high order integrator to approximate the invariant measures of a parabolic SPDE driven by additive white noise; see also [13] for the ergodic limit of a spectral Galerkin modified implicit Euler scheme of the damping stochastic Schrödinger equation driven by additive trace-class noise. Recently, exponential Euler-type schemes of full discretization and tamed semi-discretization were used in [14] and [7], respectively, to approximate the invariant measure of a stochastic Allen-Cahn type equation driven by additive colored noise. Cui *et al.* [16] investigated the spectral Galerkin drift-implicit Euler (DIE) scheme to approximate the invariant measure of the same type of equation and proposed a question of whether the invariant measure of the temporal DIE scheme is unique. Almost all of the above literature focuses on the numerical ergodicity of SPDEs driven by additive noise, the numerical ergodicity in the multiplicative noise case is more subtle and challenging.

Apart from the approximation of the invariant measure, the strong approximation, i.e. in the moments' sense, is also an important issue. There exists a general theory of strong error analysis of numerical approximations for Lipschitz SPDEs in a finite time horizon, see, e.g. [1, 11, 21] and references therein. Recently, Liu and Qiao [32] give a theory of strong error analysis of numerical approximations for monotone SPDEs in a finite time horizon. Whether there is a theory of strong error analysis in the infinite time horizon is a natural question.

The above two questions on long-time behaviors of numerical approximations for Eq. (1.1) motivate the present study. On the one hand, we aim to establish the exponential ergodicity of the numerical schemes studied in [19,32], [33] for Eq. (1.1). It should be pointed out that there is no further restriction on the time step size for the DIE scheme (DIE) in addition to the usual requirement of the implicit scheme (see Theorem 3.1), see [26] for an upper bound requirement on the time step size for numerical ergodicity of monotone SODEs. Meanwhile, applying one of our main results, Theorem 3.1, we conclude that the DIE scheme (DIE) is unique ergodic (and exponentially mixing) and thus give an affirmative answer to the question proposed in [16] (see Remark 4.1), provided that the monotone coefficient of the drift function is not too large.