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STABILIZATION-FREE VIRTUAL ELEMENT METHOD FOR THE TRANSMISSION EIGENVALUE PROBLEM ON ANISOTROPIC MEDIA*

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Abstract

In this paper, we develop the stabilization-free virtual element method for the Helmholtz transmission eigenvalue problem on anisotropic media. The eigenvalue problem is a variable-coefficient, non-elliptic, non-selfadjoint and nonlinear model. Separating the cases of the index of refraction $n \neq 1$ and $n \equiv 1$, the stabilization-free virtual element schemes are proposed, respectively. Furthermore, we prove the spectral approximation property and error estimates in a unified theoretical framework. Finally, a series of numerical examples are provided to verify the theoretical results, show the benefits of the stabilization-free virtual element method applied to eigenvalue problems, and implement the extensions to high-order and high-dimensional cases.

Mathematics subject classification: 65N25, 65N30, 65N15.
Key words: Virtual element method, Stabilization-free, Transmission eigenvalue problem, Anisotropic media, Error estimates.

1. Introduction

The eigenvalue problems arising from partial differential equations (PDEs) play a fundamental role in engineering applications. The transmission eigenvalue problem (TEP) is essential in the conventional qualitative method of inverse scattering problems from the theoretical perspective, see monographs [23,30]. In particular, the changes about the constitutive parameters of the media lead to the corresponding changes in the measured transmission eigenvalues and hence transmission eigenvalues are useful for the nondestructive testing of inhomogeneous media. For the details how this procedure works, we refer to [23] again.

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The interest of this manuscript lies in the numerical analysis for the computation of the interior TEP on anisotropic media: Find non-trivial w, u and $k \in \mathbb{C} \setminus \{0\}$ such that

$$
\begin{aligned}
-\operatorname{div}(\boldsymbol{A}\nabla w) - k^2 n w = 0 \quad & \text{in } \Omega, && \text{(1.1a)}\\
-\Delta u - k^2 u = 0 \quad & \text{in } \Omega, && \text{(1.1b)}\\
w - u = 0 \quad & \text{on } \partial\Omega, && \text{(1.1c)}\\
\frac{\partial w}{\partial \boldsymbol{\nu}_{\boldsymbol{A}}} - \frac{\partial u}{\partial \boldsymbol{\nu}} = 0 \quad & \text{on } \partial\Omega. && \text{(1.1d)}
\end{aligned}
$$

Here $\Omega \subseteq \mathbb{R}^d$ $(d = 2, 3)$ is a bounded simply connected domain with the boundary $\partial\Omega$. $\boldsymbol{A}$ is a $d \times d$ symmetric positive definite matrix with real $L^\infty(\Omega)$ entries. The index of refraction $n \in L^\infty(\Omega)$ is a positive real function, $\boldsymbol{\nu}$ is the unit outer normal to $\partial\Omega$ and $\partial w/\partial \boldsymbol{\nu}_{\boldsymbol{A}} = \boldsymbol{A}\nabla w \cdot \boldsymbol{\nu}$. The theoretical existence of transmission eigenvalues have been studied in [23, 24]. Moreover, the numerical computation is an interesting and nontrivial task since the problem possesses nonlinear, non-selfadjoint and non-elliptic properties.

Under the case of isotropic media, that is, $\boldsymbol{A}$ equals to the identity matrix $\boldsymbol{I}$, the first numerical study for the TEP refers to [31]. In recent years, various numerical methods have been applied to the TEP of isotropic media, for example, iterative method [51], finite element methods [43, 55, 57, 58] and spectral-element methods [3, 57]. Underlying the anisotropic scenario, it calls for individual numerical methods and approximation theory. A few numerical algorithms to compute transmission eigenvalues of anisotropic media have been proposed. In 2013, Ji and Sun [37] proposed a continuous finite element method and explicitly enforced the Dirichlet boundary condition to derive a large, sparse and non-Hermitian generalized matrix eigenvalue problem. Then they devised a more efficient multi-level approach to solve it. In 2017, Kleefeld and Colton [38] computed interior transmission eigenvalues for anisotropic media by using boundary integral equations and a nonlinear solver based on complex-valued contour integrals. They also considered the fundamental solutions method [39]. To date, there have been limited works in terms of the rigorous analysis of the convergence and the convergence rate of numerical schemes. Xie and Wu [56] defined the finite element approximation for the problem (1.1) and designed the multilevel correction method. Gong *et al.* [35] have formulated TEP of anisotropic media as a spectral problem of a holomorphic Fredholm operator and proved the convergence of the Lagrange linear finite element approximation by using the spectral approximation theory of holomorphic Fredholm operator, but without the optimal convergence rate. Furthermore, Meng and Mei [44] proposed the standard virtual element method to solve the TEP and applied the $\mathbb{T}$-coercivity theory to prove the a priori and a posteriori error estimates. Recently, Liu *et al.* [42] studied the convergence of the mixed finite element method for the TEP with the index of refraction $n \equiv 1$.

The virtual element method (VEM) is introduced in 2013 [8], which is the generalization of the finite element method to general polyhedral meshes. Except for the mere possibility to use polytopal meshes, the VEM is also attractive in some problems, for example, high-order PDEs, the construction of divergence-free requirement and complex geometric structures [12,25,28,40]. In the last ten years, the VEMs of elliptic problems [11], elasticity problems [26], fluidodynamics problems [12], magnetostatic problems [9] and coding aspects [10] have been widely investigated, also refer to recent monograph [4] for more details. The subject of the VEM approximation for eigenvalue problems is also an appealing field [4, Chapter 7]. To date, the VEM has been developed for different eigenvalue problems, for instance, the Steklov eigenvalue problem [40,47],