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OPTIMAL POINT-WISE ERROR ESTIMATE OF TWO SECOND-ORDER ACCURATE FINITE DIFFERENCE SCHEMES FOR THE HEAT EQUATION WITH CONCENTRATED CAPACITY*

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Abstract

In this paper, we propose and analyze two second-order accurate finite difference schemes for the one-dimensional heat equation with concentrated capacity on a computational domain $\Omega = [a, b]$. We first transform the target equation into the standard heat equation on the domain excluding the singular point equipped with an inner interface matching (IIM) condition on the singular point $x = \xi \in (a, b)$, then adopt Taylor's expansion to approximate the IIM condition at the singular point and apply second-order finite difference method to approximate the standard heat equation at the nonsingular points. This discrete procedure allows us to choose different grid sizes to partition the two sub-domains $[a, \xi]$ and $[\xi, b]$, which ensures that $x = \xi$ is a grid point, and hence the proposed schemes can be generalized to the heat equation with more than one concentrated capacities. We prove that the two proposed schemes are uniquely solvable. And through in-depth analysis of the local truncation errors, we rigorously prove that the two schemes are second-order accurate both in temporal and spatial directions in the maximum norm without any constraint on the grid ratio. Numerical experiments are carried out to verify our theoretical conclusions.

Mathematics subject classification: 65M06, 65M12.
Key words: Heat equation with concentrated capacity, Finite difference scheme, Inner interface matching condition, Unconditional convergence, Optimal error estimate.

1. Introduction

In this paper, we consider the one-dimensional heat equation with concentrated capacity (the heat system) [1, 3, 8, 12, 13]

$$[1 + K\delta(x - \xi)]\partial_t u(x,t) - \partial_{xx} u(x,t) = f(x,t), \quad (x,t) \in (a,b) \times (0,T] \tag{1.1}$$

with boundary condition

$$u(a,t) = 0, \quad u(b,t) = 0, \quad t \in (0,T], \tag{1.2}$$

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and initial condition

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1.3}$$

where $u(x,t)$ is the unknown real-valued function, $\delta(x)$ is the Dirac delta function, the positive constant K is the coefficient of the heat concentrated capacity at $x = \xi$ with $\xi \in (a,b)$ being an interior point, $u_0(x)$ and $f(x,t)$ are given real-valued functions, the terminal time T is in $(0,T^*)$ with T^* being the maximal existence time of the solution $u(x,t)$ to the heat system (1.1)-(1.3). The heat equation with concentrated capacity is a special kind of heat equation, in which the heat capacity coefficient contains a Dirac delta function. In other words, the heat capacity coefficient has the property that the jump of heat flow at the singular point is proportional to the time derivative of the temperature [7,12]. The heat system is an important class of mathematical model widely used in fluid dynamics and chemistry, including chemical reactor theory and colloid chemistry [2,21].

Extensive mathematical and numerical studies have been carried out for the heat system (1.1) in the literature. Along the mathematical front, we refer to the derivation [5], well-posedness [3,9] and stability [10] of the heat system. Along the numerical front, different efficient and accurate numerical methods are proposed. By treating the Dirac delta function either as a single point discontinuity, or a smooth function with small support, one can modify the standard finite difference scheme by using the properties of the heat capacity coefficient. Therefore, many of the finite difference schemes [3–6,8,9,18–20] and the finite element approximations [5,14–16] have been proposed. Javanović and Vulkov [8,9] derived a finite difference scheme on the uniform grid stencil and proved the convergence order of $(\tau + h^2)$ in the discrete L^2 norm but with the order of $(\tau + h^2 + h^2\sqrt{|\ln\tau|})$ in the discrete H^1 norm, besides the first-order accurate scheme, they also proposed a second-order accurate one for the heat system at the end of their paper [8]. If the exact solution of the original problem on the nonsingular points is smooth enough, Sun and Zhu [20] proposed another second-order accurate finite difference scheme (a box scheme) by using the method of order reduction on non-uniform meshes, and established the optimal error estimate in the maximum norm. Their box scheme possesses three distinct advantages: firstly, no local truncation error is introduced in discretizing the IIM condition; secondly, the scheme features overall second-order accuracy in space; thirdly, non-uniform grids can be employed in the scheme, as ensures that the singular points align precisely with grid nodes. However, the box scheme given in [20] was proposed by using an indirect method, and even at those nonsingular points with a uniform grid, enhancing the precision of the numerical solution is quite challenging. At such times, a compact finite difference scheme instead of the box scheme can be employed to improve the accuracy of the numerical solutions.

In the literature, the heat capacity coefficient containing a Dirac delta function can be viewed as an IIM condition [4,10,11]. Many numerical methods have been carried out to solve the interface problems, such as immersed interface method [17], explicit jump immersed interface method and Peskin's immersed boundary method [11]. In this paper, we first rewrite the heat equation with concentrated capacity as an equivalent interface problem, i.e. we split the computational domain into two sub-domains, and take different mesh sizes for the two sub-domains, which ensures that the singular point is a grid node. Then, we deal with the IIM condition by Taylor's expansion at the singular point, while adopting the standard discretization at the nonsingular points, resulting in two finite difference schemes. The main difference between the two schemes is to adopt different approximations of the IIM condition. In particular, we establish the optimal point-wise error estimates of the two proposed schemes by using the H^1-technique, which differs from the convergence result given in [8] where the error estimate in