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HIGH-ORDER COMPACT ADI SCHEMES FOR 2D SEMI-LINEAR REACTION-DIFFUSION EQUATIONS WITH PIECEWISE CONTINUOUS ARGUMENT IN REACTION TERM*

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Abstract

This paper deals with the numerical solutions of two-dimensional (2D) semi-linear reaction-diffusion equations (SLRDEs) with piecewise continuous argument (PCA) in reaction term. A high-order compact difference method called I-type basic scheme is developed for solving the equations and it is proved under the suitable conditions that this method has the computational accuracy $\mathcal{O}(\tau^2 + h_x^4 + h_y^4)$, where τ, h_x and h_y are the calculation stepsizes of the method in t-, x- and y-direction, respectively. With the above method and Newton linearized technique, a II-type basic scheme is also suggested. Based on the both basic schemes, the corresponding I- and II-type alternating direction implicit (ADI) schemes are derived. Finally, with a series of numerical experiments, the computational accuracy and efficiency of the four numerical schemes are further illustrated.

Mathematics subject classification: 65M06, 65M12.
Key words: Semi-linear reaction-diffusion equations, Piecewise continuous argument, High-order compact difference methods, Alternating direction implicit schemes, Computational accuracy and efficiency.

1. Introduction

To describe the heat flow in a rod with both diffusion along the rod and heat loss/gain across the lateral sides of the rod, Wiener [16] first introduced the following initial-boundary value problems (IBVPs) of linear reaction-diffusion equations with PCA:

$$\begin{cases} u_t(x,t) = \hat{a}u_{xx}(x,t) + \hat{b}u(x,\lfloor t \rfloor), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & \qquad\quad t \in (0,T], \end{cases} \tag{1.1}$$

where $u(x,t)$ denotes the temperature at point (x,t) in the rod and the lateral heat change is assumed to occur at time $\lfloor t \rfloor$, in which $\lfloor \cdot \rfloor$ is the greatest integer function. Subsequently, for depicting various diffusion phenomena, problems (1.1) were adapted into IBVPs of linear

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diffusion equations with PCA (cf. [9, 10])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_{xx}(x,\lfloor t \rfloor), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & \qquad\qquad t \in (0,T]. \end{cases} \tag{1.2}$$

IBVPs of linear forward-backward diffusion equations with PCA (cf. [15])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_{xx}(x,\lfloor t \rfloor) + \hat{a}_3 u_{xx}(x,\lfloor t+1 \rfloor), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & \qquad\qquad t \in (0,T]. \end{cases} \tag{1.3}$$

IBVPs of diffusion-convection equations with PCA (cf. [3])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_{x}(x,\lfloor t \rfloor), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & \qquad\qquad t \in (0,T]. \end{cases} \tag{1.4}$$

IBVPs of linear neutral reaction-diffusion equations with PCA (cf. [5, 6, 18])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_{xx}(x,\lfloor t \rfloor) + \hat{b}_1 u(x,t) \\ \qquad\qquad + \hat{b}_2 u(x,\lfloor t \rfloor) + \hat{c} u_t(x,\lfloor t \rfloor), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & \qquad\qquad t \in (0,T]. \end{cases} \tag{1.5}$$

IBVPs of I-type SLRDEs with PCA (cf. [2])

$$\begin{cases} u_t(x,t) = \hat{a} u_{xx}(x,t) + f\big(x,t,u(x,t),u(x,\lfloor t \rfloor)\big), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & \qquad\qquad t \in (0,T]. \end{cases} \tag{1.6}$$

IBVPs of II-type SLRDEs with PCA (cf. [8])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_{xx}(x,\lfloor t \rfloor) + f\big(x,t,u(x,t)\big), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & \qquad\qquad t \in (0,T]. \end{cases} \tag{1.7}$$

For the above partial functional differential equations (PFDEs) with PCA, some effective numerical methods and the corresponding algorithm theory have been developed. For problem (1.2), Liang *et al.* [9] and Liang *et al.* [10], respectively, proposed θ-method and Galerkin method with asymptotical stability analysis. For problem (1.3), Wang and Wen [15] generalized the θ-method in [9] and its stability theory. For problem (1.4), Esmailzadeh *et al.* [3] suggested an alternative θ-method and derived its asymptotical stability criterion. For problem (1.5), Zhang *et al.* [18] considered a class of linear approximation methods based on block boundary value methods, and Han and Zhang [5, 6] constructed the one-parameter finite element methods with error and stability analysis. Furthermore, the authors of references [2, 8] extended the above numerical approach for linear problems to nonlinear problems (1.6) and (1.7), where Esmailzadeh *et al.* [2] presented the θ-method for problem (1.6) and Hou and Zhang [8] derived a high-order compact difference method and its Richardson extrapolation scheme for problem (1.7).