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ANALYSIS OF AN EMBEDDED VARIABLE STEP IMPLICIT-EXPLICIT SCHEME FOR NATURAL CONVECTION PROBLEMS*

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Abstract

This report presents a series of implicit-explicit (IMEX) variable stepsize algorithms for natural convection equations. The presented method requires a minimally intrusive modification to an existing program, does not add to the computational complexity, and is conceptually simple. Here, IMEX means the nonlinear term is treated fully explicitly, while the remaining terms are treated implicitly. Due to the increasing demand for low memory solvers, the addition of time adaptive can improve the accuracy and efficiency of the algorithms. For the first-order algorithm, we prove the stability of the variable stepsize backward Euler scheme combined with Adams-Bashforth 2 (VSS BE-AB2) and analyze convergence. Then, the stability of Constant Timestep Filtered-BE-AB2 (BE-AB2+F) is proved. Moreover, we construct adaptive algorithms by extending the approach to variable stepsize. Finally, numerical tests confirm the convergence rates of our method and validate the theoretical results.

Mathematics subject classification: 65N12, 65N30, 65N50, 35Q79.
Key words: Time filter, Stability, Convergence, Adaptive algorithms, Natural convection.

1. Introduction

In this paper, we shall study stability and analyze convergence of a low complexity fully discrete time-stepping finite element method (FEM), then extend the method to adaptive time-stepping and higher order algorithms for natural convection (NC) problems [47]. We consider

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the numerical schemes for solving the time-dependent nonlinear NC equations

$$\begin{cases} \mathbf{u}_t - Pr\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = PrRa\mathbf{j}T & \text{in } \Omega\times(0,T_1], \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,T_1], \\ T_t + (\mathbf{u}\cdot\nabla)T - \Delta T = \gamma & \text{in } \Omega\times(0,T_1], \\ \int_\Omega p\, d\mathbf{x} = 0, \quad \mathbf{u} = 0, \quad T = 0 & \text{on } \partial\Omega, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}^0, \quad T(\mathbf{x},0) = T^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in R^d ($d = 2$ or 3) with a sufficiently smooth boundary $\partial\Omega, \mathbf{j} = (0,1)^\top$ or $\mathbf{j} = (0,0,1)^\top, Pr, Ra, T_1$ and γ represent the Prandtl number, the Rayleigh number, the given final time and the forcing function, respectively. The unknown functions are velocity vector $\mathbf{u}$, pressure p and temperature T. We consider a homogeneous Dirichlet boundary conditions in the above for simplicity and to fix the idea.

The NC convection problem (1.1) is an important system with dissipative nonlinear terms in atmospheric dynamics [3, 19]. Since this system not only contains the velocity as well as the pressure but also includes the temperature field, finding the numerical solution to problem (1.1) becomes a difficult task. For the research of problem (1.1), many scholars have proposed several kinds of efficient numerical schemes. For example, the standard Galerkin FEM [40], finite difference method [31], lattice Boltzmann method [15], the projection-based stabilized MFEM [7, 9], discontinuous Galerkin method [2, 5], domain decomposition method [37], time-stepping method [18, 52] and references therein. Moreover, NC phenomena are found in many scientific and engineering applications, and have been intensively studied in the literature, cf. [4, 16, 41, 42, 46, 54]. Hence, it is necessary to research the problem for us nowadays.

Time accuracy is critical for obtaining physically relevant solutions in the field of computational fluid dynamics (CFD). Many flow solvers use constant time steps, but there has been an expanding interest in variable step solvers [3, 19]. These methods allow for larger time steps for intervals of the simulation where the physics are stable, while allowing for smaller time steps for portions that are physically interesting. This allows for a decrease in the computational cost of the solver while simultaneously increasing its accuracy. The objective of this report is to design an efficient novel adaptive time-stepping method for NC problems based on time filters (TFs). TFs have some advantages:

(a) They can remove the overdamping of BE while remaining unconditionally energy stable [55].

(b) They can increase the time accuracy and add negligible additional computational complexity to simple, lower accuracy methods for constant time stepsize [12, 22].

(c) They give a low cost error estimator for adapting the time stepsize to ensure time accuracy [23].

The first proposed time filter was the Robert-Asselin (RA) time filter [1,3,29,50], which was used as a method to suppress the non-physical oscillation of leapfrog patterns in weather models. Williams [48, 49] improved the RA time filter, and a Robert-Asselin-Williams (RAW) time filter with third-order precision was obtained. Li and Trenchea [32] proposed a new RA time filter with third-order precision. The proposed method is easy to implement programmatically. Besides, Guzel and Layton [23] combined the backward Euler scheme and the time filter to