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THE PRESSURE-ROBUST WEAK GALERKIN FINITE ELEMENT METHOD FOR STOKES-DARCY PROBLEM*

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Abstract

In this paper, we propose a pressure-robust weak Galerkin (WG) finite element scheme to solve the Stokes-Darcy problem. To construct the pressure-robust numerical scheme, we use the divergence-free velocity reconstruction operator to modify the test function on the right side of the numerical scheme. This numerical scheme is easy to implement because it only need to modify the right side. We prove the error between the velocity function and its numerical solution is independent of the pressure function and viscosity coefficient. Moreover, the errors of the velocity function reach the optimal convergence orders under the energy norm, as validated by both theoretical analysis and numerical results.

Mathematics subject classification: 65N30, 65N15, 65N12, 35B45.
Key words: Weak Galerkin finite element methods, Coupled Stokes-Darcy problems, Pressure-robust error estimate, Divergence preserving.

1. Introduction

This paper considers the Stokes-Darcy model, which couples the Stokes equations in the free flow region with the Darcy equations in the porous medium region.

In the free flow region Ω_s, the flow is governed by the following Stokes equations:

$$-\nabla\cdot T(\mathbf{u}_s,p_s)=\mathbf{f}_s \quad \text{in } \Omega_s, \tag{1.1}$$
$$\nabla\cdot\mathbf{u}_s=g_s \quad \text{in } \Omega_s, \tag{1.2}$$
$$\mathbf{u}_s=\mathbf{0} \quad \text{on } \Gamma_s, \tag{1.3}$$

where $T(\mathbf{u}_s,p_s)=2\mu D(\mathbf{u}_s)-p_sI$ is the symmetric stress tensor and $D(\mathbf{u}_s)=(\nabla\mathbf{u}_s+(\nabla\mathbf{u}_s)^T)/2$ is the strain tensor. Define μ and I as the viscosity coefficient and the identity tensor, respectively.

In the porous medium region Ω_d, the flow is governed by the Darcy equations in the mixed formulation

$$\mu\kappa^{-1}\mathbf{u}_d+\nabla p_d=\mathbf{f}_d \quad \text{in } \Omega_d, \tag{1.4}$$
$$\nabla\cdot\mathbf{u}_d=g_d \quad \text{in } \Omega_d, \tag{1.5}$$
$$\mathbf{u}_d\cdot\mathbf{n}_d=0 \quad \text{on } \Gamma_d, \tag{1.6}$$

where κ is permeability tensor. Denote by $g=(g_s,g_d)$ the source satisfying

$$\int_{\Omega_s} g_s\,\mathrm{d}\Omega_s+\int_{\Omega_d} g_d\,\mathrm{d}\Omega_d=0.$$

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This model is coupled with three interface conditions: the mass conservation condition, the balance of forces, and the Beavers-Joseph-Saffman (BJS) condition

$$\mathbf{u}_s \cdot \mathbf{n}_s = \mathbf{u}_d \cdot \mathbf{n}_s \quad \text{on } \Gamma_{sd}, \tag{1.7}$$

$$p_s - 2\mu D(\mathbf{u}_s)\mathbf{n}_s \cdot \mathbf{n}_s = p_d \quad \text{on } \Gamma_{sd}, \tag{1.8}$$

$$-2D(\mathbf{u}_s)\mathbf{n}_s \cdot \mathbf{t}_s = \alpha\kappa^{-\frac{1}{2}}\mathbf{u}_s \cdot \mathbf{t}_s \quad \text{on } \Gamma_{sd}. \tag{1.9}$$

Here $\mathbf{n}_s$ and $\mathbf{t}_s$ are the unit outward normal vector and the unit tangent vector on the interface Γ_{sd}, respectively. And α is an empirical parameter obtained through experiments. Let $\Omega = \Omega_s \cup \Omega_d$ be an open bounded domain in $\mathbb{R}^2$. This model is shown in Fig. 1.1.

This model simulates the transport of pollutants from rivers into aquifers in environmental science. Moreover, it finds many applications in hydrology, biofluid dynamics and other fields. So far, various numerical methods have been proposed to numerically solve the Stokes-Darcy problem, including the mixed finite element method [6,9,17,24], the finite volume method [39], the discontinuous Galerkin finite element method [10,14,33,38], the virtual element method [35], the weak Galerkin finite element method [3,19,20,31,32], etc.

When using the piecewise linear polynomial combined with the lowest-order Raviart-Thomas (RT) element to discrete the velocity function in the Stokes equations and the Darcy equations, the error of the velocity function is independent of viscosity coefficient and the pressure function in [24]. We call this property of the error as pressure robustness. However, most numerical methods are not pressure-robust. Consequently, if viscosity coefficient is very small or the approximation of the pressure function is inaccurate, the approximation of the velocity function is correspondingly compromised. This non-pressure robustness also occurs when the Stokes equations are solved numerically. To obtain the pressure-robust numerical results, scholars have proposed various methods: the mixed finite element method based on an exact de Rham complex [12,15], grad-div stabilization [8,16,28–30], appropriate reconstructions of test functions [5,18,21–23,40], etc.

The WG method was first proposed in [36] for solving second-order elliptic equation. In contrast to the finite element method, the WG method employs the discontinuous weak function space as the approximate function space and replaces the classical differential operators with the weak differential operators in the variational formulation. In [37], the authors develop

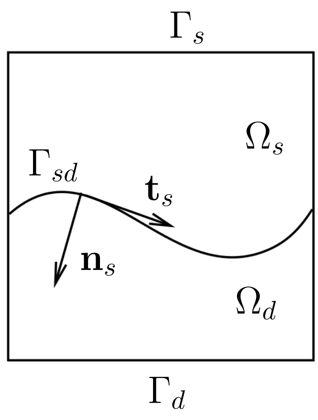

Fig. 1.1. The Stokes-Darcy model.