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STABILITY AND ERROR ESTIMATES OF ULTRA-WEAK LOCAL DISCONTINUOUS GALERKIN METHOD WITH SPECTRAL DEFERRED CORRECTION TIME-MARCHING FOR PDES WITH HIGH ORDER SPATIAL DERIVATIVES*

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Abstract

The main purpose of this paper is to give stability analysis and error estimates of the ultra-weak local discontinuous Galerkin (UWLDG) method coupled with a spectral deferred correction (SDC) temporal discretization method up to fourth order, for solving the fourth-order equation. The UWLDG method introduces fewer auxiliary variables than the local discontinuous Galerkin method and no internal penalty terms are required for stability, which is efficient for high order partial differential equations (PDEs). The SDC method we adopt in this paper is based on second-order time integration methods and the order of accuracy is increased by two for each additional iteration. With the energy techniques, we rigorously prove the fully discrete schemes are unconditionally stable. By the aid of special projections and initial conditions, the optimal error estimates of the fully discrete schemes are obtained. Furthermore, we generalize the analysis to PDEs with higher even-order derivatives. Numerical experiments are displayed to verify the theoretical results.

Mathematics subject classification: 65M60, 65M12, 35K25.
Key words: Ultra-weak local discontinuous Galerkin method, High order equations, Spectral deferred correction method, Stability, Error estimate.

1. Introduction

In this paper, we present the stability analysis and error estimates of the ultra-weak local discontinuous Galerkin method coupled with a spectral deferred correction temporal discretization methods for linear partial differential equations with high order spatial derivatives. We first consider the following fourth-order equation:

$$\begin{aligned} &u_t + u_{xxxx} = 0, \quad (x,t) \in [a,b] \times [0,T], \\ &u(x,0) = u_0(x), \quad x \in [a,b] \end{aligned} \tag{1.1}$$

with periodic boundary conditions. Then we generalize the stability analysis and error estimate to PDEs with even-order derivatives.

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The UWLDG method was first developed to solve time dependent PDEs with high order derivatives by Tao *et al.* [14]. It combines the advantages of local discontinuous Galerkin (LDG) method and ultra-weak discontinuous Galerkin (UWDG) method. The main idea of the LDG method [6, 21, 22] is to rewrite the high order equations into a first-order system, then apply the traditional DG method [11, 12] to the system and design suitable numerical fluxes to ensure stability. The UWDG method [3,4] is based on repeated integration by parts to move all spatial derivatives to the test function in the weak formulation, and ensure stability by suitable numerical fluxes. Recently, the UWLDG method has been successfully used to solve PDEs with high order derivatives, and the theoretical analysis for the semi-discrete UWLDG method [13,15] are also presented. Compared to the LDG method, the UWLDG method requires fewer auxiliary variables, thus reduces memory and computational costs. Compared to the UWDG method, no internal penalty terms are required to ensure stability.

To relax the severe time step restriction of explicit time marching methods for PDEs with high order derivatives, several semi-implicit time marching methods have been studied, such as the implicit-explicit (IMEX) time discretization [1, 2] and the SDC methods [7, 8, 10, 24]. The semi-implicit SDC method proposed in [8] is based on second-order time integration methods and the order of accuracy is increased by two for each additional iteration. In [23], the SDC method coupled with the LDG spatial discretization was studied for parabolic equations. In this paper, we consider the UWLDG method and study the stability and error estimates when coupled with the novel SDC method.

Wang *et al.* [16–18] presented the stability analysis of the Runge-Kutta type IMEX time discretization coupled with LDG method for convection-diffusion problems and fourth-order equations [20]. Further, Wang *et al.* [19] studied the analysis of the IMEX-UWDG method for convection-diffusion problems and the optimal error estimate was obtained for fully discrete schemes up to third order. The Runge-Kutta type IMEX methods have some limitations, for example, it is more difficult to construct for higher order accuracy. While for the SDC method, an advantage of this method is that it is a one step method and can be constructed easily and systematically for any order of accuracy. As far as the authors know, there is no theoretical analysis for the fully discrete IMEX-UWLDG scheme. In this paper, we perform the analysis of the stability and error estimates for some fully discrete SDC-UWLDG schemes up to fourth order both in space and time.

Our contribution here is that we first take the fourth-order equation as an example and rigorously prove that the SDC-UWLDG schemes are unconditionally stable. The main technique is the symmetric and dissipative properties of the semi-discrete UWLDG scheme, which plays a key role in establishing negative definite quadratic forms for the semi-implicit discretization of the high order derivative part. The stability results contain information about the initial numerical approximation of the second-order derivative, making the choice of the initial conditions crucial for optimal error estimates. Following the same idea as that in [9], we would like to define the special elliptic projection with respect to the semi-discrete UWLDG scheme and derive optimal error estimates of the initial numerical approximation. By carefully choosing projections, we proceed to obtain the optimal error estimates of the SDC-UWLDG schemes up to fourth order both in time and space. Moreover, we generalize the analysis to PDEs with higher even-order derivatives.

The rest of this paper is organized as follows. We first present some notations and projections in Section 2. Then we present the semi-discrete UWLDG scheme for the fourth-order equation in Section 3. In Section 4, we give the stability analysis of the second-order and fourth-