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A SPACE-TIME TREFFTZ DG METHOD FOR THE SECOND ORDER TIME-DEPENDENT MAXWELL SYSTEM IN ANISOTROPIC MEDIA*

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Abstract

The h-version analysis technique developed in [Banjai *et al.*, SIAM J. Numer. Anal., 55 (2017)] for Trefftz discontinuous Galerkin (DG) discretizations of the second order isotropic wave equation is extended to the time-dependent Maxwell equations in anisotropic media. While the discrete variational formulation and its stability and quasi-optimality are derived parallel to the acoustic wave case, the derivation of error estimates in a mesh-skeleton norm requires new transformation stabilities for the anisotropic case. The error estimates of the approximate solutions with respect to the condition number of the coefficient matrices are proved. Furthermore, we propose the global Trefftz DG method combined with local DG methods to solve the time-dependent nonhomogeneous Maxwell equations. The numerical results verify the validity of the theoretical results, and show that the resulting approximate solutions possess high accuracy.

Mathematics subject classification: 65N30, 65N55.
Key words: Time-dependent Maxwell's equation, Anisotropic, Nonhomogeneous, Trefftz method, Local discontinuous Galerkin, Error estimate.

1. Introduction

The idea at the heart of Trefftz method, which are named after the seminal work of Trefftz [36], is to choose the Trefftz approximation functions from a class of piecewise solutions of the same governing partial differential equation (PDE) without boundary conditions. Trefftz methods turned out to be particularly effective, and popular, for wave propagation problems in time-harmonic regime at medium and high frequencies, where the oscillatory nature of the solutions makes standard methods computationally too expensive, see the recent survey [15] and references therein. The Trefftz method has an important advantage over Lagrange finite elements for discretization of the Helmholtz equation and time-harmonic Maxwell equations [13–17, 28, 29, 43]: to achieve the same accuracy, relatively smaller number of degrees of

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freedom is enough in the plane wave-type methods owing to the particular choice of the basis functions that (may approximately) satisfy the considered PDE without boundary conditions.

Recently, much work has been devoted to Trefftz DG methods for time-dependent linear isotropic wave phenomena, see in particular [8–11, 22–24, 27, 30, 31, 35, 38–41, 44]. Related works [35, 38–40] proposed promising Tent Pitcher algorithms coupled with the Trefftz DG method, and obtained positive numerical results illustrating the increase of accuracy and the decrease of computational burden. A Trefftz interior-penalty second order formulation for the second order wave equation is first proposed in [1], where a best approximation result is proven for the space-time DG method with Trefftz-type basis functions and rates of convergence are proved in any dimension. Besides, for the case of one space dimension, a class of Trefftz DG method for time-dependent electromagnetic problems, written as a first order system, resulting in a two-field formulation, have been analysed in [22–24], where stability, quasi-optimality, best approximation estimates for polynomial Trefftz spaces and (fully explicit) error bounds with high order in the mesh width and in the polynomial degree are proved. Further, in [8, 9, 30] Trefftz DG methods in a first order formulation have been extended to three-dimensional time dependent Maxwell's equations, where the resulting spectral convergence order is only demonstrated by numerical tests. Currently, a space-time Trefftz discontinuous Galerkin method for the first order anisotropic Maxwell's equations in homogeneous media is proposed in [42], and a space-time Trefftz discontinuous Galerkin methods for the first order anisotropic acoustic wave equations in inhomogeneous media is proposed in [41]. Similar to frequency domain problems, compared to standard space-time DG method, Trefftz DG methods can also achieve the same convergence order and approximation properties with considerably fewer degrees of freedom, which brings more economical implementation due to integration being restricted to the space-time skeleton not space-time elements, see Remark 6.1, Section 8.1 and [1, 22].

Anisotropy can result from periodic layering of fine layers [5], preferential alignment of fractures and cracks [34], determining the response of the inclusion to an impinging acoustic or electromagnetic wave (see the textbooks [6, 21, 25]). Anisotropy may greatly influence seismic wave propagation, seismic data acquisition and subsequent data analysis and processing procedures [37]. It is therefore important to design accurate and efficient numerical methods for modeling wave propagation in anisotropic media. Besides, electromagnetic problems within this class also include the design of waveguides and antennas, scattering of electromagnetic waves from automobiles and aircraft, and the penetration and absorption of electromagnetic waves by dielectric objects [2, 33]. Under this assumption, a permittivity ε, and permeability μ, tensor can describe a linear metamaterial with no magnetoelectric coupling, where bianisotropy effects have typically played a minor role in the overall response of the experimental metamaterials, and can be mitigated by design [26]. The study [45] was devoted to computing the mathematical model for the scattering from multiple cavities both in particular transverse magnetic (TM) and transverse electric (TE) polarizations, where the cavities are required to be invariant to z-axis, namely, the permittivity ε, and permeability μ, limited to the z-axis, do not interact with other directions.

In this paper we construct a class of space-time Trefftz DG schemes for the second order linear anisotropic Maxwell equations in three-dimensional inhomogeneous media. The DG method considered is motivated by the class of interior penalty DG methods, and can be understood as the translation to Maxwell's equations of the Trefftz DG formulation for the second order wave equations in [1]. In this work, we construct and analyze a space-time interior penalty DG method, which employ space-time slabs to ensure solvability on each time step, resulting in a stable, dissipative scheme for general polynomial bases. In particular, we focus on Trefftz polynomials as local basis functions, prove quasi-optimality in the DG energy norm and that, the approximate solutions generated by the proposed method possess satisfactory error estimates with respect to meshwidth h and the condition number ρ of the coefficient matrices, respectively. Numerical results in the DG norm show that the theoretical estimates of the convergence order are optimal with respect to h. Furthermore, space-time DG methods, such as the one presented below, do not require any Courant-Friedrichs-Lewy-type (CFL-type) restrictions, owing to their implicit time-stepping interpretation. If so-called tent-pitched space-time meshes are employed, correspondingly explicit marching-type space-time schemes can be constructed (see [10, 11, 31]).

Since Trefftz basis functions on each element are solutions of the homogeneous wave equations without boundary conditions, the Trefftz methods can not be directly applied to discretizations of the nonhomogeneous wave equations. Motivated by the coupled discontinuous Galerkin formulation developed in [18], we develop the global Trefftz DG method combined with overlapping local DG method. Numerical results support the optimality of convergence rates generated by the combined numerical DG scheme just as for the homogeneous case.

The paper is organized as follows. In Section 2, we state the initial boundary value problem for the anisotropic Maxwell equations in the second order formulation. In Sections 3 and 4, we construct the space-time interior penalty method and show its stability. In Section 5, we explain how to discretize the resulting anisotropic variational problems. Section 6 provides the well-posedness, the quasi-optimality, the transformation stability with respect to mesh-dependent norms and desired error estimates for the approximate solutions. In Section 7, in order to solve the nonhomogeneous and anisotropic model, we develop a global Trefftz DG method combined with overlapping local DG method for the nonhomogeneous Maxwell equations. Finally, we report some numerical results to confirm the effectiveness of the proposed method.

2. Considered Model

Let $\mathbf{L}^2(\Sigma)$ denote the standard Sobolev space, $\Sigma \subset \mathbb{R}^3$, with the corresponding standard L^2-inner product on Σ inducing the norm $\|\cdot\|_\Sigma$. $\mathbf{H}^s(\Sigma)$ denotes the Hilbertian Sobolev space of index $s \in \mathbb{R}$ defined on Σ. $\mathbf{n}$ is the exterior unit normal to Σ. Further, we introduce the following spaces:

$$\begin{aligned}
&\mathbf{H}(\mathrm{curl};\Sigma) = \{\boldsymbol{\phi} \in \mathbf{L}^2(\Sigma) : \nabla \times \boldsymbol{\phi} \in \mathbf{L}^2(\Sigma)\},\\
&\mathbf{H}_0(\mathrm{curl};\Sigma) = \{\boldsymbol{\phi} \in \mathbf{H}(\mathrm{curl};\Sigma) : \mathbf{n} \times \boldsymbol{\phi} = \mathbf{0} \ \text{ on } \ \partial\Sigma\},\\
&\mathbf{L}^2_T(\partial\Sigma) = \{\boldsymbol{\phi} \in \mathbf{L}^2(\partial\Sigma) : \mathbf{n} \cdot \boldsymbol{\phi} = \mathbf{0} \ \text{ on } \ \partial\Sigma\}.
\end{aligned}$$

Finally, let $L^2(0,T;\mathbf{X})$ denote the standard Bochner spaces with $\mathbf{X}$ being a Banach space with norm $\|\cdot\|_{\mathbf{X}}$, and $C(0,T;\mathbf{X})$ the space of continuous functions $\boldsymbol{\phi} : [0,T] \to \mathbf{X}$ with norm

$$\|\boldsymbol{\phi}\|_{C(0,T;\mathbf{X})} := \max_{0\le t\le T} \|\boldsymbol{\phi}(t)\|_{\mathbf{X}}.$$

We consider the second order electromagnetic wave initial boundary value problem (IBVP) posed on a space-time domain $Q = \Omega \times I$, where $\Omega \subset \mathbb{R}^3$ is an open bounded Lipschitz polytope and $I = (0,T), T > 0$. $\mathbf{n}_\Omega$ is an outward-pointing unit normal vector on $\Gamma = \partial\Omega$. The second order model reads as

$$\begin{cases} \varepsilon\ddot{\mathbf{E}} + \nabla\times(\mu^{-1}\nabla\times\mathbf{E}) = \mathbf{0} & \text{in } Q, & (2.1\text{a}) \\ \mathbf{E}(\cdot,0) = \mathbf{E}_0, \quad \dot{\mathbf{E}}(\cdot,0) = \mathbf{H}_0 & \text{on } \Omega, & (2.1\text{b}) \\ \mathbf{n}_\Omega \times \mathbf{E} = \mathbf{g} & \text{on } \Gamma\times[0,T]. & (2.1\text{c}) \end{cases}$$

Here $\mathbf{E} = (E_x, E_y, E_z)^\top, \mathbf{H} = (H_x, H_y, H_z)^\top, \mathbf{E}_0 \in \mathbf{H}_0(\text{curl};\Omega)$ and $\mathbf{H}_0 \in \mathbf{L}^2(\Omega)$ are the given source data; $\mathbf{g} \in \mathbf{L}^2_T(\Gamma\times[0,T])$. The permittivity ε and the permeability μ are assumed to be of the form $\varepsilon = \varepsilon_r A$ and $\mu = \mu_r A$, where ε_r, μ_r are piecewise constant satisfying $0 < c_{\varepsilon_r} \le \varepsilon_r \le C_{\varepsilon_r}$ and $0 < c_{\mu_r} \le \mu_r \le C_{\mu_r}$, and A is assumed to be real strictly positive definite and piecewise constant symmetric matrix. If K_i and K_j are two subsets of Ω with the boundary Γ_{ij} separating them with $\varepsilon = \varepsilon_i, \mu = \mu_i$ in K_i and $\varepsilon = \varepsilon_j, \mu = \mu_j$ in K_j, we further have the transmission conditions

$$\mathbf{n}\times\mathbf{E}_i = \mathbf{n}\times\mathbf{E}_j, \quad \mathbf{n}\times\big(\mu_i^{-1}\nabla\times\mathbf{E}_i\big) = \mathbf{n}\times\big(\mu_j^{-1}\nabla\times\mathbf{E}_j\big) \quad \text{on } \Gamma_{ij}, \tag{2.2}$$

where $\mathbf{n}$ is the unit exterior normal vector to K_i (or K_j).

Let $\mathbf{E}_0 \in \mathbf{H}_0(\text{curl};\Omega), \mathbf{H}_0 \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{L}^2_T(\Gamma\times[0,T])$, then (2.1) has a unique weak solution $\mathbf{E}$ with

$$\mathbf{E} \in L^2\big([0,T];\mathbf{H}_0(\text{curl};\Omega)\big), \quad \dot{\mathbf{E}} \in L^2\big([0,T];\mathbf{L}^2(\Omega)\big), \tag{2.3}$$

see [32]. Furthermore, if $\mathbf{g} = \mathbf{0}$, the solution is continuous in time with

$$\mathbf{E} \in C\big([0,T];\mathbf{H}_0(\text{curl};\Omega)\big), \quad \dot{\mathbf{E}} \in C\big([0,T];\mathbf{L}^2(\Omega)\big). \tag{2.4}$$

We denote the space of all solutions by

$$\begin{aligned} \mathbf{V} = \big\{\mathbf{E} \,\big|\, \mathbf{E} \text{ weak solution of } (2.1),(2.2) \text{ with } \mathbf{E}_0 \in \mathbf{H}_0(\text{curl};\Omega), \\ \mathbf{H}_0 \in \mathbf{L}^2(\Omega), \mathbf{g} \in \mathbf{L}^2_T(\Gamma\times[0,T])\big\}. \end{aligned} \tag{2.5}$$

Remark 2.1. Electromagnetic wave propagation in the considered anisotropic media arises in some applications including ground penetrating radar [4] and microwave interaction with wood [12]. Also, plane wave propagation in anisotropic media is a classical topic in textbooks on electromagnetism (see e.g. [6]). Besides, the anisotropic Maxwell equations under consideration includes a medium, which is a special case of the orthotropic medium considered in [20, Section 4], where the permittivity ε and the permeability μ are assumed to be of the form

$$\varepsilon = \varepsilon_r A = \varepsilon_r \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}, \quad \mu = \mu_r B = \mu_r \begin{pmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{12} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{pmatrix}. \tag{2.6}$$

Here, the anisotropic matrix A and B are symmetric positive definite matrix. When the case of $A = B$ occurs, which cannot be ruled out in the real world, e.g. a transmission problem modeling the scattering of electromagnetic waves by an anisotropic dielectric medium (see [19, Section 5.2]),the associated model in the orthotropic medium degenerates to a special form of the model considered in this paper.

Moreover, we find that the electric and magnetic anisotropy defined by the same matrix A occurs in a non-physical electromagnetic field, induced by the use of perfectly matched layer (PML), see [19, Eqs. (17)-(18)]. It motivates us to develop the corresponding efficient numerical scheme for solving non-physical anisotropic Maxwell's equations.

For the completely anisotropic case, i.e. $A \neq B$, it needs to develop new approximation theory of anisotropic Trefftz basis functions, which will be an interesting and complicated research direction.

3. Space-time Finite Element Space

Let the time domain $(0, T)$ be divided into $N \in \mathbb{N}$ intervals I_n $(0 \le n \le N-1)$ composing a partition $\mathcal{T}_\tau$, with $I_n = (t_n, t_{n+1}), \tau_n = t_{n+1} - t_n = |I_n|, \tau = \max_{0 \le n \le N-1} \tau_n$. For each $0 \le n \le N-1$, we introduce a quasi-uniform finite element mesh $\mathcal{T}_n = \{K\}$ of the spatial domain Ω with $h_K = \text{diam}K, h_n = \max_{K \in \mathcal{T}_n} h_K$. Then the space-time domain $Q = \Omega \times (0, T)$ can be partitioned with a finite element mesh $\mathcal{T}_h$ given by

$$\mathcal{T}_h = \{K \times I_n,\ K \in \mathcal{T}_n,\ 0 \le n \le N-1\}.$$

Here $\mathcal{T}_h$ is a tensor product mesh and $h = \max_{0 \le n \le N-1} h_n$. Besides, we denote the outward-pointing unit normal vector on ∂K by $\mathbf{n}_K$.

The discrete space-time mesh will consist of piecewise polynomials on each time slab $\Omega \times I_n$, defined by the local space-time finite element space

$$\mathbf{S}_n^{h,p} := \left\{ \mathbf{E} \in \mathbf{L}^2(\Omega \times I_n) : \mathbf{E}\big|_{K \times I_n} \in \mathcal{P}_p(\mathbb{R}^{3+1})^3,\ K \in \mathcal{T}_n \right\},$$

where $\mathcal{P}_p$ is the space of polynomials of total degree p; the whole space-time finite element space on $\Omega \times [0, T]$ will be denoted by

$$\mathbf{V}^{h,p} := \left\{ \mathbf{E} \in \mathbf{L}^2(\Omega \times [0, T]) : \mathbf{E}\big|_{\Omega \times I_n} \in \mathbf{S}_n^{h,p},\ N = 0, 1, \dots, N-1 \right\}.$$

We denote by $\mathcal{F}_n = \bigcup_{K \in \mathcal{T}_n} \partial K$ the skeleton of the spatial mesh, by $\mathcal{F}_n^{\text{int}} = \bigcup_{K \in \mathcal{T}_n} \partial K \backslash \partial \Omega$ the union of the interior faces, and by $\mathcal{F}_n^{\text{bou}}$ the union of the boundary faces, respectively. Moreover, we define the union of two skeletons of two subsequent meshes by $\tilde{\mathcal{F}}_n = \mathcal{F}_n \bigcup \mathcal{F}_{n-1}$, $n = 1, \dots, N-1$; if $n = 0$, we set $\tilde{\mathcal{F}}_0 = \mathcal{F}_0$ and if $n = N$, we set $\tilde{\mathcal{F}}_N = \mathcal{F}_N$. In particular, if the spatial mesh remains fixed, all the formulas hold with $\mathcal{F} = \mathcal{F}_n = \tilde{\mathcal{F}}_n$. Finally, we set

$$\mathcal{F}_h^0 = \Omega \times \{t = 0\}, \quad \mathcal{F}_h^T = \Omega \times \{t = T\}, \quad \mathcal{F}_h^\Gamma = \Gamma \times [0, T].$$

Let $\boldsymbol{\tau}$ be a piecewise smooth vector field on $\mathcal{T}_h$. Let K^+ and K^- be two spatial elements sharing a face $e = \partial K^+ \bigcap \partial K^- \subset \mathcal{F}_n^{\text{int}}$ with respective outward normal vectors $\mathbf{n}^+$ and $\mathbf{n}^-$ on e. Let $\boldsymbol{\tau}^\pm$ be the traces on e with limits taken from $K^\pm$. We define the respective jump and average across each face $e \in \mathcal{F}_n^{\text{int}}$ by

$$\{\{\boldsymbol{\tau}\}\} := \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}, \quad [\boldsymbol{\tau}]_T := \mathbf{n}^+ \times \boldsymbol{\tau}^+ + \mathbf{n}^- \times \boldsymbol{\tau}^-,$$

if $e \subset \partial K^+ \bigcap \partial \Omega$, we set $\{\{\boldsymbol{\tau}\}\} = \boldsymbol{\tau}^+$ and $[\boldsymbol{\tau}]_T = \mathbf{n}^+ \times \boldsymbol{\tau}^+$. Further, we define the temporal full jump by $[\![\boldsymbol{\tau}(t_n)]\!] = \boldsymbol{\tau}(t_n^+) - \boldsymbol{\tau}(t_n^-), [\![\boldsymbol{\tau}(t_0)]\!] = \boldsymbol{\tau}(t_0^+)$. Here $\boldsymbol{\tau}^-$ and $\boldsymbol{\tau}^+$ denote the traces of the function $\boldsymbol{\tau}$ from the adjacent elements at lower and higher times, respectively. We will denote the spatial mesh size by $\tilde{h} : \Omega \times [0, T] \to \mathbb{R}$, defined by $\tilde{h}(\mathbf{x}, t) = \text{diam}(K)$ if $\mathbf{x} \in K$ for $K \in \mathcal{T}_n$ and $t \in (t_n, t_{n+1}]$; when $\mathbf{x} \in e = \partial K^+ \bigcap \partial K^-$, we set $\tilde{h}(\mathbf{x}, t) = \{\{h_K\}\}$ to be the average. Moreover, we assume that the mesh $\mathcal{T}_n$ satisfies the shape regularity condition, that is, there exists $c_{\mathcal{T}}$ such that

$$\frac{\text{diam}(K)}{\delta_K} \le c_{\mathcal{T}}, \quad \forall K \in \mathcal{T}_n, \quad n = 0, 1, \dots, N-1, \tag{3.1}$$

where δ_K is the radius of the inscribed circle of K.

At last, we denote by $\nabla_n \times \boldsymbol{\tau}$ the broken spatial curl operator on each time slab $\Omega \times I_n$, given by

$$(\nabla_n \times \boldsymbol{\tau})|_K := (\nabla \times \boldsymbol{\tau})|_K, \quad \forall K \in \mathcal{T}_n, \quad \boldsymbol{\tau} \in C\big(I_n; \mathbf{H}_0(\mathrm{curl}; \Omega)\big) + \mathbf{S}_n^{h,p},$$

collectively, the broken curl operator $\tilde{\nabla} \times \boldsymbol{\tau}$ on the whole space-time domain Q will be defined as

$$(\tilde{\nabla} \times \boldsymbol{\tau})|_{\Omega\times I_n} = (\nabla_n \times \boldsymbol{\tau})|_{\Omega\times I_n}, \quad n = 0, \ldots, N-1, \quad \boldsymbol{\tau} \in C\left(\prod_{n=0}^{N-1} I_n; \mathbf{H}_0(\mathrm{curl}; \Omega)\right) + \mathbf{V}^{h,p}.$$

4. A Space-time DG Method

We will follow an energy argument employed in [1] to derive the DG variational formulation for the second order linear anisotropic Maxwell equations (2.1) in three-dimensional inhomogeneous media with the nonhomogeneous boundary condition.

Assuming that $\mathbf{E}$ satisfying (2.1) is smooth enough and the test function $\boldsymbol{\phi} \in \mathbf{V} + \mathbf{V}^{h,p}$, then integrating by parts the Eq. (2.1a), and adding the symmetric term involved on the spacial boundary integral and the penalty term yields

$$\begin{aligned}&(\varepsilon\ddot{\mathbf{E}}, \dot{\boldsymbol{\phi}})_{\Omega\times I_n} + (\mu^{-1}\tilde{\nabla}\times\mathbf{E}, \tilde{\nabla}\times\dot{\boldsymbol{\phi}})_{\Omega\times I_n} - \big(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}\}\}, [\dot{\boldsymbol{\phi}}]_T\big)_{\mathcal{F}_n\times I_n}\\ &\quad - \big([\mathbf{E}]_T, \{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\phi}}\}\}\big)_{\mathcal{F}_n\times I_n} + \big(\sigma_0[\mathbf{E}]_T, [\dot{\boldsymbol{\phi}}]_T\big)_{\mathcal{F}_n\times I_n} = \big(\sigma_0\mathbf{g}, [\dot{\boldsymbol{\phi}}]_T\big)_{\Gamma\times I_n},\end{aligned} \tag{4.1}$$

where

$$\sigma_0 = C_{\sigma_0} p^2 \big(\tilde{h}(\mathbf{x}, t)\big)^{-1} \tag{4.2}$$

for a positive constant C_{σ_0} to be chosen later. Thus, we can define the following discrete energy $E_h(t, \boldsymbol{\phi})$ at time $t \in I_n$ for $\boldsymbol{\phi} \in \mathbf{V} + \mathbf{V}^{h,p}$:

$$\begin{aligned}E_h(t, \boldsymbol{\phi}) = {}& \frac{1}{2}\big\|\varepsilon^{\frac{1}{2}}\dot{\boldsymbol{\phi}}(t)\big\|_\Omega^2 + \frac{1}{2}\big\|\mu^{-\frac{1}{2}}\tilde{\nabla}\times\boldsymbol{\phi}(t)\big\|_\Omega^2 + \frac{1}{2}\big\|\sigma_0^{\frac{1}{2}}[\boldsymbol{\phi}(t)]_T\big\|_{\mathcal{F}_n}^2\\ &- \big(\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t)\}\}, [\boldsymbol{\phi}(t)]_T\big)_{\mathcal{F}_n}.\end{aligned} \tag{4.3}$$

By the existing inverse inequality $\|\boldsymbol{\phi}\|_{\partial K}^2 \le C_{\mathrm{inv}} p^2 |\partial K|/|K| \|\boldsymbol{\phi}\|_K^2$ for any $\boldsymbol{\phi} \in \mathcal{P}_p(K)^3$ (see [7]), we can obtain

$$\begin{aligned}&2\int_{\mathcal{F}_n} \big|\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t,s)\}\}\big|^2 ds\\ \le{}& \sum_{K\in\mathcal{T}_n} \int_{\partial K} |\mu^{-1}\nabla\times\boldsymbol{\phi}(t,s)|^2 ds\\ \le{}& \sum_{K\in\mathcal{T}_n} \frac{C_{\mathrm{inv}} p^2 |\partial K|}{|K|} \int_K |\mu^{-1}\nabla\times\boldsymbol{\phi}(t,\mathbf{x})|^2 d\mathbf{x}\\ \le{}& \sum_{K\in\mathcal{T}_n} \frac{C_{\mathrm{inv}} p^2 |\partial K| c_{\mu_r}^{-1} \|A^{-\frac{1}{2}}\|^2}{|K|} \int_K \big|\mu^{-\frac{1}{2}}\nabla\times\boldsymbol{\phi}(t,\mathbf{x})\big|^2 d\mathbf{x}\\ \le{}& \sum_{K\in\mathcal{T}_n} \frac{C_{\mathrm{inv}} p^2 c_{\mathcal{T}}^3 c_{\mu_r}^{-1} \|A^{-\frac{1}{2}}\|^2}{h_K} \int_K \big|\mu^{-\frac{1}{2}}\nabla\times\boldsymbol{\phi}(t,\mathbf{x})\big|^2 d\mathbf{x}.\end{aligned} \tag{4.4}$$

Therefore, provided that the penalization parameter C_{σ_0} is chosen large enough such that,

$$C_{\sigma_0} \ge C_{\mathrm{inv}} c_{\mathcal{T}}^3 c_{\mu_r}^{-1} \big\|A^{-\frac{1}{2}}\big\|^2, \tag{4.5}$$

we have that by the inequality $|xy| \le 3x^2/8 + 2y^2/3$ for any $x, y \in \mathbb{R}$,

$$\begin{aligned}
&\big|\big(\{\{\mu^{-1}\tilde{\nabla} \times \boldsymbol{\phi}(t)\}\}, [\boldsymbol{\phi}(t)]_T\big)_{\mathcal{F}_n}\big| \\
&\le \big\|\sigma_0^{\frac{1}{2}}[\boldsymbol{\phi}(t)]_T\big\|_{\mathcal{F}_n}\big\|\sigma_0^{-\frac{1}{2}}\{\{\mu^{-1}\tilde{\nabla} \times \boldsymbol{\phi}(t)\}\}\big\|_{\mathcal{F}_n} \\
&\le \frac{3}{8}\big\|\sigma_0^{\frac{1}{2}}[\boldsymbol{\phi}(t)]_T\big\|^2_{\mathcal{F}_n} + \frac{2}{3}\big\|\sigma_0^{-\frac{1}{2}}\{\{\mu^{-1}\tilde{\nabla} \times \boldsymbol{\phi}(t)\}\}\big\|^2_{\mathcal{F}_n} \\
&\le \frac{3}{8}\big\|\sigma_0^{\frac{1}{2}}[\boldsymbol{\phi}(t)]_T\big\|^2_{\mathcal{F}_n} + \frac{1}{3}\sum_{K\in\mathcal{T}_n}\big\|\mu^{-\frac{1}{2}}\nabla \times \boldsymbol{\phi}(t)\big\|^2_K,
\end{aligned} \tag{4.6}$$

ensuring the positivity of the energy $E_h(t, \boldsymbol{\phi})$ for functions in $\mathbf{V} + \mathbf{V}^{h,p}$,

$$E_h(t, \boldsymbol{\phi}) \ge \frac{1}{2}\big\|\varepsilon^{\frac{1}{2}}\dot{\boldsymbol{\phi}}(t)\big\|^2_\Omega + \frac{1}{6}\big\|\mu^{-\frac{1}{2}}\tilde{\nabla} \times \boldsymbol{\phi}(t)\big\|^2_\Omega + \frac{1}{8}\big\|\sigma_0^{\frac{1}{2}}[\boldsymbol{\phi}(t)]_T\big\|^2_{\mathcal{F}_n}. \tag{4.7}$$

Remark 4.1. We would like to point out that compared with the nonnegativity of the energy $E_h(t, v)$ of [1], the energy $E_h(t, \boldsymbol{\phi})$ defined above is positive provided $\boldsymbol{\phi}$ being nonzero owing to the improvement of (4.6).

Choosing test function as $\boldsymbol{\phi} = \mathbf{E}$ in (4.1) and summing over n, we get

$$\begin{aligned}
&\sum_{n=0}^{N-1}\int_{I_n}\frac{d}{dt}\bigg(\frac{1}{2}\big\|\varepsilon^{\frac{1}{2}}\dot{\mathbf{E}}\big\|^2_\Omega + \frac{1}{2}\big\|\mu^{-\frac{1}{2}}\tilde{\nabla} \times \mathbf{E}\big\|^2_\Omega + \frac{1}{2}\big\|\sigma_0^{\frac{1}{2}}[\mathbf{E}]_T\big\|^2_{\mathcal{F}_n} - \big(\{\{\mu^{-1}\tilde{\nabla} \times \mathbf{E}\}\}, [\mathbf{E}]_T\big)_{\mathcal{F}_n}\bigg)dt \\
&= E_h(t_N^-, \mathbf{E}) - E_h(t_0^+, \mathbf{E}) - \sum_{n=1}^{N-1}[\![E_h(t_n, \mathbf{E})]\!] = \big(\sigma_0\mathbf{g}, [\dot{\mathbf{E}}]_T\big)_{\Gamma\times I}.
\end{aligned} \tag{4.8}$$

For permitting discontinuities in time, the formulation (4.1) needs to be amended to control the terms $[\![E_h(t_n, \mathbf{E})]\!]$ having no sign in a consistent fashion. For it, we will recall the elementary algebraic identity, for some vector quantities $\mathbf{f}, \mathbf{h}$,

$$\begin{aligned}
&- \big[\!\big[\mathbf{f}\big(\mathbf{E}(t_n)\big)\cdot\mathbf{h}\big(\mathbf{E}(t_n)\big)\big]\!\big] + \big[\!\big[\mathbf{f}\big(\mathbf{E}(t_n)\big)\big]\!\big]\cdot\mathbf{h}\big(\mathbf{E}(t_n^+)\big) + \big[\!\big[\mathbf{h}\big(\mathbf{E}(t_n)\big)\big]\!\big]\cdot\mathbf{f}\big(\mathbf{E}(t_n^+)\big) \\
&= \big[\!\big[\mathbf{f}\big(\mathbf{E}(t_n)\big)\big]\!\big]\cdot\big[\!\big[\mathbf{h}\big(\mathbf{E}(t_n)\big)\big]\!\big].
\end{aligned} \tag{4.9}$$

This motivates the introduction of the additional terms into the original formulation (4.1) to change the jump of a product to the product of jumps in time in the above energy identity (4.8), without compromising its consistency.

For the first term $(\varepsilon\ddot{\mathbf{E}}, \dot{\boldsymbol{\phi}})_{\Omega\times I_n}$ in (4.1), we add the additional term $(\varepsilon[\![\dot{\mathbf{E}}(t_n)]\!], \dot{\boldsymbol{\phi}}(t_n^+))_\Omega$, $n = 0, 1, \ldots, N-1$. For $n > 0$, this does not change the consistency of (4.1) with respect to (2.1) in weak form owing to the smoothness assumption on the initial data that the exact solution satisfies $\dot{\mathbf{E}} \in C([0,T]; \mathbf{L}^2(\Omega))$. For $n = 0$, to ensure consistency, we also need to add the term $(\varepsilon\mathbf{H}_0, \dot{\boldsymbol{\phi}}(t_0^+))_\Omega$ to the right-hand side of (4.1). Using (4.9), for $\mathbf{E} = \boldsymbol{\phi} = \boldsymbol{\psi}$ with a piecewise sufficiently smooth function, we get

$$\begin{aligned}
&\sum_{n=0}^{N-1}(\varepsilon\ddot{\boldsymbol{\psi}}, \dot{\boldsymbol{\psi}})_{\Omega\times I_n} + \big(\varepsilon[\![\dot{\boldsymbol{\psi}}(t_n)]\!], \dot{\boldsymbol{\psi}}(t_n^+)\big)_\Omega \\
&= \frac{1}{2}\big\|\varepsilon^{\frac{1}{2}}\dot{\boldsymbol{\psi}}(t_N^-)\big\|^2_\Omega + \sum_{n=0}^{N-1} -\frac{1}{2}\big[\!\big[\big\|\varepsilon^{\frac{1}{2}}\dot{\boldsymbol{\psi}}(t_n)\big\|^2_\Omega\big]\!\big] + \big(\varepsilon[\![\dot{\boldsymbol{\psi}}(t_n)]\!], \dot{\boldsymbol{\psi}}(t_n^+)\big)_\Omega
\end{aligned}$$

$$= \frac{1}{2}\big\|\varepsilon^{\frac{1}{2}}\dot{\boldsymbol{\psi}}(t_N^-)\big\|_\Omega^2 + \frac{1}{2}\big\|\varepsilon^{\frac{1}{2}}\dot{\boldsymbol{\psi}}(t_0^+)\big\|_\Omega^2 + \frac{1}{2}\sum_{n=1}^{N-1}\big\|\varepsilon^{\frac{1}{2}}[\![\dot{\boldsymbol{\psi}}(t_n)]\!]\big\|_\Omega^2$$

with the extra terms contributing to energy dissipation. Similar considerations lead to the accretion of corresponding terms $([\![\mu^{-1}\tilde{\nabla}\times\mathbf{E}(t_n)]\!], \tilde{\nabla}\times\boldsymbol{\phi}(t_n^+))_\Omega$ and $(\sigma_0[\![[\mathbf{E}(t_n)]_T]\!], [\boldsymbol{\phi}(t_n^+)]_T)_{\tilde{\mathcal{F}}_n}$ to treat the second and the last terms on the left-hand side of (4.1), respectively, which induces that

$$\begin{aligned}
&\sum_{n=0}^{N-1}(\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}, \tilde{\nabla}\times\dot{\boldsymbol{\psi}})_{\Omega\times I_n} + \big([\![\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n)]\!], \tilde{\nabla}\times\boldsymbol{\psi}(t_n^+)\big)_\Omega\\
&= \frac{1}{2}\big\|\mu^{-\frac{1}{2}}\tilde{\nabla}\times\boldsymbol{\psi}(t_N^-)\big\|_\Omega^2 + \frac{1}{2}\big\|\mu^{-\frac{1}{2}}\tilde{\nabla}\times\boldsymbol{\psi}(t_0^+)\big\|_\Omega^2 + \frac{1}{2}\sum_{n=1}^{N-1}\big\|\mu^{-\frac{1}{2}}[\![\tilde{\nabla}\times\boldsymbol{\psi}(t_n)]\!]\big\|_\Omega^2,\\
&\sum_{n=0}^{N-1}\big(\sigma_0[\boldsymbol{\psi}]_T, [\dot{\boldsymbol{\psi}}]_T\big)_{\mathcal{F}_n\times I_n} + \big(\sigma_0\big[\!\big[[\boldsymbol{\psi}(t_n)]_T\big]\!\big], [\boldsymbol{\psi}(t_n^+)]_T\big)_{\tilde{\mathcal{F}}_n}\\
&= \frac{1}{2}\big\|\sigma_0^{\frac{1}{2}}[\boldsymbol{\psi}(t_N^-)]_T\big\|_{\mathcal{F}_{N-1}}^2 + \frac{1}{2}\big\|\sigma_0^{\frac{1}{2}}[\boldsymbol{\psi}(t_0^+)]_T\big\|_{\mathcal{F}_0}^2 + \frac{1}{2}\sum_{n=1}^{N-1}\big\|\big[\!\big[\sigma_0^{\frac{1}{2}}[\boldsymbol{\psi}(t_n)]_T\big]\!\big]\big\|_{\tilde{\mathcal{F}}_n}^2.
\end{aligned} \tag{4.10}$$

Hereafter, we have used that

$$\begin{aligned}
\big(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}\}\}, [\dot{\boldsymbol{\phi}}]_T\big)_{\mathcal{F}_n\times I_n} &= \big(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}\}\}, [\dot{\boldsymbol{\phi}}]_T\big)_{\tilde{\mathcal{F}}_n\times I_n},\\
\big([\mathbf{E}]_T, \{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\phi}}\}\}\big)_{\mathcal{F}_n\times I_n} &= \big([\mathbf{E}]_T, \{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\phi}}\}\}\big)_{\tilde{\mathcal{F}}_n\times I_n},
\end{aligned} \tag{4.11}$$

by the assumption (2.2) and that the spatial flux of test functions are continuous within a space-time element $K\times I_n, K\in\mathcal{T}_n$; meanwhile, for consistency, the terms $(\mu^{-1}\tilde{\nabla}\times\mathbf{E}_0, \tilde{\nabla}\times\boldsymbol{\phi}(t_0^+))_\Omega$ and $(\sigma_0[\mathbf{E}_0]_T, [\boldsymbol{\phi}(t_0^+)]_T)_{\Gamma_0}$ are also added to the right-hand side of (4.1). For the remaining third and fourth term on the left-hand side of (4.1), we introduce the additional terms $-([\![\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}(t_n)\}\}]\!], [\boldsymbol{\phi}(t_n^+)]_T)_{\tilde{\mathcal{F}}_n}$ and $-([\![[\mathbf{E}(t_n)]_T]\!], \{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\}\})_{\tilde{\mathcal{F}}_n}$, which further induces that

$$\begin{aligned}
&\sum_{n=0}^{N-1} -\big(\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}\}\}, [\dot{\boldsymbol{\psi}}]_T\big)_{\mathcal{F}_n\times I_n} - \big([\boldsymbol{\psi}]_T, \{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\psi}}\}\}\big)_{\mathcal{F}_n\times I_n}\\
&\quad - \big([\![\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n)\}\}]\!], [\boldsymbol{\psi}(t_n^+)]_T\big)_{\tilde{\mathcal{F}}_n} - \big(\big[\!\big[[\boldsymbol{\psi}(t_n)]_T\big]\!\big], \{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n^+)\}\}\big)_{\tilde{\mathcal{F}}_n}\\
&= -\big(\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_N^-)\}\}, [\boldsymbol{\psi}(t_N^-)]_T\big)_{\mathcal{F}_{N-1}} - \big(\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_0^+)\}\}, [\boldsymbol{\psi}(t_0^+)]_T\big)_{\mathcal{F}_0}\\
&\quad - \sum_{n=1}^{N-1}\big([\![\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n)\}\}]\!], \big[\!\big[[\boldsymbol{\psi}(t_n)]_T\big]\!\big]\big)_{\tilde{\mathcal{F}}_n}.
\end{aligned}$$

Meanwhile, for consistency, the terms $-(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}_0\}\}, [\boldsymbol{\phi}(t_0^+)]_T)_{\mathcal{F}_0}$ and $-([\mathbf{E}_0]_T, \{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_0^+)\}\})_{\mathcal{F}_0}$ are appended to the right-hand side of (4.1), and the term $(\sigma_1\mathbf{g}, [\boldsymbol{\phi}]_T)_{\Gamma\times I_n}$ involving the boundary condition of (2.1) is also added to the right-hand side of (4.1). Finally, for the convergence analysis, we need to add the two terms $(\sigma_1[\mathbf{E}]_T, [\boldsymbol{\phi}]_T)_{\mathcal{F}_n\times I_n}$ and $(\sigma_2[\mu^{-1}\tilde{\nabla}\times\mathbf{E}]_T, [\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T)_{\mathcal{F}_n^{\text{int}}\times I_n}$ with positive penalty parameters σ_1 and σ_2 to the left-hand side of (4.1).

Note that all the additional terms do not change the consistency due to (2.4), (2.2) and the addition of corresponding terms involving the initial condition and boundary condition to the right-hand side of (4.1).

To summarize, we can now present the space-time DG variational formulation of our method

$$\begin{aligned}
&\sum_{n=0}^{N-1}(\varepsilon\ddot{\mathbf{E}},\dot{\boldsymbol{\phi}})_{\Omega\times I_n}+\big(\varepsilon[\![\dot{\mathbf{E}}(t_n)]\!],\dot{\boldsymbol{\phi}}(t_n^+)\big)_\Omega\\
&\quad+(\mu^{-1}\tilde{\nabla}\times\mathbf{E},\tilde{\nabla}\times\dot{\boldsymbol{\phi}})_{\Omega\times I_n}+\big([\![\mu^{-1}\tilde{\nabla}\times\mathbf{E}(t_n)]\!],\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\big)_\Omega\\
&\quad-\big(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}\}\},[\dot{\boldsymbol{\phi}}]_T\big)_{\mathcal{F}_n\times I_n}-\big([\![\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}(t_n)\}\}]\!],[\boldsymbol{\phi}(t_n^+)]_T\big)_{\tilde{\mathcal{F}}_n}\\
&\quad-\big([\mathbf{E}]_T,\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\phi}}\}\}\big)_{\mathcal{F}_n\times I_n}-\big([\![[\mathbf{E}(t_n)]_T]\!],\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\}\}\big)_{\tilde{\mathcal{F}}_n}\\
&\quad+(\sigma_0[\mathbf{E}]_T,[\dot{\boldsymbol{\phi}}]_T)_{\mathcal{F}_n\times I_n}+\big(\sigma_0[\![[\mathbf{E}(t_n)]_T]\!],[\boldsymbol{\phi}(t_n^+)]_T\big)_{\tilde{\mathcal{F}}_n}\\
&\quad+(\sigma_1[\mathbf{E}]_T,[\boldsymbol{\phi}]_T)_{\mathcal{F}_n\times I_n}+\big(\sigma_2[\mu^{-1}\tilde{\nabla}\times\mathbf{E}]_T,[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n^{\mathrm{int}}\times I_n}=b^{\mathrm{init}}(\boldsymbol{\phi}),
\end{aligned}\tag{4.12}$$

where $b^{\mathrm{init}}(\boldsymbol{\phi})$ is given by

$$\begin{aligned}
b^{\mathrm{init}}(\boldsymbol{\phi})&=\big(\varepsilon\mathbf{H}_0,\dot{\boldsymbol{\phi}}(t_0^+)\big)_\Omega+\big(\mu^{-1}\tilde{\nabla}\times\mathbf{E}_0,\tilde{\nabla}\times\boldsymbol{\phi}(t_0^+)\big)_\Omega-\big(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}_0\}\},[\boldsymbol{\phi}(t_0^+)]_T\big)_{\mathcal{F}_0}\\
&\quad-\big([\mathbf{E}_0]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_0^+)\}\}\big)_{\mathcal{F}_0}+\big(\sigma_0[\mathbf{E}_0]_T,[\boldsymbol{\phi}(t_0^+)]_T\big)_{\mathcal{F}_0}\\
&\quad+(\sigma_1\mathbf{g},[\boldsymbol{\phi}]_T)_{\Gamma\times I}+\big(\sigma_0\mathbf{g},[\dot{\boldsymbol{\phi}}]_T\big)_{\Gamma\times I}.
\end{aligned}$$

We choose the flux parameters σ_1 and σ_2 depending on p and h in the following way:

$$\sigma_1|_{\partial K\bigcap\mathcal{F}_n\times I_n}=\frac{p^3}{\tilde{h}\tau_n},\qquad\sigma_2=\frac{\tilde{h}}{\tau_n}.\tag{4.13}$$

Thus, we achieve a space-time discrete method, which can also be thought of in a time-stepping method. Correspondingly, we define the following three bilinear forms:

$$\begin{aligned}
a_n(\mathbf{E},\boldsymbol{\phi})&=(\varepsilon\ddot{\mathbf{E}},\dot{\boldsymbol{\phi}})_{\Omega\times I_n}+\big(\varepsilon\dot{\mathbf{E}}(t_n^+),\dot{\boldsymbol{\phi}}(t_n^+)\big)_\Omega\\
&\quad+(\mu^{-1}\tilde{\nabla}\times\mathbf{E},\tilde{\nabla}\times\dot{\boldsymbol{\phi}})_{\Omega\times I_n}+\big(\mu^{-1}\tilde{\nabla}\times\mathbf{E}(t_n^+),\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\big)_\Omega\\
&\quad-\big(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}\}\},[\dot{\boldsymbol{\phi}}]_T\big)_{\mathcal{F}_n\times I_n}-\big(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}(t_n^+)\}\},[\boldsymbol{\phi}(t_n^+)]_T\big)_{\tilde{\mathcal{F}}_n}\\
&\quad-\big([\mathbf{E}]_T,\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\phi}}\}\}\big)_{\mathcal{F}_n\times I_n}-\big([\mathbf{E}(t_n^+)]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\}\}\big)_{\tilde{\mathcal{F}}_n}\\
&\quad+\big(\sigma_0[\mathbf{E}]_T,[\dot{\boldsymbol{\phi}}]_T\big)_{\mathcal{F}_n\times I_n}+\big(\sigma_0[\mathbf{E}(t_n^+)]_T,[\boldsymbol{\phi}(t_n^+)]_T\big)_{\tilde{\mathcal{F}}_n}\\
&\quad+\big(\sigma_1[\mathbf{E}]_T,[\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n}+\big(\sigma_2[\mu^{-1}\tilde{\nabla}\times\mathbf{E}]_T,[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n^{\mathrm{int}}\times I_n},
\end{aligned}\tag{4.14}$$

$$\begin{aligned}
b_n(\mathbf{E},\boldsymbol{\phi})&=\big(\varepsilon\dot{\mathbf{E}}(t_n^-),\dot{\boldsymbol{\phi}}(t_n^+)\big)_\Omega+\big(\mu^{-1}\tilde{\nabla}\times\mathbf{E}(t_n^-),\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\big)_\Omega\\
&\quad-\big(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}(t_n^-)\}\},[\boldsymbol{\phi}(t_n^+)]_T\big)_{\mathcal{F}_n}-\big([\mathbf{E}(t_n^-)]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\}\}\big)_{\mathcal{F}_{n-1}}\\
&\quad+\big(\sigma_0[\mathbf{E}(t_n^-)]_T,[\boldsymbol{\phi}(t_n^+)]_T\big)_{\tilde{\mathcal{F}}_n},
\end{aligned}\tag{4.15}$$

$$a(\mathbf{E},\boldsymbol{\phi})=\sum_{n=0}^{N-1}a_n(\mathbf{E},\boldsymbol{\phi})-\sum_{n=1}^{N-1}b_n(\mathbf{E},\boldsymbol{\phi}),\tag{4.16}$$

which just give the left-hand side of (4.12). Similarly, we define the following two linear forms:

$$\begin{aligned}
b_0^{\mathrm{init}}(\boldsymbol{\phi})&=\big(\varepsilon\mathbf{H}_0,\dot{\boldsymbol{\phi}}(t_0^+)\big)_\Omega+\big(\mu^{-1}\tilde{\nabla}\times\mathbf{E}_0,\tilde{\nabla}\times\boldsymbol{\phi}(t_0^+)\big)_\Omega-\big(\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}_0\}\},[\boldsymbol{\phi}(t_0^+)]_T\big)_{\mathcal{F}_0}\\
&\quad-\big([\mathbf{E}_0]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_0^+)\}\}\big)_{\mathcal{F}_0}+\big(\sigma_0[\mathbf{E}_0]_T,[\boldsymbol{\phi}(t_0^+)]_T\big)_{\mathcal{F}_0}\\
&\quad+(\sigma_1\mathbf{g},[\boldsymbol{\phi}]_T)_{\Gamma\times I_0}+\big(\sigma_0\mathbf{g},[\dot{\boldsymbol{\phi}}]_T\big)_{\Gamma\times I_0},\\
b_n^{\mathrm{init}}(\boldsymbol{\phi})&=(\sigma_1\mathbf{g},[\boldsymbol{\phi}]_T)_{\Gamma\times I_n}+\big(\sigma_0\mathbf{g},[\dot{\boldsymbol{\phi}}]_T\big)_{\Gamma\times I_n},
\end{aligned}$$

which indicates that

$$b^{\text{init}}(\boldsymbol{\phi}) = \sum_{n=0}^{N-1} b_n^{\text{init}}(\boldsymbol{\phi}).$$

Definition 4.1. *Given subspaces $X_n \subset \mathbf{S}_n^{h,p}$, the time-stepping method is described by finding $\mathbf{E}^{(n)} \in X_n, n = 1, 2, \ldots, N-1$ such that*

$$a_n(\mathbf{E}^{(n)}, \boldsymbol{\phi}) = b_n(\mathbf{E}^{(n-1)}, \boldsymbol{\phi}) + b_n^{\text{init}}(\boldsymbol{\phi}), \quad \forall\, \boldsymbol{\phi} \in X_n, \tag{4.17}$$

$$a_0(\mathbf{E}^{(0)}, \boldsymbol{\phi}) = b_0^{\text{init}}(\boldsymbol{\phi}), \qquad \forall\, \boldsymbol{\phi} \in X_0. \tag{4.18}$$

Equivalently, given a subspace $X \subset \mathbf{V}^{h,p}$, the full time-stepping discrete system can be presented as find $\mathbf{E} \in X$ such that

$$a(\mathbf{E}, \boldsymbol{\phi}) = b^{\text{init}}(\boldsymbol{\phi}), \quad \forall\, \boldsymbol{\phi} \in X. \tag{4.19}$$

Lemma 4.1. *The following identities hold for any $\boldsymbol{\psi} \in \mathbf{V} + \mathbf{V}^{h,p}$:*

$$a_n(\boldsymbol{\psi}, \boldsymbol{\psi}) = E_h(t_{n+1}^-, \boldsymbol{\psi}) + E_h(t_n^+, \boldsymbol{\psi}) + \big\|\sigma_1^{\frac{1}{2}}[\boldsymbol{\psi}]_T\big\|^2_{\mathcal{F}_n \times I_n} + \big\|\sigma_2^{\frac{1}{2}}[\mu^{-1}\tilde{\nabla} \times \boldsymbol{\psi}]_T\big\|^2_{\mathcal{F}_n^{\text{int}} \times I_n} \tag{4.20}$$

for $n = 0, 1, \ldots, N-1$, and

$$\begin{aligned} a(\boldsymbol{\psi}, \boldsymbol{\psi}) = {} & E_h(t_N^-, \boldsymbol{\psi}) + E_h(t_0^+, \boldsymbol{\psi}) \\ & + \sum_{n=1}^{N-1} \bigg(\frac{1}{2}\big\|\varepsilon^{\frac{1}{2}} [\![\dot{\boldsymbol{\psi}}(t_n)]\!]\big\|^2_{\Omega} + \frac{1}{2}\big\|\mu^{-\frac{1}{2}} [\![\tilde{\nabla} \times \boldsymbol{\psi}(t_n)]\!]\big\|^2_{\Omega} \\ & \qquad - \big([\![\{\{\mu^{-1}\tilde{\nabla} \times \boldsymbol{\psi}(t_n)\}\}]\!], [\![[\boldsymbol{\psi}(t_n)]_T]\!]\big)_{\tilde{\mathcal{F}}_n} + \frac{1}{2}\big\| [\![\sigma_0^{\frac{1}{2}}[\boldsymbol{\psi}(t_n)]_T]\!] \big\|^2_{\tilde{\mathcal{F}}_n} \bigg) \\ & + \sum_{n=0}^{N-1} \Big(\big\|\sigma_1^{\frac{1}{2}}[\boldsymbol{\psi}]_T\big\|^2_{\mathcal{F}_n \times I_n} + \big\|\sigma_2^{\frac{1}{2}}[\mu^{-1}\tilde{\nabla} \times \boldsymbol{\psi}]_T\big\|^2_{\mathcal{F}_n^{\text{int}} \times I_n} \Big). \end{aligned} \tag{4.21}$$

Proof. The identities follow from the definitions of the bilinear forms and the energy $E_h(t; \boldsymbol{\psi})$. □

Next we investigate the consistency and stability of the discrete scheme.

Theorem 4.1. *Let the space $\mathbf{V}^{h,p}$ be given. Then, the following statements hold:*

1. *Let $\mathbf{E}$ be the weak solution of (2.1), (2.2) with $\mathbf{E}_0 \in \mathbf{H}_0(\text{curl}; \Omega), \mathbf{H}_0 \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{L}_T^2(\Gamma \times [0, T])$. Then $\mathbf{E}$ satisfies (4.19).*
2. *For C_{σ_0} satisfying (4.5) and for any $\boldsymbol{\phi} \in \mathbf{V}^{h,p}$ and $t \in (0, T)$, the energy $E_h(t, \boldsymbol{\phi})$ is bounded from below by*

$$E_h(t, \boldsymbol{\phi}) \geq \frac{1}{2}\big\|\varepsilon^{\frac{1}{2}}\dot{\boldsymbol{\phi}}(t)\big\|^2_{\Omega} + \frac{1}{6}\big\|\mu^{-\frac{1}{2}}\tilde{\nabla} \times \boldsymbol{\phi}(t)\big\|^2_{\Omega} + \frac{1}{8}\big\|\sigma_0^{\frac{1}{2}}[\boldsymbol{\phi}(t)]_T\big\|^2_{\mathcal{F}_n} \tag{4.22}$$

   for $n = 0, 1, \ldots, N-1$. Further, if X is a subspace of $\mathbf{V}^{h,p}$, and $\mathbf{E}_h \in X$ is the discrete solution, then $E_h(t_N^-) \leq E_h(t_1^-)$.

Proof. The first result follows from the derivation of the formulation and the regularity of the unique solution E, see (2.2) and (2.4). We have already shown the positivity of the energy under the condition on C_{σ_0}, see (4.5).

To prove the remaining statement, combining (4.19) with (4.21) gives the energy identity

$$
\begin{aligned}
E_h(t_N^-,\mathbf{E}_h) &= b^{\mathrm{init}}(\mathbf{E}_h) - E_h(t_0^+,\mathbf{E}_h)\\
&\quad - \sum_{n=1}^{N-1}\bigg(\frac{1}{2}\big\|\varepsilon^{\frac{1}{2}}[\![\dot{\mathbf{E}}_h(t_n)]\!]\big\|_\Omega^2 + \frac{1}{2}\big\|\mu^{-\frac{1}{2}}[\![\tilde{\nabla}\times\mathbf{E}_h(t_n)]\!]\big\|_\Omega^2\\
&\qquad - \big([\![\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}_h(t_n)\}\}]\!],[\![[\mathbf{E}_h(t_n)]_T]\!]\big)_{\tilde{\mathcal{F}}_n} + \frac{1}{2}\big\|[\![\sigma_0^{\frac{1}{2}}[\mathbf{E}_h(t_n)]_T]\!]\big\|_{\tilde{\mathcal{F}}_n}^2\bigg)\\
&\quad - \sum_{n=0}^{N-1}\Big(\big\|\sigma_1^{\frac{1}{2}}[\mathbf{E}_h]_T\big\|_{\mathcal{F}_n\times I_n}^2 + \big\|\sigma_2^{\frac{1}{2}}\big[\mu^{-1}\tilde{\nabla}\times\mathbf{E}_h\big]_T\big\|_{\mathcal{F}_n^{\mathrm{int}}\times I_n}^2\Big).
\end{aligned}\tag{4.23}
$$

Expression (4.20) implies that

$$
\begin{aligned}
a_0(\mathbf{E}_h,\mathbf{E}_h) &= E_h(t_1^-,\mathbf{E}_h) + E_h(t_0^+,\mathbf{E}_h) + \big\|\sigma_1^{\frac{1}{2}}[\mathbf{E}_h]_T\big\|_{\mathcal{F}_0\times I_0}^2\\
&\quad + \big\|\sigma_2^{\frac{1}{2}}[\mu^{-1}\tilde{\nabla}\times\mathbf{E}_h]_T\big\|_{\mathcal{F}_0^{\mathrm{int}}\times I_0}^2 = b^{\mathrm{init}}(\mathbf{E}_h).
\end{aligned}\tag{4.24}
$$

Hence, the energy identity (4.23) can be written as

$$
\begin{aligned}
E_h(t_N^-,\mathbf{E}_h) &= E_h(t_1^-,\mathbf{E}_h)\\
&\quad - \sum_{n=1}^{N-1}\bigg(\frac{1}{2}\big\|\varepsilon^{\frac{1}{2}}[\![\dot{\mathbf{E}}_h(t_n)]\!]\big\|_\Omega^2 + \frac{1}{2}\big\|\mu^{-\frac{1}{2}}[\![\tilde{\nabla}\times\mathbf{E}_h(t_n)]\!]\big\|_\Omega^2\\
&\qquad - \big([\![\{\{\mu^{-1}\tilde{\nabla}\times\mathbf{E}_h(t_n)\}\}]\!],[\![[\mathbf{E}_h(t_n)]_T]\!]\big)_{\tilde{\mathcal{F}}_n} + \frac{1}{2}\big\|[\![\sigma_0^{\frac{1}{2}}[\mathbf{E}_h(t_n)]_T]\!]\big\|_{\tilde{\mathcal{F}}_n}^2\\
&\qquad + \big\|\sigma_1^{\frac{1}{2}}[\mathbf{E}_h]_T\big\|_{\mathcal{F}_n\times I_n}^2 + \big\|\sigma_2^{\frac{1}{2}}[\mu^{-1}\tilde{\nabla}\times\mathbf{E}_h]_T\big\|_{\mathcal{F}_n^{\mathrm{int}}\times I_n}^2\bigg).
\end{aligned}\tag{4.25}
$$

Procedures used to establish the positivity of the discrete energy (4.7) can also show that the above equality indicates that the discrete energy decreases at each time step. □

5. A Space-time Trefftz DG Method

The proposed Trefftz DG method for (2.1) first depends on two transformations, and then we define anisotropic Trefftz polynomial spaces. To the end, we make an additional assumption that on the mesh on ε_r,μ_r and A: ε_r,μ_r are constants and A is the constant matrix in each element $K\in\mathcal{T}_n$ for each n.

5.1. A coordinate transformation and a scaled transformation

Since A is piecewise constant and positive definite matrix in K, there exists an orthogonal matrix P and a diagonal positive definite matrix $\Lambda = \mathrm{diag}(\lambda_{\min},\lambda_{\mathrm{mid}},\lambda_{\max})$ such that $A = P^\top\Lambda P$, where $\lambda_{\min}\le\lambda_{\mathrm{mid}}\le\lambda_{\max}$ are constant. Without loss of generality, we assume that $\det(P)=1$. Meanwhile we use p_1,p_2,p_3 to denote the column vectors of P, and use $q_1^\top,q_2^\top,q_3^\top$ to denote the row vectors of P. Then each of these vectors is a unit vector, and p_1,p_2,p_3 (and q_1,q_2,q_3) are orthogonal each other.

Define the scaled field $\tilde{\mathbf{E}}$ as

$$
(E_x,E_y,E_z)^\top = G(\tilde{E}_x,\tilde{E}_y,\tilde{E}_z)^\top.\tag{5.1}
$$

Here $G = P^\top \Lambda^{-1/2}$. Set

$$m_{\max} = \sqrt{\lambda_{\text{mid}}\lambda_{\max}}, \quad m_{\text{mid}} = \sqrt{\lambda_{\max}\lambda_{\min}}, \quad m_{\min} = \sqrt{\lambda_{\min}\lambda_{\text{mid}}}.$$

Then, by direct calculation, we have

$$\nabla \times \mathbf{E} = P^\top \Lambda^{\frac{1}{2}} \Big(-\frac{q_3 \cdot \nabla \tilde{E}_y}{m_{\min}} + \frac{q_2 \cdot \nabla \tilde{E}_z}{m_{\text{mid}}}, \frac{q_3 \cdot \nabla \tilde{E}_x}{m_{\min}} - \frac{q_1 \cdot \nabla \tilde{E}_z}{m_{\max}}, -\frac{q_2 \cdot \nabla \tilde{E}_x}{m_{\text{mid}}} + \frac{q_1 \cdot \nabla \tilde{E}_y}{m_{\max}} \Big)^\top. \tag{5.2}$$

Define $M = \text{diag}(m_{\max}, m_{\text{mid}}, m_{\min})$, and construct the coordinate transformation

$$\hat{\mathbf{x}} = (\hat{x}\hat{y}\hat{z})^\top = MP(xyz)^\top \overset{\Delta}{=\!=} S\mathbf{x}, \quad S = MP. \tag{5.3}$$

Let $\hat{\Omega}, \hat{Q}, \hat{\Gamma}$ denote the images of Ω, Q, Γ, respectively, under the above coordinate transformation. With the inverse transformation S^{-1}, we define the scaled electric field

$$\hat{\mathbf{E}}(\hat{\mathbf{x}}, t) = \tilde{\mathbf{E}}(S^{-1}\hat{\mathbf{x}}, t). \tag{5.4}$$

By the transformations (5.3), (5.4) and the chain rule, the right-hand of the last equation of (5.2) can be reduced to the following equation:

$$\Big(-\frac{q_3 \cdot \nabla \tilde{E}_y}{m_{\min}} + \frac{q_2 \cdot \nabla \tilde{E}_z}{m_{\text{mid}}}, \frac{q_3 \cdot \nabla \tilde{E}_x}{m_{\min}} - \frac{q_1 \cdot \nabla \tilde{E}_z}{m_{\max}}, -\frac{q_2 \cdot \nabla \tilde{E}_x}{m_{\text{mid}}} + \frac{q_1 \cdot \nabla \tilde{E}_y}{m_{\max}} \Big)^\top = \hat{\nabla} \times \hat{\mathbf{E}}, \tag{5.5}$$

where $\hat{\nabla}\times$ denotes the curl operator with respect to $\hat{\mathbf{x}}$. Thus, by (5.2) and (5.4)-(5.5), the scaled electric field $\hat{\mathbf{E}}(\hat{\mathbf{x}}, t)$ satisfies the transformed first order equation

$$\mu^{-1}\nabla \times \mathbf{E} = \mu_r^{-1} P^\top \Lambda^{-\frac{1}{2}} \hat{\nabla} \times \hat{\mathbf{E}} = \mu_r^{-1} G\hat{\mathbf{H}}, \tag{5.6}$$

where $\hat{\mathbf{H}} = \hat{\nabla} \times \hat{\mathbf{E}}$. Set $S^\top = (S_1 S_2 S_3), G = (G_1 G_2 G_3)$. For simplicity of the presentation, set $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)^\top, \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $M = \text{diag}(M_1, M_2, M_3)$. By the property of the orthogonal matrix P and the chain rule, we have

$$\begin{aligned}
&\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) \\
&= \mu_r^{-1} \nabla \times G\hat{\mathbf{H}} = \mu_r^{-1} \sum_{j=1}^{3}\sum_{m=1}^{3} \frac{\partial \hat{H}_j}{\partial \hat{x}_m} S_m \times G_j \\
&= \mu_r^{-1} \sum_{j=1}^{3}\sum_{m=1}^{3} \frac{\partial \hat{H}_j}{\partial \hat{x}_m} M_m \lambda_j^{-\frac{1}{2}} q_m \times q_j \\
&= \mu_r^{-1} \Big(-\frac{\partial \hat{H}_1}{\partial \hat{x}_2}\lambda_3^{\frac{1}{2}} q_3 + \frac{\partial \hat{H}_1}{\partial \hat{x}_3}\lambda_2^{\frac{1}{2}} q_2 + \frac{\partial \hat{H}_2}{\partial \hat{x}_1}\lambda_3^{\frac{1}{2}} q_3 - \frac{\partial \hat{H}_2}{\partial \hat{x}_3}\lambda_1^{\frac{1}{2}} q_1 - \frac{\partial \hat{H}_3}{\partial \hat{x}_1}\lambda_2^{\frac{1}{2}} q_2 + \frac{\partial \hat{H}_3}{\partial \hat{x}_2}\lambda_1^{\frac{1}{2}} q_1 \Big) \\
&= P^\top \Lambda^{\frac{1}{2}} \hat{\nabla} \times \big(\mu_r^{-1} \hat{\nabla} \times \hat{\mathbf{E}}\big).
\end{aligned}$$

Combining with $\boldsymbol{\varepsilon}\ddot{\mathbf{E}} = P^\top \Lambda^{1/2} \varepsilon_r \ddot{\hat{\mathbf{E}}}$, yields that $\hat{\mathbf{E}}$ satisfies the isotropic second order equation

$$\varepsilon_r \ddot{\hat{\mathbf{E}}} + \hat{\nabla} \times \big(\mu_r^{-1} \hat{\nabla} \times \hat{\mathbf{E}}\big) = \mathbf{0} \quad \text{in } \hat{Q}. \tag{5.7}$$

Conversely, if the scaled electric field $\hat{\mathbf{E}}(\hat{\mathbf{x}}, t)$ satisfies the transformed isotropic Maxwell equations (5.7), the physical electric field $\mathbf{E}(\mathbf{x}, t)$

$$\mathbf{E}(\mathbf{x}, t) = G\hat{\mathbf{E}}(\hat{\mathbf{x}}, t) = G\hat{\mathbf{E}}(S\mathbf{x}, t) \tag{5.8}$$

satisfies the first equation of the original anisotropic second order problem (2.1).

Denote by $\hat{\mathcal{T}}_n = \{\hat{K}\}$ the transformed finite element mesh of the spatial domain $\hat{\Omega}$ with

$$\hat{h}_{\hat{K}} = \mathrm{diam}\hat{K}, \quad \hat{h}_n = \max_{\hat{K}\in\hat{\mathcal{T}}_n} \hat{h}_{\hat{K}}.$$

Furthermore, the transformed space-time domain $\hat{Q} = \hat{\Omega}\times(0,T)$ can be partitioned with a finite element mesh $\hat{\mathcal{T}}_{\hat{h}}$ given by

$$\hat{\mathcal{T}}_{\hat{h}} = \{\hat{K}\times I_n,\ \hat{K}\in\hat{\mathcal{T}}_n,\ 0\le n\le N-1\}, \quad \hat{h} = \max_{0\le n\le N-1} \hat{h}_n.$$

We denote by $\hat{\mathcal{F}}_n = \bigcup_{\hat{K}\in\hat{\mathcal{T}}_n} \partial\hat{K}$ the skeleton of the spatial mesh, by $\hat{\mathcal{F}}_n^{\mathrm{int}} = \bigcup_{\hat{K}\in\hat{\mathcal{T}}_n} \partial\hat{K}\backslash\partial\hat{\Omega}$ the union of the interior faces, and by $\hat{\mathcal{F}}_n^{\mathrm{bou}}$ the union of the boundary faces, respectively. Define the union of two skeletons of two subsequent meshes by $\tilde{\hat{\mathcal{F}}}_n = \hat{\mathcal{F}}_n\bigcup\hat{\mathcal{F}}_{n-1}, n = 1,\dots,N-1$; if $n=0$, we set $\tilde{\hat{\mathcal{F}}}_0 = \hat{\mathcal{F}}_0$ and if $n=N$, we set $\tilde{\hat{\mathcal{F}}}_N = \hat{\mathcal{F}}_N$. In particular, if the spatial mesh remains fixed, all the formulas hold with $\hat{\mathcal{F}} = \hat{\mathcal{F}}_n = \tilde{\hat{\mathcal{F}}}_n$. Moreover, we set $\hat{\mathcal{F}}_{\hat{h}}^0 = \hat{\Omega}\times\{t=0\}$, $\hat{\mathcal{F}}_{\hat{h}}^T = \hat{\Omega}\times\{t=T\}$ and $\hat{\mathcal{F}}_{\hat{h}}^{\hat{\Gamma}} = \hat{\Gamma}\times[0,T]$. Similarly to the definition of $\tilde{h}$, we denote the spatial mesh size by $\tilde{\hat{h}} : \hat{\Omega}\times[0,T]\to\mathbb{R}$, defined by $\tilde{\hat{h}}(\hat{\mathbf{x}},t) = \mathrm{diam}(\hat{K}) = \hat{h}_{\hat{K}}$ if $\hat{\mathbf{x}}\in\hat{K}$ for $\hat{K}\in\hat{\mathcal{T}}_n$ and $t\in(t_n,t_{n+1}]$; when $\hat{\mathbf{x}}\in\hat{\mathcal{F}}_n^{\mathrm{int}}$, we set $\tilde{\hat{h}}(\hat{\mathbf{x}},t) = \{\{\hat{h}_{\hat{K}}\}\}$ to be the average. Finally, $\hat{\mathbf{n}}_{\hat{K}}$ is an outward-pointing unit normal vector on $\partial\hat{K}$ for $\hat{K}\in\hat{\mathcal{T}}_n$.

5.2. Anisotropic Trefftz basis function spaces

We first give the definition of a discretized Trefftz space $\hat{\mathbf{T}}^{\hat{h},p}$ satisfying isotropic Maxwell's equations (5.7). For it, define a discrete local isotropic Trefftz space $\hat{\mathbf{T}}_n(\hat{K}\times I_n)$ as follows:

$$\hat{\mathbf{T}}_n(\hat{K}\times I_n) := \big\{\hat{\mathbf{E}}_{l,j}(\hat{\mathbf{x}},t)\in\mathcal{P}_p(\mathbb{R}^{3+1})^3 : \hat{\mathbf{E}}_{l,j} \text{ satisfies } \varepsilon_r\ddot{\hat{\mathbf{E}}}_{l,j} + \hat{\nabla}\times\big(\mu_r^{-1}\hat{\nabla}\times\hat{\mathbf{E}}_{l,j}\big) = \mathbf{0} \text{ in } \hat{K}\times I_n\big\}. \tag{5.9}$$

Given any basis $\{\hat{E}_l(\hat{\mathbf{x}})\}_{l=1,2,C_{p+3}^3}$ of $\mathbb{P}^p(\mathbb{R}^3)$ and $\{\tilde{E}_l(\hat{\mathbf{x}})\}_{l=1,2,C_{p-1+3}^3}$ of $\mathbb{P}^{p-1}(\mathbb{R}^3)$, where $\mathbb{P}^q(\mathbb{R}^3)$ denotes the space of polynomial of degree at most q ($q\in\mathbb{N}$) in 3 space variables, a basis for $\hat{\mathbf{T}}_n(\hat{K}\times I_n)$ is given by

$$\big\{\hat{\mathbf{E}}_{l,j}(\hat{\mathbf{x}},t)\in\hat{\mathbf{T}}_n(\hat{K}\times I_n) \text{ s.t. } \hat{\mathbf{E}}_{l,j}(\hat{\mathbf{x}},0) = \hat{E}_l(\hat{\mathbf{x}})\mathbf{e}_j,\ l=1,2,C_{p+3}^3;\ j=1,\dots,3, \text{ or } \dot{\hat{\mathbf{E}}}_{l,j}(\hat{\mathbf{x}},0) = \tilde{E}_l(\hat{\mathbf{x}})\mathbf{e}_j,\ l=1,2,C_{p-1+3}^3;\ j=1,\dots,3\big\}, \tag{5.10}$$

where $\mathbf{e}_j := (0,\cdots,1,\cdots,0)\in\mathbb{N}_0^3$ with 1 in the j-th entry, and

$$C_k^j = \frac{k!}{j!(k-j)!}, \quad j\le k\in\mathbb{N}_0.$$

As a result,

$$\dim\big(\hat{\mathbf{T}}_n(\hat{K}\times I_n)\big) = \frac{(2p+3)(p+2)(p+1)}{2}.$$

To compute explicitly the basis elements $\hat{\mathbf{E}}_{l,j}(\hat{\mathbf{x}},t)$ from $\hat{E}_l(\hat{\mathbf{x}})$ and $\tilde{E}_l(\hat{\mathbf{x}})$, we expand in monomials the general polynomial $\hat{\mathbf{E}}(\hat{\mathbf{x}},t)\in\mathbb{P}^p(\mathbb{R}^{3+1})^3$

$$\hat{\mathbf{E}}(\hat{\mathbf{x}},t) = \sum_{m=0}^{p} \hat{\mathbf{E}}_m(\hat{\mathbf{x}})t^m \tag{5.11}$$

with $\hat{\mathbf{E}}_m(\hat{\mathbf{x}}) \in \mathbb{P}^{p-m}(\mathbb{R}^3)^3$. Then $\hat{\mathbf{E}}(\hat{\mathbf{x}},t) \in \hat{\mathbf{T}}_n(\hat{K}\times I_n)$ satisfying (5.9) if and only if the recurrence relations hold

$$(m+2)(m+1)\varepsilon_r\hat{\mathbf{E}}_{m+2}(\hat{\mathbf{x}}) = -\hat{\nabla}\times\big(\mu_r^{-1}\hat{\nabla}\times\hat{\mathbf{E}}_m(\hat{\mathbf{x}})\big) \tag{5.12}$$

for all $m = 0,1,\dots,p-2$. It follows that

$$\begin{aligned}\ddot{\hat{\mathbf{E}}}(\hat{\mathbf{x}},t) &= \sum_{m=0}^{p-2}\hat{\mathbf{E}}_{m+2}(\hat{\mathbf{x}})(m+2)(m+1)t^m,\\ \hat{\nabla}\times\big(\mu_r^{-1}\hat{\nabla}\times\hat{\mathbf{E}}(\hat{\mathbf{x}},t)\big) &= \sum_{m=0}^{p-2}\hat{\nabla}\times\big(\mu_r^{-1}\hat{\nabla}\times\hat{\mathbf{E}}_m(\hat{\mathbf{x}})\big)t^m.\end{aligned} \tag{5.13}$$

These formulas allow to compute all space polynomials $\hat{\mathbf{E}}_m(\hat{\mathbf{x}})$ starting from those with index $m=0$ or $m=1$, which corresponding to the values or the first order time derivative at $t=0$ related to (5.10).

Denote by $\hat{\mathbf{T}}_n = \prod_{\hat{K}\in\hat{\mathcal{T}}_n}\hat{\mathbf{T}}_n(\hat{K}\times I_n)$ the isotropic Trefftz space defined on $\hat{\mathcal{T}}_n\times I_n$. Further, the discrete space on $\hat{\Omega}\times[0,T]$ is then defined as $\hat{\mathbf{T}}^{\hat{h},p} = \prod_{n=0}^{N-1}\hat{\mathbf{T}}_n$.

In order to derive a finite dimensional Trefftz space $\mathbf{T}^{h,p}\subset\mathbf{V}^{h,p}$ satisfying the original anisotropic wave equation (2.1), define a discrete local anisotropic Trefftz space $\mathbf{T}_n(K\times I_n)$ by

$$\begin{aligned}\mathbf{T}_n(K\times I_n) := \big\{\mathbf{E}_{l,j}(\mathbf{x},t)\in\mathcal{P}_p(\mathbb{R}^{3+1})^3 :\ &\mathbf{E}_{l,j}\ \text{ satisfies}\\ &\varepsilon\ddot{\mathbf{E}}_{l,j} + \nabla\times\big(\mu^{-1}\nabla\times\mathbf{E}_{l,j}\big) = \mathbf{0}\ \text{ in }\ K\times I_n\big\}.\end{aligned} \tag{5.14}$$

By the transformations (5.1) and (5.3), we get the basis of anisotropic Trefftz space $\mathbf{T}_n(K\times I_n)$ related to the isotropic basis (5.10) by

$$\big\{\mathbf{E}_{l,j}(\mathbf{x},t) = G\hat{\mathbf{E}}_{l,j}(S\mathbf{x},t)\in\mathbf{T}_n(K\times I_n),\ \text{ where }\ \hat{\mathbf{E}}_{l,j}\in\hat{\mathbf{T}}_n(\hat{K}\times I_n)\big\}. \tag{5.15}$$

Furthermore, denote by $\mathbf{T}_n^{h,p} = \prod_{K\in\mathcal{T}_n}\mathbf{T}_n(K\times I_n)$ the anisotropic Trefftz space defined on $\mathcal{T}_n\times I_n$, and by $\mathbf{T}^{h,p} = \prod_{n=0}^{N-1}\mathbf{T}_n^{h,p}$ on $\mathcal{T}_h$.

Then, we can obtain the discretized Trefftz DG variational formulation corresponding to (4.19): Find $\mathbf{E}_h\in\mathbf{T}^{h,p}$ such that

$$a(\mathbf{E}_h,\boldsymbol{\phi}) = b^{\text{init}}(\boldsymbol{\phi}),\quad \forall\,\boldsymbol{\phi}\in\mathbf{T}^{h,p}. \tag{5.16}$$

6. Theoretical Analysis

In this section, we will first prove the existence and uniqueness of the numerical solution generated by the discrete variational formulation (5.16). Then the quasi-optimality of the proposed method is established and the transformation stability with respect to mesh-dependent norms is proved. Finally we derive the expected rates of convergence for the proposed method.

Throughout this paper, C denotes a generic positive constant that may have different values in different occurrences, where C depends on the material coefficients ε_r,μ_r, the regularity parameter of the analytic solution $\mathbf{E}$, the shape of the elements of the mesh of $\mathcal{T}_h$, and not on A.

6.1. Well-posedness

We prove existence and uniqueness of the Trefftz DG solution of (5.16) and the bilinear form in (4.16) admits the following upper bounds. To the end, we need to prove that the DG energy norm is indeed a norm on the Trefftz polynomial space $\mathbf{T}^{h,p}$.

Lemma 6.1. *Under the definition of the above parameters C_{σ_0} as in (4.5) and $\sigma_1, \sigma_2 > 0$, bilinear forms $a_n(\cdot\,,\cdot)$ and $a(\cdot\,,\cdot)$ actually induce two seminorms*

$$\begin{aligned} |||\boldsymbol{\phi}|||_n &:= a_n(\boldsymbol{\phi},\boldsymbol{\phi})^{\frac{1}{2}}, \quad \boldsymbol{\phi} \in \mathbf{S}_n^{h,p}, \\ |||\boldsymbol{\phi}||| &:= a(\boldsymbol{\phi},\boldsymbol{\phi})^{\frac{1}{2}}, \qquad \boldsymbol{\phi} \in \mathbf{V}^{h,p}. \end{aligned}$$

These are in fact norms on Trefftz spaces $\mathbf{T}_n^{h,p}$ and $\mathbf{T}^{h,p}$.

Proof. Utilizing (4.20) and (4.22), we can obtain $|||\boldsymbol{\phi}|||_n^2 \geq 0$ and is hence a seminorm. Supposing $|||\boldsymbol{\phi}|||_n = 0$ for $\boldsymbol{\phi} \in \mathbf{T}_n^{h,p}$, then

$$\left\|\sigma_1^{\frac{1}{2}}[\boldsymbol{\psi}]_T\right\|_{\mathcal{F}_n\times I_n} = \left\|\sigma_2^{\frac{1}{2}}[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}]_T\right\|_{\mathcal{F}_n^{\text{int}}\times I_n} = 0$$

indicates that $\boldsymbol{\psi}$ and $\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}$ have no jumps across the space skeleton and that $\boldsymbol{\psi}$ satisfies the zero tangential boundary condition. In addition, $E_h(t_n^+,\boldsymbol{\psi}) = 0$ implies that $\dot{\boldsymbol{\phi}}(t_n) = \boldsymbol{\phi}(t_n) = \mathbf{0}$ on Ω. Thus, $\boldsymbol{\phi}$ is a weak solution of the homogeneous Maxwell equation on $\Omega\times I_n$ with zero initial and tangential boundary conditions, which indicates that $\boldsymbol{\phi} = \mathbf{0}$ and hence that $|||\boldsymbol{\phi}|||_n$ is a norm on $\mathbf{T}_n^{h,p}$.

Proceeding as in the first case, shows that $|||\cdot|||$ is in fact a norm on $\mathbf{T}^{h,p}$. □

Corollary 6.1. *Supposing the initial data $\mathbf{E}_0 \in H_0(\text{curl};\Omega), \mathbf{H}_0 \in L^2(\Omega)$ and the boundary data $\mathbf{g} \in L_T^2(\Gamma\times[0,T])$, the discrete system (5.16) has a unique solution.*

Proof. Existence and uniqueness of solutions to (5.16) follows from $a(\cdot\,,\cdot)$ being a norm on the Trefftz space $\mathbf{T}^{h,p}$. □

6.2. The quasi-optimality

In this section, we establish the quasi-optimality of the proposed method. To the end, we first establish the upper bounds for the bilinear form $a(\cdot\,,\cdot)$.

Lemma 6.2. *Let $\boldsymbol{\psi} \in \mathbf{V}+\mathbf{T}^{h,p}$ and $\boldsymbol{\phi} \in \mathbf{T}^{h,p}$, then*

$$|a(\boldsymbol{\psi},\boldsymbol{\phi})| \leq C|||\boldsymbol{\psi}|||_\star|||\boldsymbol{\phi}|||, \tag{6.1}$$

and

$$\begin{aligned} |||\boldsymbol{\psi}|||_\star^2 = &\sum_{n=0}^{N-1}\Big(\left\|\sigma_2^{-\frac{1}{2}}\{\{\dot{\boldsymbol{\psi}}\}\}\right\|_{\mathcal{F}_n^{\text{int}}\times I_n}^2 + \left\|\sigma_1^{-\frac{1}{2}}\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\psi}}\}\}\right\|_{\mathcal{F}_n\times I_n}^2 \\ &\quad + \left\|\sigma_0\sigma_1^{-\frac{1}{2}}[\dot{\boldsymbol{\psi}}]_T\right\|_{\mathcal{F}_n\times I_n}^2 + \left\|\sigma_1^{\frac{1}{2}}[\boldsymbol{\psi}]_T\right\|_{\mathcal{F}_n\times I_n}^2 + \left\|\sigma_2^{\frac{1}{2}}[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}]_T\right\|_{\mathcal{F}_n\times I_n}^2\Big) \\ &+ \frac{1}{2}\sum_{n=1}^{N}\Big(\left\|\varepsilon^{\frac{1}{2}}\dot{\boldsymbol{\psi}}(t_n^-)\right\|_\Omega^2 + \left\|\mu^{-\frac{1}{2}}\tilde{\nabla}\times\boldsymbol{\psi}(t_n^-)\right\|_\Omega^2 \\ &\quad + \left\|\sigma_0^{\frac{1}{2}}[\boldsymbol{\psi}(t_n^-)]_T\right\|_{\mathcal{F}_{n-1}}^2 + \left\|\sigma_0^{-\frac{1}{2}}\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n^-)\}\}\right\|_{\tilde{\mathcal{F}}_n}^2\Big). \end{aligned} \tag{6.2}$$

Proof. By the integration by parts, the Trefftz property of $\boldsymbol{\psi}$ and the "DG magic formula"

$$\begin{aligned}
&-(\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\psi}},\tilde{\nabla}\times\boldsymbol{\phi})_{\Omega\times I_n}\\
&=-\big(\dot{\boldsymbol{\psi}},\tilde{\nabla}\times(\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi})\big)_{\Omega\times I_n}-\big([\dot{\boldsymbol{\psi}}]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}\}\}\big)_{\mathcal{F}_n\times I_n}\\
&\quad+\big(\{\{\dot{\boldsymbol{\psi}}\}\},[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n^{\mathrm{int}}\times I_n}
\end{aligned}\tag{6.3}$$

give

$$\begin{aligned}
&(\varepsilon\ddot{\boldsymbol{\psi}},\dot{\boldsymbol{\phi}})_{\Omega\times I_n}+(\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi},\tilde{\nabla}\times\dot{\boldsymbol{\phi}})_{\Omega\times I_n}\\
&=-(\varepsilon\dot{\boldsymbol{\psi}},\ddot{\boldsymbol{\phi}})_{\Omega\times I_n}-(\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\psi}},\tilde{\nabla}\times\boldsymbol{\phi})_{\Omega\times I_n}\\
&\quad+\big(\varepsilon\dot{\boldsymbol{\psi}}(t_{n+1}^-),\dot{\boldsymbol{\phi}}(t_{n+1}^-)\big)_\Omega-\big(\varepsilon\dot{\boldsymbol{\psi}}(t_n^+),\dot{\boldsymbol{\phi}}(t_n^+)\big)_\Omega\\
&\quad+\big(\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_{n+1}^-),\tilde{\nabla}\times\boldsymbol{\phi}(t_{n+1}^-)\big)_\Omega-\big(\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n^+),\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\big)_\Omega\\
&=-\big([\dot{\boldsymbol{\psi}}]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}\}\}\big)_{\mathcal{F}_n\times I_n}+\big(\{\{\dot{\boldsymbol{\psi}}\}\},[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n^{\mathrm{int}}\times I_n}\\
&\quad+\big(\varepsilon\dot{\boldsymbol{\psi}}(t_{n+1}^-),\dot{\boldsymbol{\phi}}(t_{n+1}^-)\big)_\Omega-\big(\varepsilon\dot{\boldsymbol{\psi}}(t_n^+),\dot{\boldsymbol{\phi}}(t_n^+)\big)_\Omega\\
&\quad+\big(\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_{n+1}^-),\tilde{\nabla}\times\boldsymbol{\phi}(t_{n+1}^-)\big)_\Omega-\big(\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n^+),\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\big)_\Omega.
\end{aligned}\tag{6.4}$$

Further applications of integration by parts in time yield

$$\begin{aligned}
&-\big([\dot{\boldsymbol{\psi}}]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}\}\}\big)_{\mathcal{F}_n\times I_n}\\
&=\big([\boldsymbol{\psi}]_T,\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\phi}}\}\}\big)_{\mathcal{F}_n\times I_n}-\big([\boldsymbol{\psi}(t_{n+1}^-)]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_{n+1}^-)\}\}\big)_{\mathcal{F}_n}\\
&\quad+\big([\boldsymbol{\psi}(t_n^+)]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_n^+)\}\}\big)_{\mathcal{F}_n},
\end{aligned}\tag{6.5}$$

$$\begin{aligned}
&-\big(\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}\}\},[\dot{\boldsymbol{\phi}}]_T\big)_{\mathcal{F}_n\times I_n}+\big(\sigma_0[\boldsymbol{\psi}]_T,[\dot{\boldsymbol{\phi}}]_T\big)_{\mathcal{F}_n\times I_n}\\
&=\big(\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\psi}}\}\},[\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n}-\big(\sigma_0[\dot{\boldsymbol{\psi}}]_T,[\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n}\\
&\quad-\big(\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_{n+1}^-)\}\},[\boldsymbol{\phi}(t_{n+1}^-)]_T\big)_{\mathcal{F}_n}+\big(\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n^+)\}\},[\boldsymbol{\phi}(t_n^+)]_T\big)_{\mathcal{F}_n}\\
&\quad+\big(\sigma_0[\boldsymbol{\psi}(t_{n+1}^-)]_T,[\boldsymbol{\phi}(t_{n+1}^-)]_T\big)_{\mathcal{F}_n}-\big(\sigma_0[\boldsymbol{\psi}(t_n^+)]_T,[\boldsymbol{\phi}(t_n^+)]_T\big)_{\mathcal{F}_n}.
\end{aligned}\tag{6.6}$$

Substituting (6.4)-(6.6) into (4.14), we get

$$\begin{aligned}
a_n(\boldsymbol{\psi},\boldsymbol{\phi})&=\big(\{\{\dot{\boldsymbol{\psi}}\}\},[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n^{\mathrm{int}}\times I_n}-\big([\boldsymbol{\psi}(t_{n+1}^-)]_T,\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_{n+1}^-)\}\}\big)_{\mathcal{F}_n}\\
&\quad+\big(\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\psi}}\}\},[\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n}-\big(\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_{n+1}^-)\}\},[\boldsymbol{\phi}(t_{n+1}^-)]_T\big)_{\mathcal{F}_n}\\
&\quad-\big(\sigma_0[\dot{\boldsymbol{\psi}}]_T,[\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n}+\big(\sigma_0[\boldsymbol{\psi}(t_{n+1}^-)]_T,[\boldsymbol{\phi}(t_{n+1}^-)]_T\big)_{\mathcal{F}_n}\\
&\quad+\big(\varepsilon\dot{\boldsymbol{\psi}}(t_{n+1}^-),\dot{\boldsymbol{\phi}}(t_{n+1}^-)\big)_\Omega+\big(\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_{n+1}^-),\tilde{\nabla}\times\boldsymbol{\phi}(t_{n+1}^-)\big)_\Omega\\
&\quad+(\sigma_1[\boldsymbol{\psi}]_T,[\boldsymbol{\phi}]_T)_{\mathcal{F}_n\times I_n}+\big(\sigma_2[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}]_T,[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n^{\mathrm{int}}\times I_n}.
\end{aligned}\tag{6.7}$$

Thus, by setting $[\![\mathbf{f}(\mathbf{t_N^-})]\!]=\mathbf{f}(\mathbf{t_N^-})$, we have

$$\begin{aligned}
a(\boldsymbol{\psi},\boldsymbol{\phi})&=\sum_{n=0}^{N-1}a_n(\boldsymbol{\psi},\boldsymbol{\phi})-\sum_{n=1}^{N-1}b_n(\boldsymbol{\psi},\boldsymbol{\phi})\\
&=\sum_{n=0}^{N-1}\Big(\big(\{\{\dot{\boldsymbol{\psi}}\}\},[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n^{\mathrm{int}}\times I_n}+\big(\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\psi}}\}\},[\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n}
\end{aligned}$$

$$\begin{aligned}
&\quad - \big(\sigma_0[\dot{\boldsymbol{\psi}}]_T, [\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n} + \big(\sigma_1[\boldsymbol{\psi}]_T, [\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n} \\
&\quad + \big(\sigma_2[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}]_T, [\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n^{\text{int}}\times I_n}\Big) \\
&- \sum_{n=1}^{N}\Big(\big(\varepsilon\dot{\boldsymbol{\psi}}(t_n^-), [\![\dot{\boldsymbol{\phi}}(t_n)]\!]\big)_\Omega + \big(\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n^-), [\![\tilde{\nabla}\times\boldsymbol{\phi}(t_n)]\!]\big)_\Omega \\
&\quad - \big(\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\psi}(t_n^-)\}\}, [\![[\boldsymbol{\phi}(t_n)]_T]\!]\big)_{\tilde{\mathcal{F}}_n} \\
&\quad - \big([\boldsymbol{\psi}(t_n^-)]_T, [\![\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_n)\}\}]\!]\big)_{\mathcal{F}_{n-1}} \\
&\quad + \big(\sigma_0[\boldsymbol{\psi}(t_n^-)]_T, [\![[\boldsymbol{\phi}(t_n)]_T]\!]\big)_{\mathcal{F}_{n-1}}\Big).
\end{aligned} \tag{6.8}$$

The rest is using the Cauchy-Schwarz inequality to estimate most of the terms in order to obtain the stated results. For example, the first term on the right-hand side in the above sum is estimated as follows:

$$\begin{aligned}
&\big|\big(\{\{\dot{\boldsymbol{\psi}}\}\}, [\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n^{\text{int}}\times I_n}\big| \\
&\le \big\|\sigma_2^{-\frac{1}{2}}\{\{\dot{\boldsymbol{\psi}}\}\}\big\|_{\mathcal{F}_n^{\text{int}}\times I_n}\big\|\sigma_2^{\frac{1}{2}}[\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}]_T\big\|_{\mathcal{F}_n^{\text{int}}\times I_n},
\end{aligned}$$

for the second and third terms, we have

$$\begin{aligned}
&\big|\big(\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\psi}}\}\}, [\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n} - \big(\sigma_0[\dot{\boldsymbol{\psi}}]_T, [\boldsymbol{\phi}]_T\big)_{\mathcal{F}_n\times I_n}\big| \\
&\le \big\|\sigma_1^{-\frac{1}{2}}\big(\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\psi}}\}\} - \sigma_0[\dot{\boldsymbol{\psi}}]_T\big)\big\|_{\mathcal{F}_n\times I_n}\big\|\sigma_1^{\frac{1}{2}}[\boldsymbol{\phi}]_T\big\|_{\mathcal{F}_n\times I_n}.
\end{aligned}$$

The proof is complete. □

Remark 6.1. Note that (6.7)-(6.8) show that for Trefftz functions the bilinear forms can be evaluated without computing integrals over the volume terms $\Omega \times I_n$. This can bring considerable savings, especially in higher spatial dimensions, see [1, Fig. 3].

It is immediate to derive the following abstract error estimate in the $|||\cdot|||$-norm.

Theorem 6.1. *Let $\mathbf{E}_h \in \mathbf{T}^{h,p}$ be the discrete solution of the space-time Trefftz DG method and let $\mathbf{E} \in \mathbf{V}$ be the exact solution. Then, we have*

$$|||\mathbf{E} - \mathbf{E}_h||| \le \inf_{\boldsymbol{\phi}_h \in \mathbf{T}^{h,p}} \big(C|||\mathbf{E} - \boldsymbol{\phi}_h|||_\star + |||\mathbf{E} - \boldsymbol{\phi}_h|||\big). \tag{6.9}$$

Proof. By Galerkin orthogonality $a(\mathbf{E} - \mathbf{E}_h, \boldsymbol{\psi}_h) = 0$ for any $\boldsymbol{\psi}_h \in \mathbf{T}^{h,p}$, we have

$$a(\mathbf{E} - \boldsymbol{\phi}_h, \boldsymbol{\psi}_h) = a(\mathbf{E}_h - \boldsymbol{\phi}_h, \boldsymbol{\psi}_h), \quad \forall\, \boldsymbol{\phi}_h \in \mathbf{T}^{h,p}.$$

Setting $\boldsymbol{\psi}_h = \mathbf{E}_h - \boldsymbol{\phi}_h$ and using (6.1), yields

$$|||\mathbf{E}_h - \boldsymbol{\phi}_h|||^2 = a(\mathbf{E}_h - \boldsymbol{\phi}_h, \mathbf{E}_h - \boldsymbol{\phi}_h) \le C|||\mathbf{E} - \boldsymbol{\phi}_h|||_\star |||\mathbf{E}_h - \boldsymbol{\phi}_h|||.$$

Further, by the triangle inequality, we get

$$|||\mathbf{E} - \mathbf{E}_h||| \le |||\mathbf{E} - \boldsymbol{\phi}_h||| + |||\mathbf{E}_h - \boldsymbol{\phi}_h||| \le |||\mathbf{E} - \boldsymbol{\phi}_h||| + C|||\mathbf{E} - \boldsymbol{\phi}_h|||_\star.$$

The proof is complete. □

6.3. The transformation stability with respect to mesh-dependent norms

For the global Trefftz DG space $\hat{\mathbf{T}}^{\hat{h},p}$, we define the discrete energy $\hat{E}_{\hat{h}}(t,\hat{\boldsymbol{\phi}})$ at time $t\in I_n$ for $\hat{\boldsymbol{\phi}}\in\hat{\mathbf{T}}^{\hat{h},p}$,

$$\hat{E}_{\hat{h}}(t,\hat{\boldsymbol{\phi}})=\big\|\varepsilon_r^{\frac{1}{2}}\dot{\hat{\boldsymbol{\phi}}}(t)\big\|_{\hat{\Omega}}^2+\big\|\mu_r^{-\frac{1}{2}}\tilde{\hat{\nabla}}\times\hat{\boldsymbol{\phi}}(t)\big\|_{\hat{\Omega}}^2+\big\|\hat{\sigma}_0^{\frac{1}{2}}[\hat{\boldsymbol{\phi}}(t)]_T\big\|_{\hat{\mathcal{F}}_n}^2, \tag{6.10}$$

and the corresponding $|||\cdot|||_{\triangle}$-norm and the augmented $|||\cdot|||_{\nabla}$-norm, respectively,

$$\begin{aligned}
|||\hat{\boldsymbol{\phi}}|||_{\triangle}^2 &= \hat{E}_{\hat{h}}(t_N^-,\hat{\boldsymbol{\phi}})+\hat{E}_{\hat{h}}(t_0^+,\hat{\boldsymbol{\phi}})\\
&\quad+\sum_{n=1}^{N-1}\Big(\big\|\varepsilon_r^{\frac{1}{2}}[\![\dot{\hat{\boldsymbol{\phi}}}(t_n)]\!]\big\|_{\hat{\Omega}}^2+\big\|\mu_r^{-\frac{1}{2}}[\![\tilde{\hat{\nabla}}\times\hat{\boldsymbol{\phi}}(t_n)]\!]\big\|_{\hat{\Omega}}^2+\big\|[\![\hat{\sigma}_0^{\frac{1}{2}}[\hat{\boldsymbol{\phi}}(t_n)]_T]\!]\big\|_{\tilde{\hat{\mathcal{F}}}_n}^2\Big)\\
&\quad+\sum_{n=0}^{N-1}\Big(\big\|\hat{\sigma}_1^{\frac{1}{2}}[\hat{\boldsymbol{\phi}}]_T\big\|_{\hat{\mathcal{F}}_n\times I_n}^2+\big\|\hat{\sigma}_2^{\frac{1}{2}}\big[\mu_r^{-1}\tilde{\hat{\nabla}}\times\hat{\boldsymbol{\phi}}\big]_T\big\|_{\hat{\mathcal{F}}_n^{\text{int}}\times I_n}^2\Big),
\end{aligned} \tag{6.11}$$

$$\begin{aligned}
|||\hat{\boldsymbol{\phi}}|||_{\nabla}^2 &= \sum_{n=0}^{N-1}\Big(\big\|\hat{\sigma}_2^{-\frac{1}{2}}\{\{\dot{\hat{\boldsymbol{\phi}}}\}\}\big\|_{\hat{\mathcal{F}}_n^{\text{int}}\times I_n}^2+\big\|\hat{\sigma}_1^{-\frac{1}{2}}\{\{\mu_r^{-1}\tilde{\hat{\nabla}}\times\dot{\hat{\boldsymbol{\phi}}}\}\}\big\|_{\hat{\mathcal{F}}_n\times I_n}^2\\
&\qquad+\big\|\hat{\sigma}_0\hat{\sigma}_1^{-\frac{1}{2}}[\dot{\hat{\boldsymbol{\phi}}}]_T\big\|_{\hat{\mathcal{F}}_n\times I_n}^2+\big\|\hat{\sigma}_1^{\frac{1}{2}}[\hat{\boldsymbol{\phi}}]_T\big\|_{\hat{\mathcal{F}}_n\times I_n}^2+\big\|\hat{\sigma}_2^{\frac{1}{2}}\big[\mu_r^{-1}\tilde{\hat{\nabla}}\times\hat{\boldsymbol{\phi}}\big]_T\big\|_{\hat{\mathcal{F}}_n\times I_n}^2\Big)\\
&\quad+\sum_{n=1}^{N}\Big(\big\|\varepsilon_r^{\frac{1}{2}}\dot{\hat{\boldsymbol{\phi}}}(t_n^-)\big\|_{\hat{\Omega}}^2+\big\|\mu_r^{-\frac{1}{2}}\tilde{\hat{\nabla}}\times\hat{\boldsymbol{\phi}}(t_n^-)\big\|_{\hat{\Omega}}^2\\
&\qquad+\big\|\hat{\sigma}_0^{\frac{1}{2}}[\hat{\boldsymbol{\phi}}(t_n^-)]_T\big\|_{\hat{\mathcal{F}}_{n-1}}^2+\big\|\hat{\sigma}_0^{-\frac{1}{2}}\{\{\mu_r^{-1}\tilde{\hat{\nabla}}\times\hat{\boldsymbol{\phi}}(t_n^-)\}\}\big\|_{\tilde{\hat{\mathcal{F}}}_n}^2\Big),
\end{aligned} \tag{6.12}$$

where $\hat{\sigma}_0=p^2/\tilde{\hat{h}}$, $\hat{\sigma}_1=p^3/(\hat{h}\tau_n)$ and $\hat{\sigma}_2=\hat{h}/\tau_n$.

For the simplicity of notation, for each element $K\in\mathcal{T}_n$, let ρ_K denote the condition number $\text{cond}(A_K)$ of the anisotropic matrix A_K. It is easy to see that $\text{cond}(A_K)=\text{cond}(\Lambda_K)$ and $\text{cond}(S_K)=\text{cond}(M_K)=\rho_K^{1/2}$. Here $\diamondsuit_K=\diamondsuit|_K$. Set $\diamondsuit=\max_{K\in\mathcal{T}_n,n=0,\dots,N-1}\diamondsuit_K$, where $\diamondsuit=\rho,\|M^{-1}\|$ or $\|M\|$. Moreover, since $\Lambda_K=\text{diag}(\lambda_{\min}^K,\lambda_{\text{mid}}^K,\lambda_{\max}^K)$, we set

$$\begin{aligned}
\big(\lambda_{\min}^{\frac{1}{2}}\big)_{\max}&=\max_{K\in\mathcal{T}_n,\,n=0,\dots,N-1}\big(\lambda_{\min}^K\big)^{\frac{1}{2}},\\
\big(\lambda_{\min(\text{mid})}^{-\frac{1}{2}}\big)_{\max}&=\max_{K\in\mathcal{T}_n,\,n=0,\dots,N-1}\big(\lambda_{\min(\text{mid})}^K\big)^{-\frac{1}{2}}.
\end{aligned}$$

Further, we have the relation $\|M^{-1}\|^{-1}h\le\hat{h}\le\|M\|h$.

The following lemma states the transformation stability with respect to mesh-dependent norms.

Lemma 6.3. *For $\boldsymbol{\phi}\in\mathbf{T}^{h,p}$, we have*

$$\begin{aligned}
|||\boldsymbol{\phi}|||&\le C\rho^{\frac{3}{4}}\|M^{-1}\|^{\frac{3}{2}}\Big(\big(\lambda_{\min}^{\frac{1}{2}}\big)_{\max}+\big(\lambda_{\min}^{-\frac{1}{2}}\big)_{\max}\Big)|||\hat{\boldsymbol{\phi}}|||_{\triangle},\\
|||\boldsymbol{\phi}|||_{\star}&\le C\rho^{\frac{3}{4}}\|M^{-1}\|^{\frac{3}{2}}\Big(\big(\lambda_{\min}^{\frac{1}{2}}\big)_{\max}+\big(\lambda_{\min}^{-\frac{1}{2}}\big)_{\max}\Big)|||\hat{\boldsymbol{\phi}}|||_{\nabla}.
\end{aligned} \tag{6.13}$$

Proof. We will estimate each term of the norm $|||\cdot|||$. We will repeatedly use (5.2) and (5.5) induced by the transformations (5.3) and (5.4). We proceed as follows by using $|K|/|\hat{K}|=\det(M_K^{-1})\le\|M^{-1}\|^3$ and $\boldsymbol{\phi}=G\hat{\boldsymbol{\phi}}=P^{\top}\Lambda^{-1/2}\hat{\boldsymbol{\phi}}$:

$$\begin{aligned}
&\big\|\varepsilon^{\frac12}\dot{\boldsymbol\phi}(t)\big\|_\Omega = \big\|\varepsilon_r^{\frac12}P^\top\Lambda^{\frac12}PP^\top\Lambda^{-\frac12}\dot{\hat{\boldsymbol\phi}}(t)\big\|_\Omega = \big\|\varepsilon_r^{\frac12}P^\top\dot{\hat{\boldsymbol\phi}}(t)\big\|_\Omega \le \|M^{-1}\|^{\frac32}\big\|\varepsilon_r^{\frac12}\dot{\hat{\boldsymbol\phi}}(t)\big\|_{\hat\Omega},\\
&\big\|\varepsilon^{\frac12}[\![\dot{\boldsymbol\phi}(t_n)]\!]\big\|_\Omega \le \|M^{-1}\|^{\frac32}\big\|\varepsilon_r^{\frac12}[\![\dot{\hat{\boldsymbol\phi}}(t_n)]\!]\big\|_{\hat\Omega}.
\end{aligned}$$

In analogous fashion, we have

$$\begin{aligned}
\big\|\mu^{-\frac12}\tilde\nabla\times\boldsymbol\phi(t)\big\|_\Omega &= \big\|\mu_r^{-\frac12}P^\top\Lambda^{-\frac12}PP^\top\Lambda^{\frac12}\tilde{\hat\nabla}\times\hat{\boldsymbol\phi}(t)\big\|_\Omega\\
&= \big\|\mu_r^{-\frac12}P^\top\tilde{\hat\nabla}\times\hat{\boldsymbol\phi}(t)\big\|_\Omega \le \|M^{-1}\|^{\frac32}\big\|\mu_r^{\frac12}\tilde{\hat\nabla}\times\hat{\boldsymbol\phi}(t)\big\|_{\hat\Omega},\\
\big\|\mu^{-\frac12}[\![\tilde\nabla\times\boldsymbol\phi(t_n)]\!]\big\|_\Omega &\le \|M^{-1}\|^{\frac32}\big\|\mu_r^{\frac12}[\![\tilde{\hat\nabla}\times\hat{\boldsymbol\phi}(t_n)]\!]\big\|_{\hat\Omega}.
\end{aligned}$$

Next, we estimate the penalty terms. By using the relations

$$\begin{aligned}
&[\boldsymbol\phi(t)]_T = \big[|S^{-\top}\mathbf n|P^\top\Lambda^{\frac12}\hat{\boldsymbol\phi}(t)\big]_T,\\
&\|M\|^{-1}\tilde{\hat h} \le \tilde h \le \|M^{-1}\|\tilde{\hat h},\\
&\sigma_0^{\frac12} = C_{\sigma_0}^{\frac12}p\tilde h^{-\frac12} \le C\big\|A^{-\frac12}\big\|\|M\|^{\frac12}\hat\sigma_0^{\frac12},\\
&\left(\frac{|\Gamma_{ij}|}{|\hat\Gamma_{ij}|}\right)^{\frac12} \le \|M^{-1}\|,
\end{aligned}$$

we get

$$\begin{aligned}
&\big\|\sigma_0^{\frac12}[\boldsymbol\phi(t)]_T\big\|_{\mathcal F_n} \le C\rho^{\frac34}\|M^{-1}\|^{\frac32}\big\|\hat\sigma_0^{\frac12}[\hat{\boldsymbol\phi}(t)]_T\big\|_{\hat{\mathcal F}_n},\\
&\big\|\,[\![\sigma_0^{\frac12}[\boldsymbol\phi(t_n)]_T]\!]\,\big\|_{\tilde{\mathcal F}_n} \le C\rho^{\frac34}\|M^{-1}\|^{\frac32}\big\|\,[\![\hat\sigma_0^{\frac12}[\hat{\boldsymbol\phi}(t_n)]_T]\!]\,\big\|_{\tilde{\hat{\mathcal F}}_n}.
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
\big\|\sigma_1^{\frac12}[\boldsymbol\phi]_T\big\|_{\mathcal F_n\times I_n} &= \left\|\left(\frac{\hat h}{h}\right)^{\frac12}|S^{-\top}\mathbf n|P^\top\Lambda^{\frac12}\hat\sigma_1^{\frac12}[\hat{\boldsymbol\phi}]_T\right\|_{\mathcal F_n\times I_n}\\
&\le \rho^{\frac34}\|M^{-1}\|^{\frac32}\big(\lambda_{\min}^{\frac12}\big)_{\max}\big\|\hat\sigma_1^{\frac12}[\hat{\boldsymbol\phi}]_T\big\|_{\hat{\mathcal F}_n\times I_n}. \qquad (6.14)
\end{aligned}$$

Further, since

$$\begin{aligned}
\sigma_2^{\frac12}\mathbf n\times(\mu^{-1}\tilde\nabla\times\boldsymbol\phi) &= \sigma_2^{\frac12}\big(|S^{-\top}\mathbf n|\ S^\top\hat{\mathbf n}\big)\times\big(\mu_r^{-1}P^\top\Lambda^{-\frac12}\tilde{\hat\nabla}\times\hat{\boldsymbol\phi}\big)\\
&= \sigma_2^{\frac12}|S^{-\top}\mathbf n|P^\top\Lambda^{\frac12}\hat{\mathbf n}\times\big(\mu_r^{-1}\tilde{\hat\nabla}\times\hat{\boldsymbol\phi}\big)\\
&= \left(\frac{h}{\hat h}\right)^{\frac12}|S^{-\top}\mathbf n|P^\top\Lambda^{\frac12}\hat\sigma_2^{\frac12}\hat{\mathbf n}\times\big(\mu_r^{-1}\tilde{\hat\nabla}\times\hat{\boldsymbol\phi}\big),
\end{aligned}$$

we have

$$\big\|\sigma_2^{\frac12}[\mu^{-1}\tilde\nabla\times\boldsymbol\phi]_T\big\|_{\mathcal F_n^{\rm int}\times I_n} \le \rho^{\frac12}\|M^{-1}\|^{\frac32}\big(\lambda_{\rm mid}^{-\frac12}\big)_{\max}\big\|\hat\sigma_2^{\frac12}\big[\mu_r^{-1}\tilde{\hat\nabla}\times\hat{\boldsymbol\phi}\big]_T\big\|_{\hat{\mathcal F}_n^{\rm int}\times I_n}.$$

Combining the above estimates yields the first result of (6.13).

Since

$$\sigma_0\sigma_1^{-\frac12}[\dot{\boldsymbol\phi}]_T = C_{\sigma_0}\left(\frac{\hat h}{h}\right)^{\frac12}|S^{-\top}\mathbf n|P^\top\Lambda^{\frac12}\hat\sigma_0\hat\sigma_1^{-\frac12}[\dot{\hat{\boldsymbol\phi}}]_T,$$

we have

$$\big\|\sigma_0\sigma_1^{-\frac12}[\dot{\boldsymbol\phi}]_T\big\|_{\mathcal F_n\times I_n} \le C\rho^{\frac34}\|M^{-1}\|^{\frac32}\big(\lambda_{\min}^{-\frac12}\big)_{\max}\big\|\hat\sigma_0\hat\sigma_1^{-\frac12}[\dot{\hat{\boldsymbol\phi}}]_T\big\|_{\hat{\mathcal F}_n\times I_n}.$$

Similarly, we have

$$\begin{aligned}
\|\sigma_2^{-\frac{1}{2}}\{\{\dot{\boldsymbol{\psi}}\}\}\|_{\mathcal{F}_n^{\rm int}\times I_n} &= \left\|\left(\frac{\tilde{h}}{\tau_n}\right)^{-\frac{1}{2}} P^\top \Lambda^{-\frac{1}{2}}\{\{\dot{\hat{\boldsymbol{\phi}}}\}\}\right\|_{\mathcal{F}_n^{\rm int}\times I_n}\\
&\le \rho^{\frac{3}{4}}\|M^{-1}\|^{\frac{3}{2}}\big(\lambda_{\min}^{\frac{1}{2}}\big)_{\max}\|\hat{\sigma}_2^{-\frac{1}{2}}\{\{\dot{\hat{\boldsymbol{\phi}}}\}\}\|_{\hat{\mathcal{F}}_n^{\rm int}\times I_n},\\
\|\sigma_0^{-\frac{1}{2}}\{\{\mu^{-1}\tilde{\nabla}\times\boldsymbol{\phi}(t_n^-)\}\}\|_{\tilde{\mathcal{F}}_n} &\le C\|M^{-1}\|^{\frac{3}{2}}\|\hat{\sigma}_0^{-\frac{1}{2}}\{\{\mu_r^{-1}\tilde{\hat{\nabla}}\times\hat{\boldsymbol{\phi}}(t_n^-)\}\}\|_{\tilde{\hat{\mathcal{F}}}_n},\\
\|\sigma_1^{-\frac{1}{2}}\{\{\mu^{-1}\tilde{\nabla}\times\dot{\boldsymbol{\phi}}\}\}\|_{\mathcal{F}_n\times I_n} &\le \|M^{-1}\|^{\frac{3}{2}}\big(\lambda_{\min}^{-\frac{1}{2}}\big)_{\max}\|\hat{\sigma}_1^{-\frac{1}{2}}\{\{\mu_r^{-1}\tilde{\hat{\nabla}}\times\dot{\hat{\boldsymbol{\phi}}}\}\}\|_{\hat{\mathcal{F}}_n\times I_n}.
\end{aligned}$$

Combining the above estimates yields the second estimate of (6.13). □

6.4. Error estimates of Trefftz discontinuous Galerkin approximations

We show next the Trefftz basis is sufficient to deliver the expected rates of convergence for the proposed method. Without causing confusion, we rewrite

$$\lambda_{\min} = (\lambda_{\min})_{\max}, \quad \lambda^{-1}_{\min(\rm mid)} = \big(\lambda^{-1}_{\min(\rm mid)}\big)_{\max},$$

Lemma 6.4. *Let the setting of Theorem* 6.1 *hold, let $\boldsymbol{\phi}_h \in \mathbf{T}^{h,p}$ be an arbitrary function in the discrete space, and let $\boldsymbol{\eta} = \mathbf{E} - \boldsymbol{\phi}_h$ and $\hat{\boldsymbol{\eta}} = G^{-1}\boldsymbol{\eta}(S^{-1}\hat{\mathbf{x}}, t)$. Then*

$$\begin{aligned}
|||\mathbf{E}-\mathbf{E}_h||| \le\; & C\rho^{\frac{3}{4}}\|M^{-1}\|^{\frac{3}{2}}\Big(\lambda_{\min}^{\frac{1}{2}} + \lambda_{\min}^{-\frac{1}{2}}\Big)\\
&\times \sum_{n=0}^{N-1}\sum_{\hat{K}\in\hat{\mathcal{T}}_n}\Bigg(\frac{p^4}{\hat{h}_{\hat{K}}^2\tau_n}\|\hat{\boldsymbol{\eta}}\|^2_{\hat{K}\times I_n} + \frac{p^2}{\tau_n}\Bigg(\max\Bigg\{1, \frac{\tau_n^2}{\hat{h}_{\hat{K}}^2}\Bigg\}\|\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n} + \|\hat{\nabla}\hat{\boldsymbol{\eta}}\|^2_{\hat{K}\times I_n}\Bigg)\\
&\qquad + \tau_n\Bigg(\|\hat{\nabla}\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n} + \max\Bigg\{1, \frac{\hat{h}_{\hat{K}}^2}{\tau_n^2}\Bigg\}p^{-1}\|D^2\hat{\boldsymbol{\eta}}\|^2_{\hat{K}\times I_n}\Bigg)\\
&\qquad + \frac{\hat{h}_{\hat{K}}^2\tau_n}{p^4}\|\hat{\nabla}(\hat{\nabla}\times\dot{\hat{\boldsymbol{\eta}}})\|^2_{\hat{K}\times I_n}\Bigg)^{\frac{1}{2}},
\end{aligned} \tag{6.15}$$

where $D^2\hat{\boldsymbol{\eta}}$ denotes second order Hessian matrix of $\hat{\boldsymbol{\eta}}$ with respect to spatial variables.

Proof. By the quasi-optimality (6.9) and stability estimate (6.13), we obtain

$$\begin{aligned}
|||\mathbf{E}-\mathbf{E}_h||| &\le (C|||\boldsymbol{\eta}|||_\star + |||\boldsymbol{\eta}|||)\\
&\le C\rho^{\frac{3}{4}}\|M^{-1}\|^{\frac{3}{2}}\Big(\big(\lambda_{\min}^{\frac{1}{2}}\big)_{\max} + \big(\lambda_{\min}^{-\frac{1}{2}}\big)_{\max}\Big)(|||\hat{\boldsymbol{\eta}}|||_\bigtriangledown + |||\hat{\boldsymbol{\eta}}|||_\triangle).
\end{aligned} \tag{6.16}$$

We need to estimate each term of the norms on the right-hand side. We will repeatedly use the weighted trace estimate [3, Theorem 1.6.6] for $\boldsymbol{\psi} \in H^1(D)^3, D \subset \mathbb{R}^k, k = 1, \ldots, 3+1$,

$$\|\boldsymbol{\psi}\|^2_{\partial D} \le C\big(\epsilon^{-1}\|\boldsymbol{\psi}\|^2_D + \epsilon\|\nabla\boldsymbol{\psi}\|^2_D\big), \quad \epsilon \in (0,1). \tag{6.17}$$

We proceed as follows for all the terms in $|||\cdot|||_\bigtriangledown$. The remaining terms in $|||\cdot|||_\triangle$ are treated completely analogously

$$\begin{aligned}
&\sum_{n=1}^{N} \big\|\varepsilon_r^{\frac{1}{2}} \dot{\hat{\boldsymbol{\eta}}}(t_n^-)\big\|_{\hat{\Omega}}^2 \\
&= \sum_{n=1}^{N} \sum_{\hat{K}\in\hat{\mathcal{T}}_{n-1}} \big\|\varepsilon_r^{\frac{1}{2}} \dot{\hat{\boldsymbol{\eta}}}(t_n^-)\big\|_{\hat{K}}^2 \\
&\leq \sum_{n=0}^{N-1} \sum_{\hat{K}\in\hat{\mathcal{T}}_n} \left(\frac{p}{\tau_n} \big\|\varepsilon_r^{\frac{1}{2}} \dot{\hat{\boldsymbol{\eta}}}\big\|_{\hat{K}\times I_n}^2 + \frac{\tau_n}{p} \big\|\varepsilon_r^{\frac{1}{2}} \ddot{\hat{\boldsymbol{\eta}}}\big\|_{\hat{K}\times I_n}^2 \right) \\
&\leq \sum_{n=0}^{N-1} \sum_{\hat{K}\in\hat{\mathcal{T}}_n} \left(\frac{p}{\tau_n} \varepsilon_r \|\dot{\hat{\boldsymbol{\eta}}}\|_{\hat{K}\times I_n}^2 + \frac{\tau_n}{p} \varepsilon_r \big\|\varepsilon_r^{-1}\mu_r^{-1} \hat{\nabla} \times \big(\hat{\nabla} \times \hat{\boldsymbol{\eta}}\big)\big\|_{\hat{K}\times I_n}^2 \right) \\
&\leq C \sum_{n=0}^{N-1} \sum_{\hat{K}\in\hat{\mathcal{T}}_n} \left(\frac{p}{\tau_n} \|\dot{\hat{\boldsymbol{\eta}}}\|_{\hat{K}\times I_n}^2 + \frac{\tau_n}{p} \big\|\hat{\nabla} \times \big(\hat{\nabla} \times \hat{\boldsymbol{\eta}}\big)\big\|_{\hat{K}\times I_n}^2 \right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\sum_{n=1}^{N} \big\|\mu_r^{-\frac{1}{2}} \tilde{\hat{\nabla}} \times \hat{\boldsymbol{\eta}}(t_n^-)\big\|_{\hat{\Omega}}^2 \\
&\leq C \sum_{n=0}^{N-1} \sum_{\hat{K}\in\hat{\mathcal{T}}_n} \left(\frac{p}{\tau_n} \|\hat{\nabla} \times \hat{\boldsymbol{\eta}}\|_{\hat{K}\times I_n}^2 + \frac{\tau_n}{p} \|\hat{\nabla} \times \dot{\hat{\boldsymbol{\eta}}}\|_{\hat{K}\times I_n}^2 \right).
\end{aligned}$$

Next, we estimate the penalty term

$$\begin{aligned}
&\sum_{n=1}^{N} \big\|\hat{\sigma}_0^{\frac{1}{2}} [\hat{\boldsymbol{\eta}}(t_n^-)]_T\big\|_{\hat{\mathcal{F}}_{n-1}}^2 \\
&\leq C \sum_{n=0}^{N-1} \sum_{\hat{K}\in\hat{\mathcal{T}}_n} \big\|\hat{\sigma}_0^{\frac{1}{2}} \hat{\boldsymbol{\eta}}(t_{n+1}^-)\big\|_{\partial\hat{K}}^2 \\
&\leq C \sum_{n=0}^{N-1} \sum_{\hat{K}\in\hat{\mathcal{T}}_n} \left(\frac{p^3}{\tau_n \hat{h}_{\hat{K}}} \|\hat{\boldsymbol{\eta}}\|_{\partial\hat{K}\times I_n}^2 + \frac{p\tau_n}{\hat{h}_{\hat{K}}} \|\dot{\hat{\boldsymbol{\eta}}}\|_{\partial\hat{K}\times I_n}^2 \right) \\
&\leq C \sum_{n=0}^{N-1} \sum_{\hat{K}\in\hat{\mathcal{T}}_n} \left(\frac{p^4}{\tau_n \hat{h}_{\hat{K}}^2} \|\hat{\boldsymbol{\eta}}\|_{\hat{K}\times I_n}^2 + \frac{p^2}{\tau_n} \|\hat{\nabla}\hat{\boldsymbol{\eta}}\|_{\hat{K}\times I_n}^2 \right. \\
&\qquad\qquad \left. + \frac{p^2\tau_n}{\hat{h}_{\hat{K}}^2} \|\dot{\hat{\boldsymbol{\eta}}}\|_{\hat{K}\times I_n}^2 + \tau_n \|\hat{\nabla}\dot{\hat{\boldsymbol{\eta}}}\|_{\hat{K}\times I_n}^2 \right).
\end{aligned} \tag{6.18}$$

In analogous fashion, we also have

$$\begin{aligned}
&\sum_{n=1}^{N} \big\|\hat{\sigma}_0^{-\frac{1}{2}} \{\{\mu_r^{-1} \tilde{\hat{\nabla}} \times \hat{\boldsymbol{\eta}}(t_n^-)\}\}\big\|_{\tilde{\hat{\mathcal{F}}}_n}^2 \\
&\leq C \sum_{n=0}^{N-1} \sum_{\hat{K}\in\tilde{\hat{\mathcal{T}}}_{n+1}} \big\|\hat{\sigma}_0^{-\frac{1}{2}} \{\{\mu_r^{-1} \hat{\nabla} \times \hat{\boldsymbol{\eta}}(t_{n+1}^-)\}\}\big\|_{\partial\hat{K}}^2
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\tilde{\hat{\mathcal T}}_{n+1}}\left(\frac{\hat h_{\hat K}}{p\tau_n}\|\hat\nabla\times\hat{\boldsymbol\eta}\|^2_{\partial\hat K\times I_n}+\frac{\hat h_{\hat K}\tau_n}{p^3}\|\hat\nabla\times\dot{\hat{\boldsymbol\eta}}\|^2_{\partial\hat K\times I_n}\right)\\
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\tilde{\hat{\mathcal T}}_{n+1}}\left(\frac{1}{\tau_n}\|\hat\nabla\times\hat{\boldsymbol\eta}\|^2_{\hat K\times I_n}+\frac{\hat h^2_{\hat K}}{p\tau_n}\|\hat\nabla(\hat\nabla\times\hat{\boldsymbol\eta})\|^2_{\hat K\times I_n}\right.\\
&\qquad\left.+\frac{\tau_n}{p^2}\|\hat\nabla\times\dot{\hat{\boldsymbol\eta}}\|^2_{\hat K\times I_n}+\frac{\hat h^2_{\hat K}\tau_n}{p^4}\|\hat\nabla(\hat\nabla\times\dot{\hat{\boldsymbol\eta}})\|^2_{\hat K\times I_n}\right)\\
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\hat{\mathcal T}_{n}}\left(\frac{1}{\tau_n}\|\hat\nabla\times\hat{\boldsymbol\eta}\|^2_{\hat K\times I_n}+\frac{\hat h^2_{\hat K}}{p\tau_n}\|\hat\nabla(\hat\nabla\times\hat{\boldsymbol\eta})\|^2_{\hat K\times I_n}\right.\\
&\qquad\left.+\frac{\tau_n}{p^2}\|\hat\nabla\times\dot{\hat{\boldsymbol\eta}}\|^2_{\hat K\times I_n}+\frac{\hat h^2_{\hat K}\tau_n}{p^4}\|\hat\nabla(\hat\nabla\times\dot{\hat{\boldsymbol\eta}})\|^2_{\hat K\times I_n}\right).
\end{aligned}
$$

Next, recalling the definition of $\hat\sigma_1$ and $\hat\sigma_2$, we estimate

$$
\begin{aligned}
&\sum_{n=0}^{N-1}\left\|\hat\sigma_1^{\frac12}[\hat{\boldsymbol\eta}]_T\right\|^2_{\hat{\mathcal F}_n\times I_n}\\
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\hat{\mathcal T}_n}\left\|\hat\sigma_1^{\frac12}\hat{\boldsymbol\eta}\right\|^2_{\partial\hat K\times I_n}\\
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\hat{\mathcal T}_n}\hat\sigma_1\left(\frac{p}{\hat h_{\hat K}}\|\hat{\boldsymbol\eta}\|^2_{\hat K\times I_n}+\frac{\hat h_{\hat K}}{p}\|\hat\nabla\hat{\boldsymbol\eta}\|^2_{\hat K\times I_n}\right)\\
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\hat{\mathcal T}_n}\left(\frac{p^4}{\hat h^2_{\hat K}\tau_n}\|\hat{\boldsymbol\eta}\|^2_{\hat K\times I_n}+\frac{p^2}{\tau_n}\|\hat\nabla\hat{\boldsymbol\eta}\|^2_{\hat K\times I_n}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=0}^{N-1}\left\|\hat\sigma_2^{\frac12}[\mu_r^{-1}\tilde{\hat\nabla}\times\hat{\boldsymbol\eta}]_T\right\|^2_{\hat{\mathcal F}_n\times I_n}\\
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\hat{\mathcal T}_n}\left\|\hat\sigma_2^{\frac12}\mu_r^{-1}\tilde{\hat\nabla}\times\hat{\boldsymbol\eta}\right\|^2_{\partial\hat K\times I_n}\\
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\hat{\mathcal T}_n}\hat\sigma_2\left(\frac{p^2}{\hat h_{\hat K}}\|\hat\nabla\times\hat{\boldsymbol\eta}\|^2_{\hat K\times I_n}+\frac{\hat h_{\hat K}}{p^2}\|\hat\nabla(\hat\nabla\times\hat{\boldsymbol\eta})\|^2_{\hat K\times I_n}\right)\\
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\hat{\mathcal T}_n}\left(\frac{p^2}{\tau_n}\|\hat\nabla\times\hat{\boldsymbol\eta}\|^2_{\hat K\times I_n}+\frac{\hat h^2_{\hat K}}{p^2\tau_n}\|\hat\nabla(\hat\nabla\times\hat{\boldsymbol\eta})\|^2_{\hat K\times I_n}\right).
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
&\sum_{n=0}^{N-1}\left\|\hat\sigma_2^{-\frac12}\{\{\dot{\hat{\boldsymbol\eta}}\}\}\right\|^2_{\hat{\mathcal F}_n^{\rm int}\times I_n}\\
&\le C\sum_{n=0}^{N-1}\sum_{\hat K\in\hat{\mathcal T}_n}\left\|\hat\sigma_2^{-\frac12}\dot{\hat{\boldsymbol\eta}}\right\|^2_{\partial\hat K\times I_n}
\end{aligned}
$$

$$\leq C\sum_{n=0}^{N-1}\sum_{\hat{K}\in\hat{\mathcal{T}}_n}\hat{\sigma}_2^{-1}\left(\frac{p^2}{\hat{h}_{\hat{K}}}\|\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}+\frac{\hat{h}_{\hat{K}}}{p^2}\|\hat{\nabla}\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}\right)$$
$$\leq C\sum_{n=0}^{N-1}\sum_{\hat{K}\in\hat{\mathcal{T}}_n}\left(\frac{p^2\tau_n}{\hat{h}^2_{\hat{K}}}\|\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}+\frac{\tau_n}{p^2}\|\hat{\nabla}\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}\right).$$

Besides, we have

$$\sum_{n=0}^{N-1}\left\|\hat{\sigma}_1^{-\frac{1}{2}}\{\{\mu_r^{-1}\tilde{\hat{\nabla}}\times\dot{\hat{\boldsymbol{\eta}}}\}\}\right\|^2_{\hat{\mathcal{F}}_n\times I_n}$$
$$\leq C\sum_{n=0}^{N-1}\sum_{\hat{K}\in\hat{\mathcal{T}}_n}\left\|\hat{\sigma}_1^{-\frac{1}{2}}\mu_r^{-1}\hat{\nabla}\times\dot{\hat{\boldsymbol{\eta}}}\right\|^2_{\partial\hat{K}\times I_n}$$
$$\leq C\sum_{n=0}^{N-1}\sum_{\hat{K}\in\hat{\mathcal{T}}_n}\hat{\sigma}_1^{-1}\left(\frac{p}{\hat{h}_{\hat{K}}}\|\hat{\nabla}\times\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}+\frac{\hat{h}_{\hat{K}}}{p}\|\hat{\nabla}(\hat{\nabla}\times\dot{\hat{\boldsymbol{\eta}}})\|^2_{\hat{K}\times I_n}\right)$$
$$\leq C\sum_{n=0}^{N-1}\sum_{\hat{K}\in\hat{\mathcal{T}}_n}\left(\frac{\tau_n}{p^2}\|\hat{\nabla}\times\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}+\frac{\hat{h}^2_{\hat{K}}\tau_n}{p^4}\|\hat{\nabla}(\hat{\nabla}\times\dot{\hat{\boldsymbol{\eta}}})\|^2_{\hat{K}\times I_n}\right).$$

Finally, we estimate

$$\sum_{n=0}^{N-1}\left\|\hat{\sigma}_0\hat{\sigma}_1^{-\frac{1}{2}}[\dot{\hat{\boldsymbol{\eta}}}]_T\right\|^2_{\hat{\mathcal{F}}_n\times I_n}$$
$$\leq C\sum_{n=0}^{N-1}\sum_{\hat{K}\in\hat{\mathcal{T}}_n}\hat{\sigma}_0^2\hat{\sigma}_1^{-1}\|\dot{\hat{\boldsymbol{\eta}}}\|^2_{\partial\hat{K}\times I_n}$$
$$\leq C\sum_{n=0}^{N-1}\sum_{\hat{K}\in\hat{\mathcal{T}}_n}\hat{\sigma}_0^2\hat{\sigma}_1^{-1}\left(\frac{p}{\hat{h}_{\hat{K}}}\|\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}+\frac{\hat{h}_{\hat{K}}}{p}\|\hat{\nabla}\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}\right)$$
$$\leq C\sum_{n=0}^{N-1}\sum_{\hat{K}\in\hat{\mathcal{T}}_n}\left(\frac{p^2\tau_n}{\hat{h}^2_{\hat{K}}}\|\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}+\tau_n\|\hat{\nabla}\dot{\hat{\boldsymbol{\eta}}}\|^2_{\hat{K}\times I_n}\right).$$

Using

$$\|\hat{\nabla}\times(\hat{\nabla}\times\hat{\boldsymbol{\eta}})\|_{\hat{K}\times I_n}+\|\hat{\nabla}(\hat{\nabla}\times\hat{\boldsymbol{\eta}})\|_{\hat{K}\times I_n}\leq C\|D^2\hat{\boldsymbol{\eta}}\|_{\hat{K}\times I_n},$$
$$\|\hat{\nabla}\times\dot{\hat{\boldsymbol{\eta}}}\|_{\hat{K}\times I_n}+\|\hat{\nabla}\times\hat{\boldsymbol{\eta}}\|_{\hat{K}\times I_n}\leq C\|\hat{\nabla}\hat{\boldsymbol{\eta}}\|_{\hat{K}\times I_n},$$

and combining the above estimates yields the desired result. □

The next lemma is the direct application to the linear time-dependent Maxwell equations of [27, Lemma 1] (also see [1, Proposition 17]).

Lemma 6.5. *Denote the Trefftz space by*

$$\hat{\mathbf{V}}_n(\hat{K}\times I_n)=\left\{\hat{\mathbf{E}}:\hat{\mathbf{E}}\ \textit{satisfies}\ \varepsilon_r\ddot{\hat{\mathbf{E}}}+\hat{\nabla}\times\left(\mu_r^{-1}\hat{\nabla}\times\hat{\mathbf{E}}\right)=\mathbf{0}\ \textit{in}\ \hat{K}\times I_n\in\hat{\mathcal{T}}_n\times I_n\right\}.$$

Assume that each space-time element $\hat{K}\times I_n$ is star-shaped with respect to a ball. For $p,s\in\mathbb{N}_0$, let

$$\hat{\mathbf{E}}\in\hat{\mathbf{V}}_n(\hat{K}\times I_n)\bigcap\mathbf{H}^{s+1}(\hat{K}\times I_n)^3.$$

Then there exists $\hat{\boldsymbol{\phi}}_{\hat{h}} \in \hat{\mathbf{T}}_n(\hat{K} \times I_n)$ *such that, for all* $0 \le j \le m := \min\{p, s\}$,

$$\left|\hat{\mathbf{E}} - \hat{\boldsymbol{\phi}}_{\hat{h}}\right|_{H^j(\hat{K}\times I_n)} \le C\hat{h}_{\hat{K}}^{m+1-j} |\hat{\mathbf{E}}|_{H^{m+1}(\hat{K}\times I_n)}. \tag{6.19}$$

Theorem 6.2. *Let* $\mathbf{E}$ *and* $\mathbf{E}_h$ *be the solutions of the IBVP* (2.1) *and of the Trefftz DG formulation* (5.16)*, respectively. For each element* $K \times I_n \in \mathcal{T}_h$*, assume* $h_K \sim \tau_n$ *and local regularity* $\mathbf{E} \in H^{s_K+1}(K \times I_n)^3$ *for some* $s_K \in \mathbb{N}_0$*, and define* $m_K := \min\{p, s_K\}$*. Then we have the error bound in the skeleton-based norm*

$$|||\mathbf{E} - \mathbf{E}_h||| \le C \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_n} \rho^{\frac{m_K}{2}+\frac{7}{4}} \lambda_{\text{mid}}^{-\frac{3}{4}} \left(\lambda_{\min}^{-\frac{1}{4}} + \lambda_{\min}^{\frac{3}{4}}\right) h^{m_K-\frac{1}{2}} \|\mathbf{E}\|_{H^{m_K+1}(K\times I_n)}. \tag{6.20}$$

Proof. We define $\boldsymbol{\phi}_h = G\hat{\boldsymbol{\phi}}_{\hat{h}}$, where $\hat{\boldsymbol{\phi}}_{\hat{h}}$ satisfying (6.19) denotes the plane wave approximation of the scaled electromagnetic field $\hat{\mathbf{E}}$. This, together with (6.15), leads to

$$\begin{aligned} |||\mathbf{E} - \mathbf{E}_h||| \le C \sum_{n=0}^{N-1} \sum_{\hat{K}\in\hat{\mathcal{T}}_n} & \rho^{\frac{3}{4}} \|M^{-1}\|^{\frac{3}{2}} \left(\lambda_{\min}^{\frac{1}{2}} + \lambda_{\min}^{-\frac{1}{2}}\right) \\ & \times \left(\hat{h}^{m_K}\tau_n^{-\frac{1}{2}} + \hat{h}^{m_K-1}\tau_n^{\frac{1}{2}}\right) \|\hat{\mathbf{E}}\|_{H^{m_K+1}(\hat{K}\times I_n)}. \end{aligned} \tag{6.21}$$

With the transformations (5.1) and (5.3), and by the scaling argument, we can obtain

$$\|\hat{\mathbf{E}}\|_{H^{m_K+1}(\hat{K}\times I_n)} \le \left\|\Lambda_K^{\frac{1}{2}}\right\| \det(M_K)^{\frac{1}{2}} \left\|M_K^{-1}\right\|^{m_K+1} \|\mathbf{E}\|_{H^{m_K+1}(K\times I_n)}. \tag{6.22}$$

Combining (6.21) and (6.22), and using

$$\begin{aligned} &\left\|\Lambda_K^{\frac{1}{2}}\right\| \det(M_K)^{\frac{1}{2}} \left\|M_K^{-1}\right\| \le \rho\lambda_{\min}, \\ &\|M\|^{-1}\hat{h} \le h \le \|M^{-1}\|\hat{h}, \end{aligned}$$

yields

$$\begin{aligned} |||\mathbf{E} - \mathbf{E}_h||| &\le C \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_n} \rho^{\frac{7}{4}} \|M^{-1}\|^{m_K+\frac{3}{2}} \lambda_{\min} \left(\lambda_{\min}^{\frac{1}{2}} + \lambda_{\min}^{-\frac{1}{2}}\right) \\ &\quad \times \left(\hat{h}^{m_K}\tau_n^{-\frac{1}{2}} + \hat{h}^{m_K-1}\tau_n^{\frac{1}{2}}\right) \|\mathbf{E}\|_{H^{m_K+1}(K\times I_n)} \\ &\le C \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_n} \rho^{\frac{m_K}{2}+\frac{7}{4}} \lambda_{\text{mid}}^{-\frac{3}{4}} \left(\lambda_{\min}^{-\frac{1}{4}} + \lambda_{\min}^{\frac{3}{4}}\right) h^{m_K-\frac{1}{2}} \|\mathbf{E}\|_{H^{m_K+1}(K\times I_n)}. \end{aligned} \tag{6.23}$$

The proof is complete. □

Remark 6.2. We would like to point out that the proposed method does not require any CFL-type restrictions for stability and convergence: as long as $h_K \sim \tau_n$, the above optimal convergence can be obtained, owing to their implicit time-stepping interpretation. Indeed, an important advantage in using such space-time methods is that they do not require any CFL-type restrictions.

7. A Nonhomogeneous Model

The model reads as

$$\begin{cases} \varepsilon\ddot{\mathbf{E}} + \nabla\times(\mu^{-1}\nabla\times\mathbf{E}) = \mathbf{J} & \text{in } Q, \quad (7.1a)\\ \mathbf{E}(\cdot,0) = \mathbf{E}_0, \quad \dot{\mathbf{E}}(\cdot,0) = \mathbf{H}_0 & \text{on } \Omega, \quad (7.1b)\\ \mathbf{n}_\Omega\times\mathbf{E} = \mathbf{g} & \text{on } \Gamma\times[0,T]. \quad (7.1c)\end{cases}$$

Here $\mathbf{J}\in\mathbf{L}^2(\Omega)$ are the given source data.

In the framework proposed by [18] of the global Trefftz DG method combined with overlapping local DG method, we decompose the solution $\mathbf{E}$ of the problem (7.1) into $\mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(2)}$, where $\mathbf{E}^{(1)}$ is a particular local solution of the Eq. (7.1a) on each fictitious domain with homogeneous boundary and initial conditions, and $\mathbf{E}^{(2)}$ satisfies the locally homogeneous Maxwell equations.

7.1. Nonhomogeneous local problems

For each space-time element $K_n = K\times I_n\in\mathcal{T}_h, K\in\mathcal{T}_n$, let K^* be a fictitious domain that contains K as its subdomain. Set the fictitious domain

$$K_n^* = K^*\times I_n, \quad \mathcal{F}_{K^*}^{t_n} = K^*\times\{t=t_n\}, \quad \mathcal{F}_{K^*,n}^{\partial} = \partial K^*\times I_n,$$

and the local space

$$\mathbf{V}_{K^*}^{(1)} = \{\boldsymbol{\phi}\in\mathbf{H}^{1+\delta_0}(K_n^*) \text{ s.t. } \nabla\times\boldsymbol{\phi}\in\mathbf{H}^{1+\delta_0}(K_n^*) \text{ with } \delta_0>0,\ \ddot{\boldsymbol{\phi}}\in L^2(I_n;K_n^*).\}$$

The particular solution $\mathbf{E}^{(1)}\in(L^2(Q))^3$ is defined as $\mathbf{E}^{(1)}|_{K_n} = \mathbf{E}_{K_n}^{(1)} = \mathbf{E}_{K_n^*}^{(1)}|_K$, where $\mathbf{E}_{K_n^*}^{(1)}\in\mathbf{V}_{K^*}^{(1)}$ satisfies the nonhomogeneous local Maxwell equations on the fictitious domain K_n^*

$$\begin{cases} \varepsilon\ddot{\mathbf{E}}^{(1)} + \nabla\times(\mu^{-1}\nabla\times\mathbf{E}^{(1)}) = \mathbf{J} & \text{in } K_n^*,\\ \mathbf{E}^{(1)}(\cdot,t_n) = \mathbf{0}, \quad \dot{\mathbf{E}}^{(1)}(\cdot,t_n) = \mathbf{0} & \text{on } \mathcal{F}_{K^*}^{t_n},\\ \mathbf{n}_K\times\mathbf{E}^{(1)} = \mathbf{0} & \text{on } \mathcal{F}_{K^*,n}^{\partial}.\end{cases} \tag{7.2}$$

Similarly to the derivation of the initial time-stepping discrete system (4.18) (note that the solution of the local problem (7.2) defined on the current time interval I_n is somewhat equivalent to that of (2.1) defined on the initial time interval I_0), we can obtain the overlapping local space-time DG variational formulation of our combined method

$$\begin{aligned} a^{(1)}(\mathbf{E}^{(1)},\boldsymbol{\phi}) = {} & \big(\varepsilon\ddot{\mathbf{E}}^{(1)},\dot{\boldsymbol{\phi}}\big)_{K^*\times I_n} + \big(\varepsilon\dot{\mathbf{E}}^{(1)}(t_n^+),\dot{\boldsymbol{\phi}}(t_n^+)\big)_{K^*}\\ & + \big(\mu^{-1}\nabla\times\mathbf{E}^{(1)},\nabla\times\dot{\boldsymbol{\phi}}\big)_{K^*\times I_n} + \big(\mu^{-1}\nabla\times\mathbf{E}^{(1)}(t_n^+),\nabla\times\boldsymbol{\phi}(t_n^+)\big)_{K^*}\\ & - \big(\mu^{-1}\nabla\times\mathbf{E}^{(1)},\mathbf{n}\times\dot{\boldsymbol{\phi}}\big)_{\mathcal{F}_{K^*,n}^{\partial}} - \big(\mu^{-1}\nabla\times\mathbf{E}^{(1)}(t_n^+),\mathbf{n}\times\boldsymbol{\phi}(t_n^+)\big)_{\partial K^*}\\ & - \big(\mathbf{n}\times\mathbf{E}^{(1)},\mu^{-1}\nabla\times\dot{\boldsymbol{\phi}}\big)_{\mathcal{F}_{K^*,n}^{\partial}} - \big(\mathbf{n}\times\mathbf{E}^{(1)}(t_n^+),\mu^{-1}\nabla\times\boldsymbol{\phi}(t_n^+)\big)_{\partial K^*}\\ & + \big(\sigma_0\mathbf{n}\times\mathbf{E}^{(1)},\mathbf{n}\times\dot{\boldsymbol{\phi}}\big)_{\mathcal{F}_{K^*,n}^{\partial}} + \big(\sigma_0\mathbf{n}\times\mathbf{E}^{(1)}(t_n^+),\mathbf{n}\times\boldsymbol{\phi}(t_n^+)\big)_{\partial K^*}\\ & + \big(\sigma_1\mathbf{n}\times\mathbf{E}^{(1)},\mathbf{n}\times\boldsymbol{\phi}\big)_{\mathcal{F}_{K^*,n}^{\partial}} = b^{(1)}(\boldsymbol{\phi}), \end{aligned} \tag{7.3}$$

where $b^{(1)}(\boldsymbol{\phi})$ is given by

$$b^{(1)}(\boldsymbol{\phi}) = \big(\mathbf{J},\dot{\boldsymbol{\phi}}\big)_{K^*\times I_n}, \quad \forall\,\boldsymbol{\phi}\in\mathbf{V}_{K^*}^{(1)}. \tag{7.4}$$

7.2. Global residual problem

Set local Trefftz space for any $K \times I_n \in \mathcal{T}_h$,

$$\begin{aligned}\mathbf{Tre}(K \times I_n) = \big\{ & \boldsymbol{\phi} \in \mathbf{H}^{1+\delta_0}(K \times I_n) \text{ s.t. } \varepsilon\ddot{\boldsymbol{\phi}} + \nabla \times (\mu^{-1}\nabla \times \boldsymbol{\phi}) = \mathbf{0}, \\ & \text{and } \nabla \times \boldsymbol{\phi} \in \mathbf{H}^{1+\delta_0}(\times I_n) \text{ with } \delta_0 > 0,\ \ddot{\boldsymbol{\phi}} \in L^2(I_n; K)\big\},\end{aligned}$$

and the global Trefftz space

$$\mathbf{Tre}(\mathcal{T}_h) = \big\{ \boldsymbol{\phi} \in \mathbf{L}^2(Q) \text{ s.t. } \boldsymbol{\phi}|_{K\times I_n} \in \mathbf{Tre}(K \times I_n),\ \forall\, K \times I_n \in \mathcal{T}_h \big\}. \tag{7.5}$$

Similarly to the derivation of (4.12), we can obtain the Trefftz-DG variational formulation: Find $\mathbf{E}^{(2)} \in \mathbf{Tre}(\mathcal{T}_h)$ such that

$$a(\mathbf{E}^{(2)}, \boldsymbol{\phi}) = b^{\text{init}}(\boldsymbol{\phi}) + \sum_{n=0}^{N-1} (\mathbf{J}, \dot{\boldsymbol{\phi}})_{\Omega\times I_n} - a(\mathbf{E}^{(1)}, \boldsymbol{\phi}), \quad \forall\, \boldsymbol{\phi} \in \mathbf{Tre}(\mathcal{T}_h). \tag{7.6}$$

7.3. Discretization of the variational problems

We decompose the discrete solution $\mathbf{E}_h$ of the problem (7.1) into $\mathbf{E}_h = \mathbf{E}_h^{(1)} + \mathbf{E}_h^{(2)}$, where $\mathbf{E}_h^{(1)}$ is the discrete solution of continuous variational formulations (7.3), and $\mathbf{E}_h^{(2)} \in \mathbf{T}^{h,p}$ is the discrete solution of continuous Trefftz DG variational formulation (7.6).

Let q be a positive integer and D be a bounded and connected domain in $\mathbb{R}^{3+1}$. Let $S_q(D)$ denote the set of polynomials defined on D, whose orders are less or equal to q. Set $\mathbf{S}_q(D) = (S_q(D))^3$. Define $\mathbf{S}^n_{K^*}$ by

$$\mathbf{S}^n_{K^*} := \big\{ \mathbf{E} \in \mathbf{L}^2(K^* \times I_n) : \mathbf{E}\big|_{K^*\times I_n} \in \mathbf{S}_q(K^* \times I_n),\ K \in \mathcal{T}_n \big\}.$$

Example 7.1. The following is a basis for $\mathbf{S}^n_{K^*}$ with $q = 2$:

$$\{1, x, y, z, t, x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt\}.$$

Then a discretized version of the continuous variational problem (7.3) is: Find $\mathbf{E}^{(1)}_{h,K_n^*} \in \mathbf{S}^n_{K^*}$ such that

$$a^{(1)}\big(\mathbf{E}^{(1)}_{h,K_n^*}, \boldsymbol{\phi}\big) = b^{(1)}(\boldsymbol{\phi}), \quad \forall\, \boldsymbol{\phi} \in \mathbf{S}^n_{K^*}. \tag{7.7}$$

Further, define $\mathbf{E}_h^{(1)}$ by $\mathbf{E}_h^{(1)}|_K = \mathbf{E}^{(1)}_{h,K_n^*}|_K$. A natural way is to choose K^* as the geometric sphere. Then the variational problems (7.7) can be solved easily by using the polar coordinate transformation for the calculation of the involved integrations. We would like to emphasize that the discrete problems (7.7) are local and independent each other for $K \in \mathcal{T}_n, n = 0, \ldots, N-1$, so they can be explicitly solved in parallel and the cost is small.

A discretized version associated with variational problem (7.3) can be described as follows: Find $\mathbf{E}_h^{(2)} \in \mathbf{T}^{h,p}$ such that

$$a\big(\mathbf{E}_h^{(2)}, \boldsymbol{\phi}\big) = b^{\text{init}}(\boldsymbol{\phi}) + \sum_{n=0}^{N-1} (\mathbf{J}, \dot{\boldsymbol{\phi}})_{\Omega\times I_n} - a\big(\mathbf{E}_h^{(1)}, \boldsymbol{\phi}\big), \quad \forall\, \boldsymbol{\phi} \in \mathbf{T}^{h,p}. \tag{7.8}$$

8. Numerical Experiments

In this section, we apply the proposed methods to solve the wave propagation in anisotropic media, and we report numerical results to verify the efficiency of the method.

In our tests, we set $\varepsilon_r = \mu_r = 1$, and use a uniform mesh with $h_K \approx h_t = 2^{-l}, l \in \mathbb{N}$. Meanwhile, we choose the same number p of basis functions for every elements $K \times I_n$, and estimate the convergence orders of the approximations by measuring the relative errors in the energy norm at a given time T

$$\text{error} = \frac{\left(\left\|\varepsilon^{\frac{1}{2}}\left(\dot{\mathbf{E}}(T) - \dot{\mathbf{E}}_h(T)\right)\right\|_\Omega^2 + \left\|\mu^{-\frac{1}{2}}\tilde{\nabla} \times \left(\mathbf{E}(T) - \mathbf{E}_h(T)\right)\right\|_\Omega^2\right)^{\frac{1}{2}}}{\left(\left\|\varepsilon^{\frac{1}{2}}\dot{\mathbf{E}}(T)\right\|_\Omega^2 + \left\|\mu^{-\frac{1}{2}}\tilde{\nabla} \times \mathbf{E}(T)\right\|_\Omega^2\right)^{\frac{1}{2}}}$$

and the errors in $|||\cdot|||$-norm, respectively. We choose the space-time domain $Q = (0,1)^3 \times (0,1)$, and set the anisotropic matrix

$$A = \begin{pmatrix} \lambda_{\min} a^2 + \lambda_{\text{mid}} b^2 & ab(\lambda_{\text{mid}} - \lambda_{\min}) & 0 \\ ab(\lambda_{\text{mid}} - \lambda_{\min}) & \lambda_{\min} b^2 + \lambda_{\text{mid}} a^2 & 0 \\ 0 & 0 & \lambda_{\max} \end{pmatrix},$$

where the diagonal matrix $\Lambda = \text{diag}(1, \rho, \rho)$,

$$P = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a = \frac{1}{\sqrt{2}}, \quad b = \frac{1}{\sqrt{2}}.$$

The penalization parameter C_{σ_0} is chosen by (4.5). Particularly, we choose $C_{\sigma_0} = 2^4\|A^{-1/2}\|^2$.

In each experiment, the spatial meshes are kept fixed $\mathcal{T} = \mathcal{T}_n$ and a uniform time step is used. In the Tables 8.1-8.3, it is given for the mesh level $l = 4$ and $\rho = 2$. All of the computations have been done in MATLAB, and the system matrix was computed by numerical integration.

8.1. Homogeneous case

Consider the exact smooth solution (see [9, Section 6.2])

$$\hat{\mathbf{E}} = \begin{pmatrix} 0 & 0 & \hat{E}_3(\hat{x}, \hat{y}, t) \end{pmatrix}^\top,$$

where the material parameters and the fields are independent of $\hat{z}$, and choose

$$\hat{E}_3 = \sin(\pi\hat{x})\sin(\pi\hat{y})\cos(\sqrt{2}\pi t).$$

Then with the transformations (5.1) and (5.3), we define $\mathbf{E}(\mathbf{x}, t) = G\,\hat{\mathbf{E}}(S\mathbf{x}, t)$.

The convergence rates with respect to h generated by the proposed method and DG method (4.19) employing the full polynomials are given in Table 8.1 and Fig. 8.1.

It can be seen from Table 8.1 that, the convergence rates of relative errors in the energy norm are $\mathcal{O}(h^p)$, and the convergence rates of errors in $|||\cdot|||$-norm are $\mathcal{O}(h^{p-1/2})$, which both demonstrate numerically optimal convergence rates for the space-time scheme with the proposed mesh refinement on the spatial domain. This optimal behavior of the Trefftz method has already been observed for acoustic waves problems in two space dimensions [1]. Besides, note that the

Table 8.1: Convergence rates of the TDG schemes with respect to h.

p		Energy-norm error	Rate	$\vert\vert\vert \cdot \vert\vert\vert$-norm error	Rate
2	TDG	4.15e-3	1.99	1.06e-2	1.50
	DG	4.12e-3	1.98	1.04e-2	1.49
3	TDG	4.32e-4	3.02	1.37e-3	2.52
	DG	4.33e-4	3.01	1.39e-3	2.50
4	TDG	2.46e-5	4.03	1.18e-4	3.51
	DG	2.44e-5	4.01	1.19e-4	3.50

errors obtained by the full and the Trefftz spaces are very similar for the same order, but the Trefftz spaces require fewer degrees of freedom and cheaper implementation, see Remark 6.1.

The behavior of the errors on conditional numbers ρ are shown in Fig. 8.2.

We can obtain that the approximate solutions generated by the proposed method indeed have high accuracies, and the behavior of relative errors in the energy norm and of the errors in $\vert\vert\vert \cdot \vert\vert\vert$-norm on conditional numbers ρ is superior to that given in Theorem 6.2, which is still under consideration.

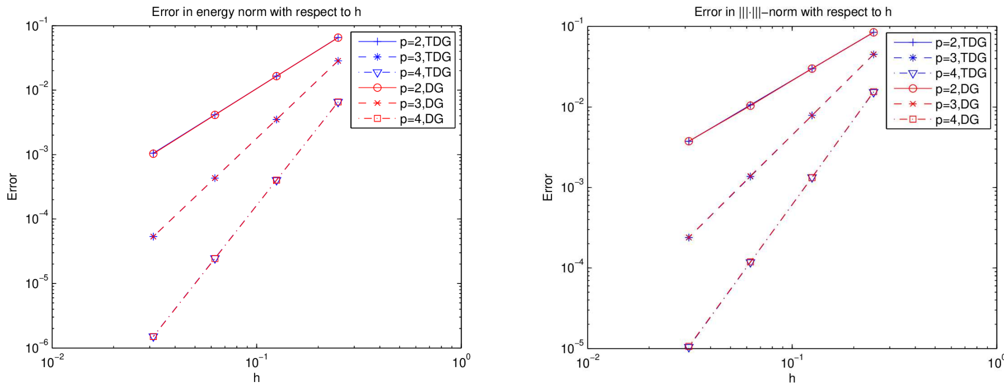

Fig. 8.1. Left: Relative energy-norm error vs h, Right: $\vert\vert\vert \cdot \vert\vert\vert$-norm error vs h.

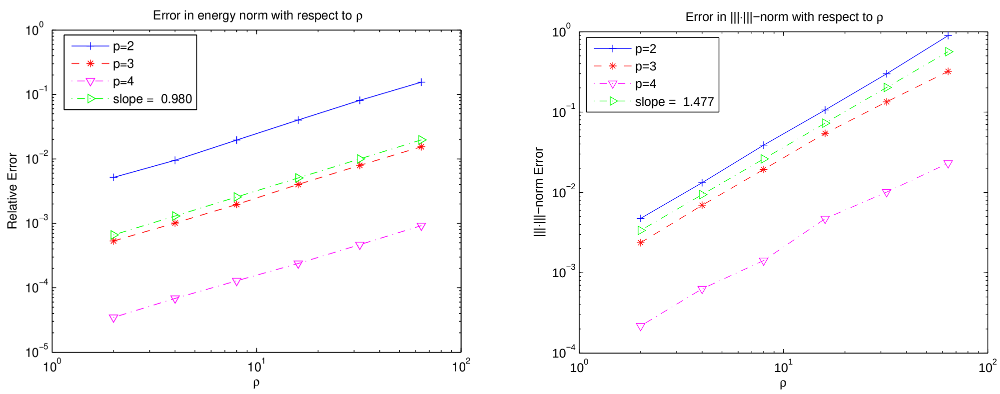

Fig. 8.2. Left: Relative energy-norm error vs ρ, Right: $\vert\vert\vert \cdot \vert\vert\vert$-norm error vs ρ.

8.2. Nonhomogeneous case

Consider the exact smooth solution for the nonhomogeneous model (7.1)

$$\mathbf{E}(\mathbf{x}, t) = G\,\hat{\mathbf{E}}(S\mathbf{x}, t),$$

where

$$\hat{\mathbf{E}} = \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{pmatrix} = \begin{pmatrix} \sin(\pi\hat{y})\sin(\pi\hat{z})\cos(\pi t) \\ \sin(\pi\hat{x})\sin(\pi\hat{z})\sin(\pi t) \\ \cos(\pi\hat{x})\sin(\pi\hat{y})\cos(\pi t) \end{pmatrix}.$$

The source term $\mathbf{J}$ satisfies the Eq. (7.1a). Set $q = p$.

The convergence rates with respect to h are given in Table 8.2 and Fig. 8.3. In Table 8.2, we can see that, the convergence orders of the errors in the energy norm are $\mathcal{O}(h^p)$. Besides, the convergence rates of errors in $|||\cdot|||$-norm are $\mathcal{O}(h^{p-1/2})$, which support convergence rate optimality of the combined numerical DG scheme just as for the homogeneous case. The behavior of the errors on conditional numbers ρ are shown in Fig. 8.4.

We can also see that the approximate solutions generated by the proposed method indeed have high accuracies, and the relative errors in the energy norm and the errors in $|||\cdot|||$-norm on conditional numbers ρ behave as $\mathcal{O}(\rho)$ and $\mathcal{O}(\rho^{3/2})$, respectively.

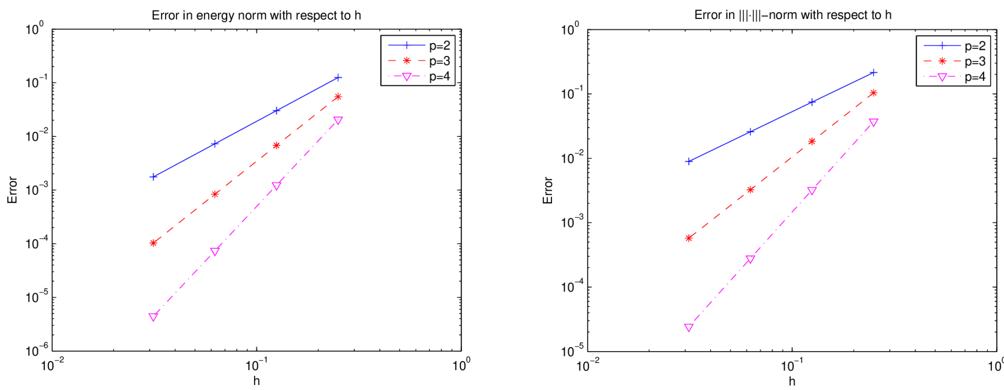

Fig. 8.3. Left: Relative energy-norm error vs h, Right: $|||\cdot|||$-norm error vs h.

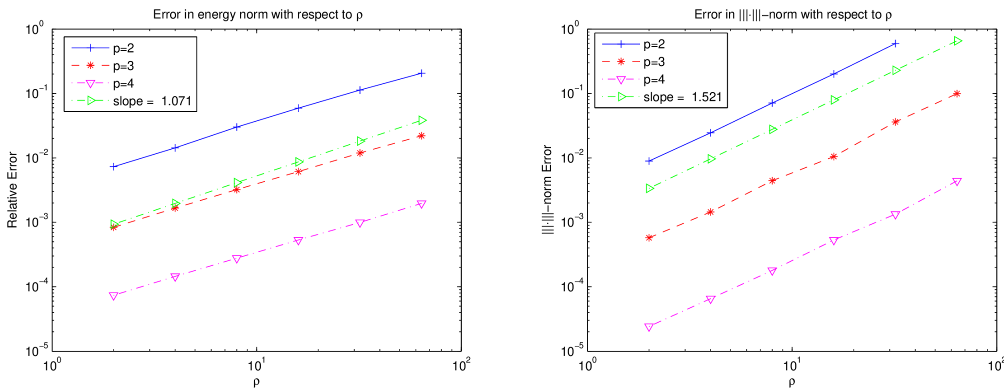

Fig. 8.4. Left: Relative energy-norm error vs ρ, Right: $|||\cdot|||$-norm error vs ρ.

Table 8.2: Convergence rates of the combined space-time DG scheme with respect to h.

(p,q)	Energy-norm error	Rate	$\lVert\cdot\rVert$-norm error	Rate
(2,2)	7.29e-3	2.05	2.58e-2	1.53
(3,3)	8.36e-4	3.02	3.25e-3	2.50
(4,4)	7.37e-5	4.06	2.79e-4	3.53

8.3. Inhomogeneous case

Set $\mathbf{a} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^\top$. We compute the electric field due to an electric dipole source at the point $\mathbf{x}_0 = (-0.6, -0.6, -0.6)$. Choose

$$\mathbf{E}_0 = -\mathrm{i}\omega\phi(\mathbf{x}, \mathbf{x}_0) G\,\mathbf{a} + \frac{1}{\mathrm{i}\omega\varepsilon_r} G\,\nabla_{\hat{h}}(\nabla_{\hat{h}}\phi \cdot \mathbf{a}), \tag{8.1}$$

where

$$\phi(\mathbf{x}, \mathbf{x}_0) = \frac{\exp(\mathrm{i}\omega\sqrt{\varepsilon_r}|S\,\mathbf{x} - \mathbf{x}_0|)}{4\pi|S\,\mathbf{x} - \mathbf{x}_0|}.$$

Set $\mathbf{H}_0 = \nabla \times \mathbf{E}_0$. The boundary data $\mathbf{g}$ can be computed by $\mathbf{n} \times \mathbf{a}\phi$.

Consider the case of inhomogeneous media

$$\Lambda = \begin{cases} \Lambda_1 = \mathrm{diag}(1, 1, \rho), & \Omega_1 : z < 0.5, \\ \Lambda_2 = \mathrm{diag}(1, 1, 2\rho), & \Omega_2 : z > 0.5. \end{cases} \tag{8.2}$$

The above two subdomains correspond to the orthogonal matrix

$$P_1 = \frac{1}{7}\begin{pmatrix} 6 & 2 & 3 \\ 3 & -6 & -2 \\ 2 & 3 & -6 \end{pmatrix} \quad \text{on subdomain} \quad \Omega_1,$$

$$P_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{on subdomain} \quad \Omega_2.$$

It is hard to construct an analytic solution for the case of layered media. We consider the actual measurements of the solution error by taking the numerical solution on the finest mesh and on the largest number of basis functions per element as a reference solution.

The convergence rates with respect to h are given in Table 8.3 and Fig. 8.5. In Table 8.3, we can see that, the convergence orders of the errors in the energy norm are $\mathcal{O}(h^p)$. Besides,

Table 8.3: Convergence rates of the combined space-time DG scheme with respect to h.

(p,q)	Energy-norm error	Rate	$\lVert\cdot\rVert$-norm error	Rate
(2,2)	1.01e-2	1.95	3.78e-2	1.47
(3,3)	1.14e-3	2.97	4.85e-3	2.45
(4,4)	1.11e-4	4.01	4.14e-4	3.49

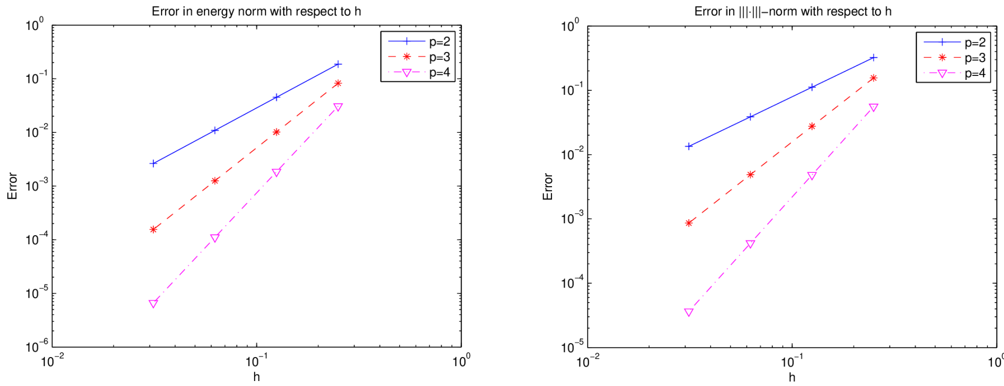

Fig. 8.5. Left: Relative energy-norm error vs h, Right: $|||\cdot|||$-norm error vs h.

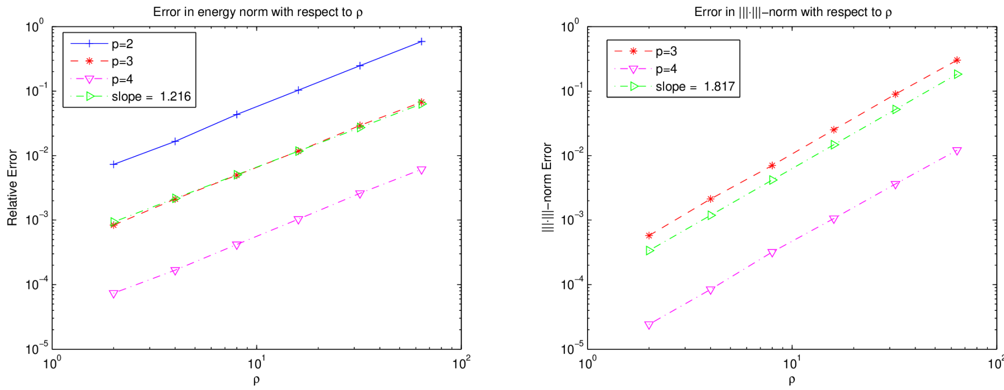

Fig. 8.6. Left: Relative energy-norm error vs ρ, Right: $|||\cdot|||$-norm error vs ρ.

the convergence rates of errors in $|||\cdot|||$-norm are $\mathcal{O}(h^{p-1/2})$, which support convergence rate optimality of the combined numerical DG scheme just as for the homogeneous case.

The behavior of the errors on conditional numbers ρ are shown in Fig. 8.6. We can also see that the approximate solutions generated by the proposed method indeed have high accuracies, and the relative errors in the energy norm and the errors in $|||\cdot|||$-norm on conditional numbers ρ behave as $\mathcal{O}(\rho^{1.2})$ and $\mathcal{O}(\rho^{1.8})$, respectively.

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