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A SPACE-TIME TREFFTZ DG METHOD FOR THE SECOND ORDER TIME-DEPENDENT MAXWELL SYSTEM IN ANISOTROPIC MEDIA*

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Abstract

The h-version analysis technique developed in [Banjai *et al.*, SIAM J. Numer. Anal., 55 (2017)] for Trefftz discontinuous Galerkin (DG) discretizations of the second order isotropic wave equation is extended to the time-dependent Maxwell equations in anisotropic media. While the discrete variational formulation and its stability and quasi-optimality are derived parallel to the acoustic wave case, the derivation of error estimates in a mesh-skeleton norm requires new transformation stabilities for the anisotropic case. The error estimates of the approximate solutions with respect to the condition number of the coefficient matrices are proved. Furthermore, we propose the global Trefftz DG method combined with local DG methods to solve the time-dependent nonhomogeneous Maxwell equations. The numerical results verify the validity of the theoretical results, and show that the resulting approximate solutions possess high accuracy.

Mathematics subject classification: 65N30, 65N55.
Key words: Time-dependent Maxwell's equation, Anisotropic, Nonhomogeneous, Trefftz method, Local discontinuous Galerkin, Error estimate.

1. Introduction

The idea at the heart of Trefftz method, which are named after the seminal work of Trefftz [36], is to choose the Trefftz approximation functions from a class of piecewise solutions of the same governing partial differential equation (PDE) without boundary conditions. Trefftz methods turned out to be particularly effective, and popular, for wave propagation problems in time-harmonic regime at medium and high frequencies, where the oscillatory nature of the solutions makes standard methods computationally too expensive, see the recent survey [15] and references therein. The Trefftz method has an important advantage over Lagrange finite elements for discretization of the Helmholtz equation and time-harmonic Maxwell equations [13–17, 28, 29, 43]: to achieve the same accuracy, relatively smaller number of degrees of

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freedom is enough in the plane wave-type methods owing to the particular choice of the basis functions that (may approximately) satisfy the considered PDE without boundary conditions.

Recently, much work has been devoted to Trefftz DG methods for time-dependent linear isotropic wave phenomena, see in particular [8–11, 22–24, 27, 30, 31, 35, 38–41, 44]. Related works [35, 38–40] proposed promising Tent Pitcher algorithms coupled with the Trefftz DG method, and obtained positive numerical results illustrating the increase of accuracy and the decrease of computational burden. A Trefftz interior-penalty second order formulation for the second order wave equation is first proposed in [1], where a best approximation result is proven for the space-time DG method with Trefftz-type basis functions and rates of convergence are proved in any dimension. Besides, for the case of one space dimension, a class of Trefftz DG method for time-dependent electromagnetic problems, written as a first order system, resulting in a two-field formulation, have been analysed in [22–24], where stability, quasi-optimality, best approximation estimates for polynomial Trefftz spaces and (fully explicit) error bounds with high order in the mesh width and in the polynomial degree are proved. Further, in [8, 9, 30] Trefftz DG methods in a first order formulation have been extended to three-dimensional time dependent Maxwell's equations, where the resulting spectral convergence order is only demonstrated by numerical tests. Currently, a space-time Trefftz discontinuous Galerkin method for the first order anisotropic Maxwell's equations in homogeneous media is proposed in [42], and a space-time Trefftz discontinuous Galerkin methods for the first order anisotropic acoustic wave equations in inhomogeneous media is proposed in [41]. Similar to frequency domain problems, compared to standard space-time DG method, Trefftz DG methods can also achieve the same convergence order and approximation properties with considerably fewer degrees of freedom, which brings more economical implementation due to integration being restricted to the space-time skeleton not space-time elements, see Remark 6.1, Section 8.1 and [1, 22].

Anisotropy can result from periodic layering of fine layers [5], preferential alignment of fractures and cracks [34], determining the response of the inclusion to an impinging acoustic or electromagnetic wave (see the textbooks [6, 21, 25]). Anisotropy may greatly influence seismic wave propagation, seismic data acquisition and subsequent data analysis and processing procedures [37]. It is therefore important to design accurate and efficient numerical methods for modeling wave propagation in anisotropic media. Besides, electromagnetic problems within this class also include the design of waveguides and antennas, scattering of electromagnetic waves from automobiles and aircraft, and the penetration and absorption of electromagnetic waves by dielectric objects [2, 33]. Under this assumption, a permittivity ε, and permeability μ, tensor can describe a linear metamaterial with no magnetoelectric coupling, where bianisotropy effects have typically played a minor role in the overall response of the experimental metamaterials, and can be mitigated by design [26]. The study [45] was devoted to computing the mathematical model for the scattering from multiple cavities both in particular transverse magnetic (TM) and transverse electric (TE) polarizations, where the cavities are required to be invariant to z-axis, namely, the permittivity ε, and permeability μ, limited to the z-axis, do not interact with other directions.

In this paper, we construct a class of space-time Trefftz DG schemes for the second order linear anisotropic Maxwell equations in three-dimensional inhomogeneous media. The DG method considered is motivated by the class of interior penalty DG methods, and can be understood as the translation to Maxwell's equations of the Trefftz DG formulation for the second order wave equations in [1]. In this work, we construct and analyze a space-time interior penalty DG method, which employ space-time slabs to ensure solvability on each time step, resulting