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WEAK CONVERGENCE ANALYSIS OF A SPLITTING-UP METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

The weak convergence analysis plays an important role in error estimates for stochastic differential equations, which concerns with the approximation of the probability distribution of solutions. In this paper, we investigate the weak convergence order of a splitting-up method for stochastic differential equations. We first construct a splitting-up approximation, based on which we also set up a splitting-up numerical solution. We prove both of these two approximation methods are of first order of weak convergence with the help of Malliavin calculus. Finally, we present several numerical experiments to illustrate our theoretical analysis.

Mathematics subject classification: 60H10, 60H35, 65C30.
Key words: Stochastic differential equation, Splitting-up method, Weak convergence, Malliavin calculus.

1. Introduction

Stochastic differential equations play an important role in many fields such as physics [18], biology [28], finance [5], medicine [20], etc. Most stochastic differential equations (SDEs) arising in practice cannot be solved explicitly. Thus, the construction of efficient numerical methods is of great importance.

The weak error, sometimes more relevant in various fields such as finance and engineering, concerns with the approximation of the probability distribution of solutions [23,31]. It measures the error made by sampling from an approximate probability law of the exact solution at a fixed time, rather than the deviation from trajectory of the exact solution, as for the strong error [2]. There are different strategies on studying weak convergence of numerical schemes for SDEs. Kohatsu-Higa [24] study the weak convergence of Euler-Maruyama scheme for nonlinear SDEs via integration by parts formula in Malliavin calculus. Zygalakis [42] study the weak convergence of numerical schemes for SDEs via weak Taylor expansion. Usually, the weak convergence order is twice of the strong convergence order. Meanwhile, there are also many studies on weak convergence to various types of stochastic differential equations. Buckwar and Shardlow [11] study the weak convergence rate of a forward Euler approximation to stochastic differential delay equations. Kohatsu-Higa *et al.* [25] study weak convergence of Euler-Maruyama scheme for SDEs with irregular drift coefficient. Zhao and Wang [41] study the weak convergence order of numerical schemes for SDEs with super-linearly growing coefficients. Cui *et al.* [14, 16] discretize stochastic partial differential equations (SPDEs) with non-globally Lipschitz coefficients by Galerkin finite element method (FEM) spatial semi-discretization and backward Euler

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temporal semi-discretization and analyze the weak convergence order of the numerical scheme. Cai *et al.* [12] discretize stochastic Allen-Cahn equation by spectral Galerkin spatial semi-discretization and tamed exponential Euler temporal semi-discretization and analyze its weak convergence order. Cai *et al.* [13] discretize SPDEs with fractional noise by spectral Galerkin spatial semi-discretization and exponential Euler temporal semi-discretization and analyze its weak convergence order. The temporal weak convergence order of the above numerical schemes is twice of the strong convergence order.

Splitting-up methods [30,35] can convert a complicated equation into several easily solvable ones and improve computational efficiency. Splitting-up methods have been widely applied to various fields, such as Hamilton system [3], Maxwell equation [26], nonlinear Schrödinger equation [15], nonlinear filtering [40], etc. There are a few achievement on studying the splitting-up method of stochastic differential equations, and many scholars focus on the strong convergence analysis for splitting-up numerical approximation [1,4,7–10,19]. For SDEs with global Lipschitz coefficients, Wang and Li [36] study the mean-square convergence of split-step forward methods for autonomous SDEs and obtain convergence order of $1/2$. Ding *et al.* [17] construct a split-step theta method for SDEs with global Lipschitz coefficients and obtain mean-square convergence order of $1/2$. Singh [34] study the strong convergence of split-step forward Milstein methods for autonomous SDEs and obtain convergence order of 1.

Meanwhile, as for numerical approximation for SDEs with coefficients under more relaxed conditions, splitting-up backward methods are applied. Higham *et al.* [21] study p-th-moments strong convergence of split-step backward methods for autonomous SDEs which drift coefficient satisfies one-sided Lipschtiz condition and obtain convergence order of $1/2$. Huang [22] constructs a split-step θ-method for SDEs with one-sided Lipschtiz drift coefficient and analyze its exponential mean-square stability. Wu and Gan [38] analyze the mean-square convergence rate of the split-step θ-method for SDEs with non-globally Lipschitz diffusion coefficients and obtain convergence order of $1/2$. Liu *et al.* [29] analyze the split-step balanced theta method for autonomous SDEs under a non-globally Lipschtiz condition and obtain the strong convergence order of $1/2$. Yang and Zhao [39] analyze a split-step theta scheme for nonlinear SDEs with jump and obtain the strong convergence order of $1/2$. Beyn *et al.* [6] study the split-step backward Milstein scheme for SDEs with super-linearly growing drift and diffusion coefficients and get the strong convergence of order 1. Wu and Gan [37] study the split-step theta Milstein scheme for SDEs with super-linearly growing drift and diffusion coefficients and get the strong convergence of order 1. The splitting-up method have been successfully applied in solving various SDEs, which results that the analysis of weak convergence rate becomes a hot topic.

The main work of this paper is to analyze the weak convergence properties of a numerical approximation to SDE based on splitting-up technique. We divide an SDE into two equations: one is an SDE without drift term and the other one is a determined ordinary differential equation (ODE). Based on these two equations we construct a splitting-up approximation. Furthermore, applying Euler-Maruyama and Euler method to these two equations, respectively, we construct a splitting-up numerical approximation. With the help of theory of Malliavin calculus, we prove that both of these two approximations converge to the exact solution of SDE with weak convergence order of 1.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and notations used in this paper, and gives a brief introduction of Malliavin calculus. In Section 3, we construct a splitting-up approximation to an SDE and prove its weak convergence order is of 1. In Section 4, we construct a splitting-up numerical solution and obtain its