

ANALYSIS OF TWO NEW PARAREAL ALGORITHMS BASED ON DIAGONALIZATION FOR TIME-PERIODIC PROBLEMS*

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Abstract

Numerical simulation of time-periodic problems is a special area of research, since the time periodicity modifies the problem structure, and then it is desirable to use parallel methods to solve such problems. The classical parareal algorithm for time-periodic problems, which is parallel in time, solving an initial value coarse problem, called the periodic parareal algorithm with initial value coarse problem (PP-IC), usually converges very slowly, and even diverges for wave propagation problems. In this paper, we first present a new PP-IC algorithm based on a diagonalization technique proposed recently. In this new algorithm, we approximate the coarse propagator G in the classical PP-IC algorithm with a head-tail coupled condition such that G can be parallelized using diagonalization in time. We analyze the convergence factors of the diagonalization-based PP-IC algorithm for both the linear and nonlinear cases. Then, we further design and analyze a new parallel-in-time algorithm for time-periodic problems by combining the Krylov subspace method with the diagonalization-based PP-IC algorithm to accelerate the convergence. Finally, we also determine an appropriate choice of the parameter α in the head-tail coupling condition, and illustrate our theoretical results with several numerical experiments, both for model problems and the realistic application of Maxwell's equations.

Mathematics subject classification: 65L20, 65L70, 65Y05.

Key words: Parareal algorithm, Diagonalization technique, Krylov subspace, Time-periodic problem, Convergence analysis.

1. Introduction

In this paper, we consider some numerical aspects on parallel-in-time algorithms of the following time-periodic model problem:

$$\begin{aligned} \mathbf{u}'(t) &= \mathbf{f}(\mathbf{u}(t), t), \quad t \in [0, T], \\ \mathbf{u}(0) &= \mathbf{u}(T), \end{aligned} \tag{1.1}$$

where $\mathbf{f} : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$ satisfies $\mathbf{f}(\cdot, 0) = \mathbf{f}(\cdot, T)$, and $\mathbf{u} : [0, T] \rightarrow \mathbb{R}^d$. Time-periodic problems arise from various practical engineering problems, such as eddy current simulations [1], fluid-structure interaction and cardiac flow problems [34], periodically forced reactors [53, 54]. Different from initial-value problems, since the time periodicity modifies the problem structure

* Received March 5, 2024 / Revised version received August 8, 2024 / Accepted December 23, 2024 /

Published online February 24, 2025 /

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and solution methods significantly, the numerical simulation of time-periodic problems is quite special and difficult, see for example [4, 45, 52].

When the scale of time-periodic problems increases, it is desirable to use parallel methods to solve such problems. Classical parallel methods proposed to solve time-periodic problems usually use the spatial parallelism, such as multigrid methods [32] and waveform relaxation methods [35, 55], which have proved to be quite effective. While since the parallelization in space often saturates, parallel-in-time methods have been widely concerned and developed rapidly over the last twenty years. The parareal algorithm is a parallel-in-time method that was first proposed by Lions, Maday and Turinici [39], where the initial value problems are solved on time subintervals in parallel on fine grids, and through iterations the initial values on each subinterval are corrected to converge to the correct values of the overall solution. A precise convergence analysis has been systematically studied for linear ordinary and partial differential equations [30], and for nonlinear problems [19]. The parareal algorithm has also been applied to many different kinds of problems and application areas, such as dynamical problems [6, 12], quantum control problems [42, 43], time-periodic problems [25], optimal control problems governed by PDEs [27], fractional PDEs [58], highly oscillatory systems [57], and so on. Besides, other new parallel-in-time algorithms, e.g. PFASST [11, 44], MGRIT [10, 13], and several direct time parallel methods, e.g. diagonalization technique [21, 23, 28, 41], PARAEXP [18], have also been presented. More recently, several new space-time parallel methods have also been investigated, which are in parallel in both space and time directions, combining the parareal algorithm with waveform relaxation methods [38, 40] or Schwarz waveform relaxation methods [5, 24]. For a complete overview of historical development of parallel-in-time methods, see [17].

The classical parareal algorithm can produce a speed-up for first order ODEs [16], but when applied to some second order ODEs [2, 14, 15] or most hyperbolic problems [51], such as wave equation and Hamilton system [20], the method usually becomes unstable, resulting in less than ideal convergence performance or even non convergence. Therefore in [29], a Krylov subspace enhanced parareal algorithm based on the modified PITA framework in [15] was presented and analyzed, as a quite effective remedy for a beating phenomenon when the parareal algorithm is applied to second order ODEs. The parareal algorithm with projection is also considered in [9] and proved to be stable for hyperbolic systems. A direct time parallel method using diagonalization has been presented for the wave equation [22], which is based on the idea first introduced by Maday and Rønquist [41], and further the parareal algorithm using diagonalization as the coarse propagator is also effective for both dissipative and wave propagation problems [31].

Parallel-in-time methods have also been developed for time-periodic problems in recent years. In [25], two parareal algorithms for time-periodic problems were proposed and analyzed: The first one is called the periodic parareal algorithm with periodic coarse problem, and the second one is called the periodic parareal algorithm with initial value coarse problem (PP-IC). Furthermore, by using waveform relaxation as the fine propagator on each subinterval in the parareal framework for time-periodic problems, two new space-time parallel algorithms have been developed based on the PP-PC [48] and PP-IC [49] algorithms. The parareal algorithm for time-periodic problems with discontinuous inputs has also been developed [26]. The ideas based on PP-PC and PP-IC have also been widely used to solve eddy current simulations [37] and cardiac flow problems [34]. Meanwhile, solving time-periodic wave problems can usually use solving the corresponding time-harmonic problems by some fast and parameter-robust preconditioned solvers, see [36, 46]. As a guiding example, solving the Helmholtz equation is also a difficult research field, which is concerned and developed gradually. In this paper, we mainly