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ERROR ESTIMATES OF A CLASS OF SERENDIPITY VIRTUAL ELEMENT METHODS FOR SEMILINEAR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS ON CURVED DOMAINS*

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Abstract

The rigorous error analysis of a class of serendipity virtual element methods applied to numerically solve semilinear parabolic integro-differential equations on curved domains is the focus of this study. Different from the standard virtual element method, the serendipity virtual element method eliminates all the internal-moment degrees of freedom only under certain conditions of the mesh and the degree of approximation. Consequently, if the interpolation operators are utilized to approximate the nonlinear terms, the implementation of Newton's iteration algorithm can be simplified. Nonhomogeneous Dirichlet boundary conditions are considered in this paper. The strategy of approximating curved domains with polygonal domains is taken into consideration, and to overcome the issue of suboptimal convergence caused by enforcing Dirichlet boundary conditions strongly, Nitsche-based projection method is employed to impose the boundary conditions weakly. For time discretization, Crank-Nicolson scheme incorporating trapezoidal quadrature rule is adopted. Based on the concrete formulation of Nitsche-based projection method, a Ritz-Volterra projection is introduced and its approximation properties are rigorously analyzed. Building upon these approximation properties, error estimates are derived for the fully discrete scheme. Additionally, the extension of the fully discrete scheme to 3D case is also included. Finally, we present two numerical experiments to corroborate the theoretical findings.

Mathematics subject classification: 65M15, 65M60.
Key words: Serendipity virtual element method, Curved domain, Nitsche-based projection method, Semilinear parabolic integro-differential equation.

1. Introduction

Parabolic integro-differential equations (PIDEs) are widely used in various scientific disciplines to model phenomena with memory effects, such as heat transfer in materials with memory [29], nonlocal flow in porous media [20], temperature changes within nuclear reactors [32], and the dynamics of epidemic spread and evolution [18].

In this study, we focus on the following semilinear PIDEs with nonhomogeneous initial-boundary value conditions:

$$a_0 u_t - \nabla\cdot(a_1\nabla u) - \int_0^t \nabla\cdot\big(a_2\,\nabla u(\tau)\big)\mathrm{d}\tau + b(u) = f \quad \text{in } \Omega\times(0,T), \tag{1.1a}$$

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$$u = g \qquad \text{on } \partial\Omega \times (0,T), \tag{1.1b}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{1.1c}$$

where we consider Ω as a bounded, convex, and open subset of $\mathbb{R}^d$ ($d \in \{2,3\}$), with its boundary $\partial\Omega$ consisting of a finite number of curves (surfaces) $\{(\partial\Omega)_i\}_{i=1}^{N_\Omega}$. It is assumed that each curve (surface) $(\partial\Omega)_i$ is sufficiently smooth, and the overall boundary $\partial\Omega$ possesses Lipschitz continuity. The coefficients a_0, a_1 and a_2 are dependent solely on the spatial variable $\boldsymbol{x}$. The finite terminal time is denoted by T. The source function f, boundary value g, and initial value u_0 are provided as given information. The term $b(u)$ represents the nonlinearity in the equation.

Research interest in the field of mathematics has been significantly driven by the practical importance of PIDEs. For the analysis of PIDEs regarding existence, uniqueness, and regularity, some relevant results can be found in [23, 31, 40] as well as the cited references therein. Nevertheless, due to the intricate nature of the shape of the domain Ω and the presence of nonlinear terms, it is challenging to provide an explicit solution for the problem (1.1). Hence, it is imperative to consider efficient and accurate numerical methods for (1.1). The methods for discretization in temporal direction of PIDEs include backward-Euler scheme with rectangular quadrature rule [25], Crank-Nicolson scheme with trapezoidal quadrature rule [35], discontinuous Galerkin time-stepping scheme [30], and Laplace transformation [28]. Spatial discretization approaches range from finite element methods [36] to weak Galerkin finite element methods [41], mixed finite element methods [34] and hybridizable discontinuous Galerkin methods [26].

Despite the increasing interest in numerical methods for PIDEs, there has been a notable lack of focus on methods that can effectively handle polygonal or polyhedral meshes. These types of meshes offer more flexibility and accuracy in representing complex geometries compared to traditional triangular or quadrilateral meshes. As a result, the field of computational mathematics has experienced a growing interest in developing the numerical methods that are able to accommodate polygonal (polyhedral) meshes in recent years. Among these approaches, one particularly effective method is virtual element method (VEM) [4]. This method has gained popularity in various scientific and engineering applications, as discussed comprehensively in [8]. The use of L^2-projection operator is a common strategy in virtual element methods for handling nonlinear terms. It has been employed in a wide variety of initial boundary value problems, such as semilinear parabolic equations [1], N-coupled nonlinear Schrödinger-Boussinesq equations [27] and nematic liquid crystal flows [42]. However, this strategy comes with increased computational complexity of Newton's iteration algorithms for the resulting system, due to the need for element-wise integral computations, see [2].

Recently, a new strategy was proposed in [22] to address the aforementioned limitations by employing interpolation operators in serendipity virtual element method (SVEM) spaces to approximate the nonlinear terms [7]. As a modified version of VEM, SVEM strives to minimize the dimension of approximation spaces. SVEM has demonstrated its applicability in a wide range of physics problems, including linear magneto-static models [5], nonlinear elasticity [38], and general second order elliptic equations [6], among others. This study specifically focuses on a specialized type of SVEM that imposes specific requirements on the relationship between the mesh element shape and the SVEM order, thereby enabling the complete elimination of the internal-moment degrees of freedom. This specific method, also known as SVEM in the ideal case [22], simplifies the implementation of Newton's iteration algorithms if the interpolation operators are utilized to approximate the nonlinear terms, in comparison with the strategies utilizing L^2-projection operators, as discussed in [2, 22].