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A NEW ANALYTICAL STUDY FOR MULTI-DIMENSIONAL NAVIER-STOKES EQUATIONS WITH TIME-FRACTIONAL ORDER*

Hegagi Mohamed Ali
Department of Mathematics, College of Science, University of Bisha, Bisha 61922, Saudi Arabia;
Department of Mathematics, Faculty of Science, Aswan University, Aswan 81528, Egypt
Emails: hegagi_math@aswu.edu.eg, he.ahmed@ub.edu.sa

Abstract

In this research article, we present convenient analytical-approximate solutions for fluid flow models known as multi-dimensional Navier-Stokes equations containing time-fractional order by using a relatively new analytical method called modified generalized Mittag-Leffler function method. The Caputo fractional derivative is used to describe fractional mathematical formalism. The approximate solutions for five problems are implemented to demonstrate the validity and accuracy of the proposed method. It is also demonstrated that the solutions obtained from our method when $\alpha = 1$ coincide with the exact solutions, this is displayed by using some 2D and 3D plots for each problem. Moreover, the comparison between our outcomes with given exact solutions and results obtained by other methods in the literature besides absolute error is provided in some tables. Additionally, we offer some plots when α has different values to present the effect of fractional order on the solution of each suggested problem. The numerical simulation presented in this work indicates that the proposed method is efficient, reliable, accurate and easy which has less computational ability to give analytical-approximate solution form. So, this method can be extended to implement on different related problems arising in various areas of innovation and research.

Mathematics subject classification: 35Q30, 35R11, 33E12, 74H10.
Key words: Navier-Stokes equations, Fractional partial differential equations, Mittag-Leffler function, Nonlinear problems, Analytic approximate solutions.

1. Introduction

The Navier-Stokes equations (NSEs) are considered famous partial differential equations (PDEs) that govern the motion of viscous fluid flow and describe different geophysical fluid mechanical problems. NSEs are the primary PDEs that relate momentum, continuity and energy equations and the first appearance of NSEs in the year 1822 [34]. Then, these equations have been used to describe many various interesting applications in theoretical studies and physical phenomena such as liquid flow in pipes, managing climate estimating, ocean currents, airflow around the wings of aircraft, blood flow and examination of contamination [33, 40].

The first formulation of NSEs in fractional order was done by El-Shahed and Salem [17] in 2005, they solved this model by using finite Fourier sine transform, finite Hankel transforms and Laplace transform. After that many researchers have tried to solve NSEs by using different methods for example: in [14], authors used the variational iteration transform technique to evaluate multi-dimensional fractional NSEs. In [23], the residual power series method has been

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applied for the solution of the fractional NSEs with 2 and 3 dimensions. In [22], a hybrid approach of decomposition method and Elzaki transformation has been applied to solve fractional NSEs. In [36], authors have adopted a fractional reduced differential transformation method as a semi-analytical scheme to get an approximate solution of fractional NSEs and many other research papers in the literature (see, e.g. [16, 19–21, 25]).

The classical Navier-Stokes and continuity equations are given by

$$\begin{aligned} &\frac{\partial \Theta}{\partial t} + (\Theta \cdot \triangledown)\Theta = K \triangledown^2 \Theta - \frac{1}{\rho} \triangledown P, \\ &\triangledown \cdot \Theta = 0, \\ &\Theta = 0 \quad \text{on} \;\; \Omega \times (0, T), \end{aligned} \tag{1.1}$$

where $\Theta = (U, V, W)$ and P represent the fluid vector of velocity and pressure, respectively, ρ is the density, $K = \phi/\rho$ is the kinematics viscosity, ϕ is the dynamic viscosity, (x, y, z) are spatial components in Ω and t is the time variable.

Fractional calculus (FC) is a branch of mathematics concerned with integrals and derivatives of arbitrary orders. Thus, it is considered a generalization of the integer-order calculus. Furthermore, the differential equations whether partial or ordinary that have fractional order are called fractional differential equations (FDEs). Recently, FDEs have acquired popularity, importance and attracted many famous researchers and scientists [9, 31, 38, 39]. This is due to its prominent applications in many fields of science and engineering, such as reaction-diffusion, electric networks, electrochemistry of corrosion, polymer physics, fluid flow, chemical physics, propagation of waves and many other important applications [5, 6, 11–13, 26, 27, 29, 37].

The most advantage of using FC in describing applications is that it predicts the future and gives a good description of the behavior of the system from the integer case due to its properties like non-locality, non-singularity, kernel, memory effect and so on. This means that the fractional models are featured because they do not depend on the current situation, but rather depend on all genetic and historical cases (memory effect).

The main goal and motivation are to investigate on solution of time fractional order multi-dimensional NSEs using modified generalized Mittag-Leffler function method (MGMLFM). The generalization of the classical problem Eq. (1.1) with fractional order $0 < \alpha \leq 1$ is defined as

$$\begin{aligned} &{}_0^C D_t^\alpha \Theta + (\Theta \cdot \triangledown)\Theta = K \triangledown^2 \Theta - \frac{1}{\rho} \triangledown P, \\ &\triangledown \cdot \Theta = 0, \\ &\Theta = 0 \quad \text{on} \;\; \Omega \times (0, T), \end{aligned} \tag{1.2}$$

where ${}_0^C D_t^\alpha$ is Caputo fractional derivative (CFD).

The novelty and our contribution to this work are evident in presenting the MGMLFM as an analytical technique to give the appropriate solution for multi-dimensional NSEs within fractional order. The analytical solutions for five different problems of fractional NSEs are calculated by using the proposed method. The obtained results are displayed and verified with some 2D and 3D plots for each problem. Moreover, the comparison of these outcomes with given exact solutions and others obtained by different techniques in the literature besides absolute error is provided in some tables to show the validity and accuracy of the MGMLFM. The solution to these illustrative problems proved that the MGMLFM has less computing costs, straightforward steps and higher convergence rates. Therefore, the MGMLFM is constructive to