Journal of Computational Mathematics
Vol.xx, No.x, 2025, 1–16.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2503-m2024-0225

CONVERGENCE ANALYSIS OF A CLASS OF REGULARIZATION METHODS WITH A NOVEL DISCRETE SCHEME FOR SOLVING INVERSE PROBLEMS*

M.P. Rajan
School of Mathematics, Indian Institute of Science Education and Research Thiruvananthapuram, Thiruvananthapuram 695551, Kerala, India
Email: rajanmp@iisertvm.ac.in

Abstract

Many inverse problems that appear in applications can be modeled as an operator equation. In practice, most of these problems are ill-posed, and computing solutions to such problems in an efficient manner is challenging and has been of greatest interest among researchers in the recent past. While many approaches are developed within infinite-dimensional Hilbert space settings, practical applications often require solutions in finite-dimensional spaces, and we need to discretize the problem. In this manuscript, we study a novel discretization scheme along with a class of regularization techniques for solving linear ill-posed problems and obtain the optimal order error estimates under an a priori parameter choice strategy. We illustrate the computational efficacy of the proposed scheme through numerical examples, and the results demonstrate that the proposed scheme is more economical due to the amount of discrete information needed to solve the problem is significantly lower than the traditional finite-dimensional approach.

Mathematics subject classification: 65J10, 65J20, 65J22, 47L10.
Key words: Ill-posed problem, Inverse problem, Regularization.

1. Introduction

Let $A : X \to Y$ be a bounded linear operator acting between Hilbert spaces X and Y with range $R(A)$ not necessarily be closed. In such situations, it is very well known in the literature that the problem of solving the linear operator equation

$$Ax = y \tag{1.1}$$

is ill-posed (cf. [4]) and many inverse problems appearing in the scientific applications are models of the form (1.1) and they are basically ill-posed in nature. A renowned example and a prototype of such an ill-posed problem is the Fredholm integral equation of the first kind, which is a model for many practical applications and for an excellent reference to the theoretical treatment of these problems, see $[1, 4, 8, 9]$. In order to get stable approximate solutions for the ill-posed equation (1.1), a class of regularization methods can be generated by a family $\{\alpha : \alpha > 0\}$ of piecewise continuous functions on $[0, b]$ for certain $b > 0$, by taking

$$x_\alpha := g_\alpha(A^*A)A^*y, \quad \alpha > 0, \tag{1.2}$$

* Received October 10, 2024 / Revised version received January 1, 2025 / Accepted March 7, 2025 / Published online April 16, 2025 /

as candidates for the approximation of $A^\dagger y$ (cf. [1]), where $A^\dagger$ is the Moore-Penrose generalized inverse of A. If only approximate data $\tilde{y}$ with $\|y - \tilde{y}\| < \delta, \delta > 0$, is available, then we consider

$$\tilde{x}_\alpha := g_\alpha(A^*A)A^*\tilde{y}, \quad \alpha > 0. \tag{1.3}$$

For the convergence and error analysis, we impose the following conditions on g_α.

Assumption 1.1. For some $\nu_0 > 0$ and for $0 \le \nu \le \nu_0$, there exists $c_\nu > 0$ such that

$$\sup_{0 \le \lambda \le b} \lambda^\nu |1 - \lambda g_\alpha(\lambda)| \le c_\nu \alpha^\nu, \quad \forall\, \alpha > 0.$$

Assumption 1.2. There exists $d > 0$ such that

$$\sup_{0 \le \lambda \le b} \lambda^{\frac{1}{2}} |g_\alpha(\lambda)| \le d\, \alpha^{-\frac{1}{2}}, \quad \forall \alpha > 0.$$

These assumptions are general enough to include many regularization methods such as the ones given below (cf. [7]).

(a) Tikhonov regularization

$$(A^*A + \alpha I)x_\alpha = A^*y.$$

Here,

$$g_\alpha(\lambda) = \frac{1}{\lambda + \alpha},$$

Assumptions 1.1 and 1.2 hold true for $\nu_0 = 1$.

(b) Generalized Tikhonov regularization

$$\big((A^*A)^{q+1} + \alpha^{q+1} I\big)x_\alpha = (A^*A)^q A^*y.$$

Here

$$g_\alpha(\lambda) = \frac{\lambda^q}{\lambda^{q+1} + \alpha^{q+1}},$$

Assumptions 1.1 and 1.2 hold true for $\nu_0 = q + 1, q \ge -1/2$.

(c) Iterated Tikhonov regularization. In this method, the k-th iterated approximation x_k is computed from

$$(A^*A + \alpha)x_k = \alpha x_{k-1} + A^*y, \quad k = 1, 2, \dots .$$

Here

$$g_\alpha(\lambda) = \frac{1}{\lambda}\left[1 - \left(\frac{\alpha}{\alpha + \lambda}\right)^k\right],$$

Assumptions 1.1 and 1.2 hold true for any $\nu_0 \ge k$ and $\lambda \ne 0$.

(d) Method of successive approximations (explicit scheme). For $0 < \mu < 2/\|A\|^2$,

$$x_k = (1 - \mu A^*A)x_{k-1} + \mu A^*y, \quad k = 1, 2, \dots, \quad x_0 = 0.$$

Here,

$$g_\alpha(\lambda) = \frac{1}{\lambda}[1 - (1 - \mu\lambda)^k], \quad \alpha = \frac{1}{k},$$

and the Assumptions 1.1, 1.2 hold true for any $\nu_0 > 0, \lambda \ne 0$.

(e) Method of successive approximations (implicit scheme). For $0 < \mu$,

$$(A^*A+\mu)x_k = \mu x_{k-1} + A^*y, \quad k = 1,2,\ldots, \quad x_0 = 0.$$

Here,

$$g_\alpha(\lambda) = \frac{1}{\lambda}\left[1 - \left(\frac{\mu}{\mu+\lambda}\right)^k\right],$$

where $\alpha = 1/k, k = 1,2,\ldots$. Assumptions 1.1 and 1.2 hold true for any $\nu_0 > 0$ and $\lambda \neq 0$.

While applying a regularization method in practical situations, what we have at our disposal are approximations $A_m, \tilde{y}$ of A, y, respectively. In such situations, one considers

$$\tilde{x}_{\alpha,m} := g_\alpha(A_m^*A_m)A_m^*\tilde{y}. \tag{1.4}$$

For the numerical implementation of (1.3) as well, the above approximation is followed with A_m replacing A. The traditional approach is to use AP_m or P_mAP_m as approximations of A, where $P_m : X \to X$ is the orthogonal projection onto a finite-dimensional space X_m. Suppose that X is separable and $\{e_1, e_2, \ldots\}$ is an orthonormal basis of X, then P_m is defined by

$$P_m x = \sum_{j=1}^{m} \langle x, e_j\rangle e_j, \quad x \in X.$$

Since each P_m is of finite rank, this approximate equation reduces to a matrix equation, that can be solved numerically.

In the recent past, many researchers have been interested in developing approximation schemes that are computationally more efficient and that need significantly less discrete information for computing the solution without compromising the optimal property of the method (cf. [5,6,10,12]). This approach is basically termed an information-based approximation analysis in computing solutions. In this manuscript, we consider one such scheme studied in [10,11] and analyzed with respect to Tikhonov regularization for solving the linear ill-posed problems. Our objective is to investigate the applicability of this scheme to a wider class of regularization methods (1.4) and illustrate the advantages of such types of discrete schemes. In [10,11], the approximate solution is achieved by solving

$$(A_m^*A_m + \alpha I)\tilde{x}_{\alpha,m} = A_m^*\tilde{y}$$

with

$$A_m = P_0AP_m + \sum_{k=1}^{m}(P_k - P_{k-1})AP_{m-k}. \tag{1.5}$$

Here, (P_j) is a sequence of orthogonal projections on the finite dimensional subspaces $V_0 \subset V_1 \subset \cdots V_j \subset \cdots X$ with $\dim V_j \sim 2^{sj}, s \geq 1$. Also, the projections satisfy the approximation property,

$$\|I - P_j\|_{r,0} \leq \kappa_r 2^{-rj}, \quad \forall\, j \in N, \quad r > 0, \tag{1.6}$$

where $\kappa_r \geq 1$ is independent of j and each pair (i,j) relates to (e_i, Ae_j) and $(e_i, \tilde{y})$ in a finite plane

$$\Omega_m = \{0\} \times [0,m] \cup_{k=1}^{m} (k-1,k] \times [0, m-k].$$

The norm of an operator from the space $X_r \to X$ is denoted by $\|.\|_{r,0}$ as described in (1.7).

The salient feature of the above scheme is that one needs only a significantly smaller portion of discrete information on a plane with $m \times m$ points for the computation of the solution. In traditional approaches, one requires all the discrete information of a $m \times m$ points on a plane to compute the solution. The theoretical and numerical analysis of [10, 11] with respect to Tikhonov regularization unveils the supremacy of (1.5) over other traditional projection schemes AP_m and P_mAP_m. Although in [10], the analysis has been done with respect to different smoothness properties, in this manuscript, we only analyze the problem with respect to a single smoothness property. However, for a different set of smoothness properties, we supply the results without proof. To make the paper self-contained, we avoid as much as possible the dependency on [10]. Analogous to [10], we consider the following: For $r > 0$, let X_r be a dense subspace of the Hilbert space X with the norm

$$\|f\|_r := \|f\| + \|L_r f\|, \quad f \in X_r,$$

where $L_r : X_r \to X$ is a closed linear operator. If $A : X \to X, B : X_r \to X, C : X \to X_r$ are bounded operators, then we shall denote their respective norms by

$$\|A\|, \quad \|B\|_{r,0}, \quad \|C\|_{0,r}. \tag{1.7}$$

Let $A : X \to X$ be a linear compact operator that has specific smoothness properties namely,

$$(S_1) \qquad \|A\|_{0,r} \le \gamma_1, \quad \|A^*\|_{0,r} \le \gamma_2. \tag{1.8}$$

It is established in [10] that if

$$\hat{x} \in R\big((A^*A)^\nu\big), \quad 0 < \nu \le 1,$$

and the regularization parameter α is chosen in a priori manner that depends on m and δ, say $\alpha = \alpha(\delta, m)$, then, one achieves the optimal result

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = \mathcal{O}\big(\delta^{\frac{2\nu}{2\nu+1}}\big). \tag{1.9}$$

The notation,

$$\beta_1(\delta, m) = \mathcal{O}\big(\beta_2(\delta, m)\big)$$

for functions $\beta_1(\delta, m)$ and $\beta_2(\delta, m)$, we mean that there exist a constant $c > 0$, independent of (δ, m) such that

$$\beta_1(\delta, m) \le c\ \beta_2(\delta, m).$$

Similarly, the notation

$$\beta_1(\delta, m) \sim \beta_2(\delta, m)$$

we mean there exists positive $c_1, c_2 > 0$, independent of (δ, m) such that

$$c_1\beta_1(\delta, m) \le \beta_2(\delta, m) \le c_2\beta_1(\delta, m).$$

Our objective is to carry out the convergence analysis of the proposed scheme and derive the optimal order result (1.9) with respect to (1.4) with A_m in (1.5). To achieve this, the paper is structured as follows: Section 2 presents the convergence analysis of the scheme and the convergence rate based on an a priori parameter choice rule. We also make a comparative study by presenting the computational complexity needed to solve the problem for traditional and proposed schemes. Finally, Section 3 provides numerical examples to demonstrate the theoretical findings.

2. Error Estimate

Let $A : X \to X$ be a compact operator having the smoothness properties specified by (1.8) and (P_m) be a sequence of orthogonal projections having the approximation property (1.6). For each $m \in \mathbf{N}$, let A_m be defined by (1.5). We assume that $y \in R(A)$. Let $\{g_\alpha : \alpha > 0\}$ be a family of piecewise continuous functions defined on $(0, b]$, where

$$b \geq \max\left\{\|A\|^2, \|A_m\|^2\right\}, \quad \forall\, m \in \mathbf{N}$$

satisfies the Assumptions 1.1, 1.2 for some $\nu_0 > 0$. Let $\hat{x} := A^\dagger y$ and $x_\alpha, \tilde{x}_{\alpha,m}$ are defined accordingly (1.2) and (1.4), and

$$x_{\alpha,m} := g_\alpha(A_m^* A_m) A_m^* y, \tag{2.1}$$

where

$$y \in D(A^\dagger), \quad \|y - \tilde{y}\| \leq \delta > 0.$$

Further, we assume that $\hat{x} \in R((A^*A)^\nu)$ for some $\nu \in (0, \nu_0]$. Let $\hat{u} \in X$ be such that $\hat{x} = (A^*A)^\nu \hat{u}$. By the definition of x_α and spectral theory, we have

$$\begin{aligned}\hat{x} - x_\alpha &= \hat{x} - g_\alpha(A^*A)A^*y \\ &= [I - g_\alpha(A^*A)A^*A]\hat{x} \\ &= (A^*A)^\nu [I - g_\alpha(A^*A)A^*A]\hat{u}.\end{aligned}$$

Therefore, using the Assumptions 1.1 and 1.2 on g_α, we get

$$\begin{aligned}\|\hat{x} - x_\alpha\| &= \|(A^*A)^\nu [I - g_\alpha(A^*A)A^*A]\hat{u}\| \\ &\leq \sup_{0 \leq \lambda \leq b} \lambda^\nu |1 - \lambda g_\alpha(\lambda)|\, \|\hat{u}\| \leq c_\nu \|\hat{u}\| \alpha^\nu,\end{aligned}$$

$$\begin{aligned}\|x_{\alpha,m} - \tilde{x}_{\alpha,m}\| &= \|A_m^* g_\alpha(A_m A_m^*)(y - \tilde{y})\| \\ &= \left\|(A_m A_m^*)^{\frac{1}{2}} g_\alpha(A_m A_m^*)(y - \tilde{y})\right\| \\ &\leq \sup_{0 \leq \lambda \leq b} \lambda^{\frac{1}{2}} |g_\alpha(\lambda)|\, \|y - \tilde{y}\| \leq d \frac{\delta}{\sqrt{\alpha}}.\end{aligned}$$

Now we derive an estimate for $\|\hat{x} - \tilde{x}_{\alpha,m}\|$. First we observe that

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \leq \|\hat{x} - x_{\alpha,m}\| + \|x_{\alpha,m} - \tilde{x}_{\alpha,m}\|.$$

Since

$$\|x_{\alpha,m} - \tilde{x}_{\alpha,m}\| \leq d \frac{\delta}{\sqrt{\alpha}},$$

we have

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \leq \|\hat{x} - x_{\alpha,m}\| + d \frac{\delta}{\sqrt{\alpha}}. \tag{2.2}$$

The Proposition 2.1 given below computes an estimate for $\|\hat{x} - x_{\alpha,m}\|$ and for proving it, we use a particular case of Lemma 2.1.

Lemma 2.1 (cf. [13]). *Suppose $A : X \to X, B : X \to X$ are bounded linear operators. If $\ell > 0$, then*

$$\left\||A|^\ell - |B|^\ell\right\| \leq a_\ell \|A - B\|^{\min\{\ell,\, 1\}},$$

where

$$a_\ell = \begin{cases} \big|\ln\|A-B\|\big|, & \text{if} \quad \ell = 1, \\ 1, & \text{if} \quad \ell \neq 1. \end{cases}$$

We recall from spectral theory that for $A \in BL(X)$, $|A|$ is defined by $|A| := (A^*A)^{1/2}$. Note that suppose $A : X \to X$ and $B : X \to X$ are positive, self adjoint, bounded operators, then the Lemma 2.1 implies:

$$\|A^\ell - B^\ell\| \le \|A - B\|^{\min\{1,\ell\}}, \quad \ell > 0. \tag{2.3}$$

Proposition 2.1.

$$\|\hat{x} - x_{\alpha,m}\| \le c\big(\alpha^\nu + \|A^*A - A_m^*A_m\|^{\min\{1,\nu\}} + \alpha^{-\frac{1}{2}}\big\|(A_m - P_mA)(A^*A)^\nu\big\|\big).$$

Proof. We observe that

$$\begin{aligned} \hat{x} - x_{\alpha,m} &= \hat{x} - g_\alpha(A_m^*A_m)A_m^*A\hat{x} \\ &= [I - g_\alpha(A_m^*A_m)A_m^*A_m]\hat{x} \\ &\quad + g_\alpha(A_m^*A_m)A_m^*(A_m - A)\hat{x}, \end{aligned}$$

so that

$$\|\hat{x} - x_{\alpha,m}\| \le \|[I - g_\alpha(A_m^*A_m)A_m^*A_m]\hat{x}\| + \|g_\alpha(A_m^*A_m)A_m^*(A_m - A)\hat{x}\|. \tag{2.4}$$

Since, $\hat{x} = (A^*A)^\nu\hat{u}$,

$$\begin{aligned} &\big\|[I - g_\alpha(A_m^*A_m)A_m^*A_m]\hat{x}\big\| \\ &= \big\|[I - A_m^*A_mg_\alpha(A_m^*A_m)](A^*A)^\nu\hat{u}\big\| \\ &\le \big\|[I - A_m^*A_mg_\alpha(A_m^*A_m)]\big[(A^*A)^\nu - (A_m^*A_m)^\nu\big]\hat{u}\big\| \\ &\quad + \big\|[I - A_m^*A_mg_\alpha(A_m^*A_m)](A_m^*A_m)^\nu\hat{u}\big\|. \end{aligned}$$

We set $r_\alpha(\lambda) := I - \lambda g_\alpha(\lambda)$. Now, using the Assumption 1.1 on g_α,

$$\begin{aligned} &\big\|[I - A_m^*A_mg_\alpha(A_m^*A_m)](A_m^*A_m)^\nu\hat{u}\big\| \\ &\le \sup_{0\le\lambda\le b} \lambda^\nu|1 - \lambda g_\alpha(\lambda)|\,\|\hat{u}\| \le c_\nu\|\hat{u}\|\alpha^\nu, \end{aligned}$$

and by Assumption 1.1 on g_α and the result (2.3)

$$\begin{aligned} &\big\|[I - A_m^*A_mg_\alpha(A_m^*A_m)]\big[(A^*A)^\nu - (A_m^*A_m)^\nu\big]\hat{u}\big\| \\ &= \big\|r_\alpha(A_m^*A_m)\big[(A^*A)^\nu - (A_m^*A_m)^\nu\big]\hat{u}\big\| \\ &\le \|r_\alpha(A_m^*A_m)\|\,\big\|\big[(A^*A)^\nu - (A_m^*A_m)^\nu\big]\big\|\,\|\hat{u}\| \\ &\le c_0\|\hat{u}\|\,\big\|\big[(A^*A)^\nu - (A_m^*A_m)^\nu\big]\big\| \\ &\le c_0\|\hat{u}\|\,\|A^*A - A_m^*A_m\|^{\min\{1,\nu\}}. \end{aligned}$$

Since, $A_m^*P_m = A_m^*$, $\hat{x} = (A^*A)^\nu\hat{u}$ and using the Assumption 1.2 on g_α we have

$$\begin{aligned} &\|g_\alpha(A_m^*A_m)A_m^*(A_m - A)\hat{x}\| \\ &= \|g_\alpha(A_m^*A_m)A_m^*(A_m - P_mA)\hat{x}\| \\ &= \big\|(A_mA_m^*)^{\frac{1}{2}}g_\alpha(A_mA_m^*)(A_m - P_mA)\hat{x}\big\| \\ &\le \big\|(A_mA_m^*)^{\frac{1}{2}}g_\alpha(A_mA_m^*)\big\|\,\big\|(A_m - P_mA)(A^*A)^\nu\hat{u}\big\| \\ &\le d\|\hat{u}\|\,\alpha^{-\frac{1}{2}}\big\|(A_m - P_mA)(A^*A)^\nu\big\|. \end{aligned}$$

Using the above estimates for $\|[I - g_\alpha(A_m^*A_m)A_m^*A_m]\hat{x}\|$ and $\|g_\alpha(A_m^*A_m)A_m^*(A - A_m)\hat{x}\|$ in the relation (2.4) we get the required result. □

In view of the relation (2.2) and the above proposition, we have to find estimates for the quantities $\|A^*A - A_m^*A_m\|$, $\|(A_m - P_mA)(A^*A)^\nu\|$. To derive the estimates for the above quantities, we require some preliminary results, as given below, which can be easily established using the approximation property and smoothness assumption.

Lemma 2.2. *If the operator A is having the smoothness property S_1 in* (1.8) *and the projection P_m is having the approximation property* (1.6)*, then for $m \in N$,*

$$\|(I - P_m)A\| = \mathcal{O}(2^{-mr}), \tag{2.5}$$

$$\|A(I - P_m)\| = \mathcal{O}(2^{-mr}). \tag{2.6}$$

Following result can be obtained from [10]. However, we shall supply the proof as well for completeness.

Lemma 2.3 (cf. [10]). *If the operator A is having the smoothness property S_1* (1.8)*, then*

$$\|A^*A - A_m^*A_m\| = \mathcal{O}(m2^{-mr}).$$

Proof. We note that

$$\|A^*A - A_m^*A_m\| \le \|A^*(I - P_m)A\| + \|A^*P_mA - A_m^*A_m\|. \tag{2.7}$$

By Lemma 2.2, we can see that

$$\|A^*(I - P_m)A\| \le \|(I - P_m)A\|^2 = \mathcal{O}(2^{-mr}). \tag{2.8}$$

From the expression of A_m, we can have

$$A_m^* = P_mA^*P_0 + \sum_{k=1}^{m} P_{m-k}A^*(P_k - P_{k-1}),$$

$$A_m^*A_m = P_mA^*P_0AP_m + \sum_{k=1}^{m} P_{m-k}A^*(P_k - P_{k-1})AP_{m-k}.$$

Hence,

$$\begin{aligned} &\|A^*P_mA - A_m^*A_m\| \\ &\le \|A^*P_0A - P_mA^*P_0AP_m\| \\ &\quad + \sum_{k=1}^{m} \|A^*(P_k - P_{k-1})A - P_{m-k}A^*(P_k - P_{k-1})AP_{m-k}\|. \end{aligned} \tag{2.9}$$

Each term in the above expression can be estimated as

$$\|A^*P_0A - P_mA^*P_0AP_m\| \le 2\|(I - P_m)A^*P_0A\| = \mathcal{O}(2^{-mr}),$$
$$\|A^*(P_k - P_{k-1})A - P_{m-k}A^*(P_k - P_{k-1})AP_{m-k}\| \le 2\|(P_{m-k} - I)A^*(P_k - P_{k-1})A\|.$$

Thus, by Lemma 2.2, we have

$$\|(P_{m-k} - I)A^*(P_k - P_{k-1})A\|$$

$$
\begin{aligned}
&= \|A^*(P_k - P_{k-1})A(P_{m-k} - I)\| \\
&\le \|A^*(P_k - I)A(P_{m-k} - I)\| \\
&\qquad + \|A^*(I - P_{k-1})A(P_{m-k} - I)\| \\
&\le \|A^*(P_k - I)\|\|A(P_{m-k} - I)\| \\
&\qquad + \|A^*(I - P_{k-1})\|\|A(P_{m-k} - I)\| \\
&= \|(P_k - I)A\|\|A(P_{m-k} - I)\| \\
&\qquad + \|(I - P_{k-1})A\|\|A(P_{m-k} - I)\| = \mathcal{O}(2^{-mr}).
\end{aligned}
$$

Hence, from (2.9), we get

$$\|A^*P_mA - A_m^*A_m\| = \mathcal{O}(m2^{-mr}). \tag{2.10}$$

Therefore, substituting (2.8) and (2.10) in (2.7) will yield $\|A^*A - A_m^*A_m\| = \mathcal{O}(m2^{-mr})$. □

Thus, by Lemma 2.3, we have

$$\|A^*A - A_m^*A_m\|^{\min\{1,\nu\}} = \mathcal{O}\big((m2^{-mr})^{\nu_1}\big), \quad \nu_1 = \min\{\nu, 1\}. \tag{2.11}$$

To get an estimate for $\|(A_m - P_mA)(A^*A)^\nu\|$, we shall use the below results on fractional powers.

Lemma 2.4 (cf. [13]). *Let $A : X \to X$ be any bounded operator and $Q_m : X \to X$ be an orthogonal projection. Then*

$$\big\|(I - Q_m)|A|^p\big\| = \mathcal{O}\big(\|A(I - Q_m)\|^{\min\{p,1\}}\big). \tag{2.12}$$

Lemma 2.5. *If A is having the smoothness property S_1 (1.8), then*

$$\big\|(A_m - P_mA)(A^*A)^\nu\big\| = \begin{cases} \mathcal{O}(2^{-2\nu mr}), & \text{if} \quad \nu < \dfrac{1}{2}, \\ \mathcal{O}(m2^{-mr}), & \text{if} \quad \nu \ge \dfrac{1}{2}. \end{cases}$$

Proof. We have

$$P_mA - A_m = P_0A(I - P_m) + \sum_{k=1}^{m}(P_k - P_{k-1})A(I - P_{m-k}).$$

Therefore, by Lemmas 2.2 and 2.4, we obtain

$$
\begin{aligned}
&\big\|(P_mA - A_m)(A^*A)^\nu\big\| \\
&\le \big\|A(I - P_m)(A^*A)^\nu\big\| + \sum_{k=1}^{m}\big\|(I - P_{k-1})A(I - P_{m-k})(A^*A)^\nu\big\| \\
&\le \|A(I - P_m)\|\,\big\|(I - P_m)(A^*A)^\nu\big\| \\
&\qquad + \sum_{k=1}^{m}\|(I - P_{k-1})A\|\,\big\|(I - P_{m-k})(A^*A)^\nu\big\| \\
&\le 2^{-mr}(2^{-mr})^{\min\{2\nu,1\}} + \sum_{k=1}^{m}2^{-(k-1)r}(2^{-(m-k)r})^{\min\{2\nu,1\}}
\end{aligned}
$$

$$= 2^{-(1+\min\{2\nu,1\})mr} + 2^{-mr(\min\{2\nu,1\})}2^r\sum_{k=1}^{m}2^{-kr(1-\min\{2\nu,1\})}$$

$$= \begin{cases} 2^{-mr(1+2\nu)} + 2^{-2\nu mr}\,2^r\,\sum_{k=1}^{m}2^{-kr(1-2\nu)}, & \text{if} \quad \nu < \dfrac{1}{2}, \\ 2^{-2mr} + m2^{-mr}\,2^r, & \text{if} \quad \nu \geq \dfrac{1}{2} \end{cases}$$

$$= \begin{cases} 2^{-mr(1+2\nu)} + c2^{-2\nu mr}\,2^r, & \text{if} \quad \nu < \dfrac{1}{2}, \\ 2^{-2mr} + m2^{-mr}\,2^r, & \text{if} \quad \nu \geq \dfrac{1}{2} \end{cases}$$

$$= \begin{cases} \mathcal{O}(2^{-2\nu mr}), & \text{if} \quad \nu < \dfrac{1}{2}, \\ \mathcal{O}(m2^{-mr}), & \text{if} \quad \nu \geq \dfrac{1}{2}. \end{cases}$$

The proof is complete. □

Now, the estimates in (2.11) and Lemma 2.5 together with Proposition 2.1 and the relation (2.2) gives the following result.

Theorem 2.1. *Suppose that* $\hat{x} \in R(A^*A)^\nu, 0 < \nu \leq \nu_0$ *and* $y \in R(A)$. *If* A *is having the smoothness property* S_1 *in* (1.8), *then*

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \leq \begin{cases} c\left(\alpha^\nu + (m2^{-mr})^\nu + \dfrac{2^{-2mr\nu}}{\sqrt{\alpha}} + \dfrac{\delta}{\sqrt{\alpha}}\right), & \textit{if} \quad \nu < \dfrac{1}{2}, \\ c\left(\alpha^\nu + (m2^{-mr})^\nu + \dfrac{m2^{-mr}}{\sqrt{\alpha}} + \dfrac{\delta}{\sqrt{\alpha}}\right), & \textit{if} \quad \dfrac{1}{2} \leq \nu < 1, \\ c\left(\alpha^\nu + m2^{-mr} + \dfrac{m2^{-mr}}{\sqrt{\alpha}} + \dfrac{\delta}{\sqrt{\alpha}}\right), & \textit{if} \quad 1 \leq \nu \leq \nu_0. \end{cases}$$

From the above theorem, we derive the following result with an a priori parameter choice rule of the regularization parameter.

Theorem 2.2. *Let the assumptions in Theorem* 2.1 *be satisfied. For* $0 < \nu < 1/2$, *if* $\alpha := \alpha(\delta)$ *and* $m := m(\delta)$ *are such that*

$$\alpha \sim \delta^{\frac{2}{2\nu+1}}, \quad m2^{-mr} \sim \delta^{\frac{2}{2\nu+1}},$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = \mathcal{O}\big(\delta^{\frac{2\nu}{2\nu+1}}\big).$$

For $1/2 \leq \nu \leq \nu_0$, *if* $\alpha := \alpha(\delta)$ *and* $m := m(\delta)$ *are such that*

$$\alpha \sim \delta^{\frac{2}{2\nu+1}}, \quad m2^{-mr} \sim \delta,$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = \mathcal{O}(\delta^{\frac{2\nu}{2\nu+1}}).$$

Proof. Proof follows from the estimate given in Theorem 2.1. □

Remark 2.1. Suppose that A satisfies a set of the smoothness assumptions S_2 instead of S_1 in (1.8), where

$$(S_2) \qquad \|A\|_{0,r} \le \gamma_1, \quad \|A^*\|_{0,r} \le \gamma_2, \quad \|(L_r A)^*\|_{0,r} \le \gamma_3, \tag{2.13}$$

$\gamma_1, \gamma_2, \gamma_3$ are positive real numbers, then one would be able to derive the following result.

Theorem 2.3. *Suppose that $y \in R(A)$ and A is having the smoothness property S_2 in* (2.13)*. If $\hat{x} \in R((A^*A)^\nu)$, $0 < \nu \le \nu_0$, then*

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \le c \left(\alpha^\nu + \frac{\delta}{\sqrt{\alpha}} + (2^{-mr})^{\min\{\nu,1\}} + \frac{2^{-mr}}{\sqrt{\alpha}} \right).$$

In particular, if $0 < \nu \le 1/2$, and $\alpha := \alpha(\delta)$ and $m := m(\delta)$ are such that

$$\alpha \sim \delta^{\frac{2}{2\nu+1}}, \quad 2^{-mr} \sim \delta^{\frac{2}{2\nu+1}},$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = \mathcal{O}(\delta^{\frac{2\nu}{2\nu+1}}).$$

If $1/2 \le \nu \le 1$, and $\alpha := \alpha(\delta)$ and $m := m(\delta)$ are such that

$$\alpha \sim \delta^{\frac{2}{2\nu+1}}, \quad 2^{-mr} \sim \delta,$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = \mathcal{O}(\delta^{\frac{2\nu}{2\nu+1}}).$$

We leave its proof as an exercise and the reader may refer [10] for more details.

2.1. Computational complexity analysis

In this subsection, we analyse the amount of discrete information in terms of the number of inner products needed to compute the solution from (1.4) with different A_m's. Let $Card\, I.P$ denotes the number of inner products. The $Card\, I.P$ for traditional schemes $A_m = P_m A P_m$ and $A_m = AP_m$, can be estimated as

$$Card\, I.P \sim \dim(V_m).\dim(V_m) \sim 2^{2ms}.$$

If $\hat{x} \in R((T^*T)^\nu)$, then from [7, Theorem 3.1 and Lemma 4.3], [2, relation (2.10)] and Lemma 2.2, we can see that for getting the order $\mathcal{O}(\delta^{2\nu/(2\nu+1)})$ in both the cases, the condition required on m with A satisfying the property (1.8) is

$$2^m \sim \delta^{-\frac{2\nu}{r(2\nu+1)p}}, \quad p = \min\{2\nu, 1\}.$$

If $0 < \nu \le 1/2$, then $p = 2\nu$ so that $Card\, I.P \sim \delta^{-2s/((2\nu+1)r)}$. When $1/2 < \nu \le \nu_0, p = 1$ so that $Card\, I.P \sim \delta^{-4s\nu/((2\nu+1)r)}$. Therefore, for solving (1.4) with $P_m A P_m$ and AP_m

$$Card\, I.P \sim \begin{cases} \delta^{-\frac{2s}{(2\nu+1)r}}, & \text{if} \quad 0 < \nu \le \dfrac{1}{2}, \\ \delta^{-\frac{4s\nu}{(2\nu+1)r}}, & \text{if} \quad \dfrac{1}{2} < \nu \le \nu_0. \end{cases}$$

In the new discretization scheme case,

$$A_m = P_0 A P_m + \sum_{k=1}^{m} (P_k - P_{k-1}) A P_{m-k},$$

the inner products involved are $\langle e_i, \tilde{y} \rangle$ and $\langle e_i, Ae_j \rangle$ for

$$(i,j) \in \Omega_m = \{0\} \times [0,m] \cup_{k=1}^{m} (k-1,k] \times [0, m-k],$$

and hence,

$$\begin{aligned} Card\, I.P &\sim \sum_{k=0}^{m} \dim(V_k).\dim(V_{m-k}) + \dim(V_m) \\ &\sim \sum_{k=0}^{m} 2^{ks}.2^{(m-k)s} + 2^{ms} \sim m2^{ms}. \end{aligned}$$

If $\hat{x} \in R((A^*A)^\nu)$, then by Theorem 2.2, the order $\mathcal{O}(\delta^{2\nu/(2\nu+1)})$ can be achieved by imposing the condition on m as $m2^{-mr} \sim \delta^{2/(2\nu+1)}$, for $0 < \nu \leq 1/2$ and $m2^{-mr} \sim \delta$, for $1/2 \leq \nu \leq \nu_0$. Therefore, for solving (1.4) with A_m in (1.5), the requirements are

$$Card\, I.P \sim \begin{cases} \delta^{-\frac{2s}{(2\nu+1)r}} \left(\log(\delta^{-1})\right)^{1+\frac{s}{r}}, & \text{if} \quad 0 < \nu \leq \dfrac{1}{2}, \\ \delta^{-\frac{s}{r}} \left(\log(\delta^{-1})\right)^{1+\frac{s}{r}}, & \text{if} \quad \dfrac{1}{2} < \nu \leq \nu_0. \end{cases}$$

In case A is having the property S_2, one can estimate the $Card\, I.P$ in a similar manner by means of Theorem 2.3.

3. Numerical Experiments

In this section, we present the numerical implementation details of the proposed scheme to illustrate the significant advantages of the scheme compared to traditional approaches. We consider both Tikhonov regularization (TR) and iterated Tikhonov regularization (ITR) for the illustration. We consider $X = L^2[0,1]$ and $\{e_1, e_2, \ldots, \}$, as the Haar orthonormal basis, where $e_1(s) = 1$ for all $s \in [0,1]$, and for $m = 2^{k-1} + j, k = 1, 2, \ldots, j = 1, 2, \ldots, 2^{k-1}$,

$$e_m(s) = \begin{cases} 2^{\frac{k-1}{2}}, & \text{if} \quad s \in \left[\dfrac{j-1}{2^{k-1}}, \dfrac{j-1/2}{2^{k-1}}\right), \\ -2^{\frac{k-1}{2}}, & \text{if} \quad s \in \left[\dfrac{j-1/2}{2^{k-1}}, \dfrac{j}{2^{k-1}}\right), \\ 0, & \text{if} \quad s \notin \left[\dfrac{j-1}{2^{k-1}}, \dfrac{j}{2^{k-1}}\right]. \end{cases}$$

Let $A : X \to X$ be the integral operator,

$$(Ax)(s) = \int_0^1 k(s,t) x(t)\, dt, \quad s \in [0,1]$$

with

$$k(s,t) = \begin{cases} t(1-s), & \text{if} \quad t \leq s, \\ s(1-t), & \text{if} \quad t > s. \end{cases}$$

Let X^r with $r = 1$ be the Sobolev space having functions f with derivative $f' \in L^2[0,1]$. We take two examples for illustration purposes, as given below.

Example 3.1. Let

$$y(s) = \frac{1}{24}(s - 2s^3 + s^4).$$

Then, we have

$$\hat{x}(t) = \frac{1}{2}(t - t^3), \quad t \in [0, 1].$$

From [3], it is clear that $\hat{x} \in R((T^*T)^\nu)$ for all $\nu < 5/8$.

Example 3.2. Let

$$y(s) = \frac{1}{30}(3s - 5s^3 + 3s^5 - s^6).$$

Then,

$$\hat{x}(t) = t - 2t^3 + t^4, \quad t \in [0, 1]$$

and from [3], $\hat{x} \in R((T^*T)^\nu)$ for all $\nu < 9/8$.

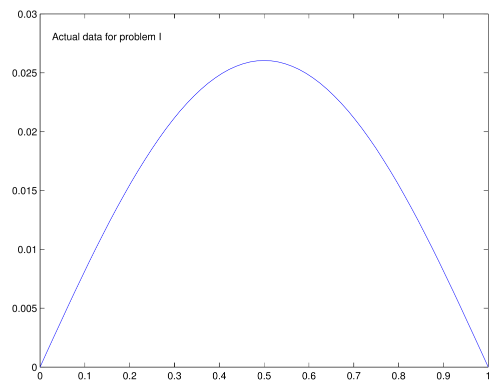
In Example 3.1, we take $\nu = 1/2$ and thus by Theorem 2.2, choosing $\alpha \sim \delta$ and $m2^{-m} \sim \delta$, we get the rate

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = \mathcal{O}(\delta^{\frac{1}{2}}).$$

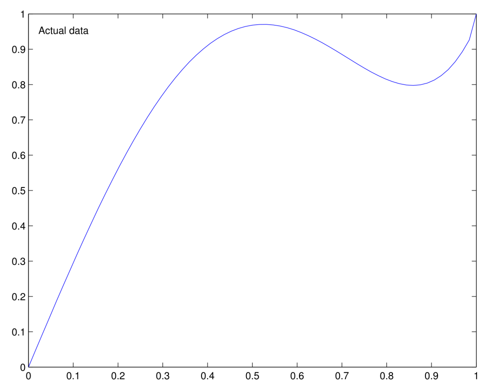
In Example 3.2, we take $\nu = 1$ and thus by Theorem 2.2, choosing $\alpha \sim \delta^{2/3}$ and $m2^{-m} \sim \delta$, we get

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = \mathcal{O}(\delta^{\frac{2}{3}}).$$

We consider randomly perturbed data $\tilde{y}$ with $\|\tilde{y} - y\| \leq \delta$ with data errors 5% and 10%, respectively. Let $\tilde{e}_{\alpha,m}$ be the numerically computed term $\|\hat{x} - \tilde{x}_{\alpha,m}\|$. The exact data for Examples 3.1 and 3.2 are presented in Fig. 3.1. The perturbed data with 5% and 10% errors for Examples 3.1 and 3.2 are respectively presented in Figs. 3.2 and 3.3. The numerical results with different schemes for Examples 3.1 and 3.2 are summarised in Tables 3.1 and 3.2 respectively. $Card\, I.P$ for solving a 16×16 system using the proposed scheme is 48 whereas that of the other two traditional methods is 256. Similarly, for solving a 64×64 system, $Card\, I.P$ for the proposed scheme is 256, and in other cases, it is 4096. The solution obtained through ITR for the new scheme, P_mAP_m and AP_m when data error is 5% and $k = 1$, and $k = 2$ is shown in Fig. 3.4. We note that these solutions are obtained by solving a system of size 64×64. The

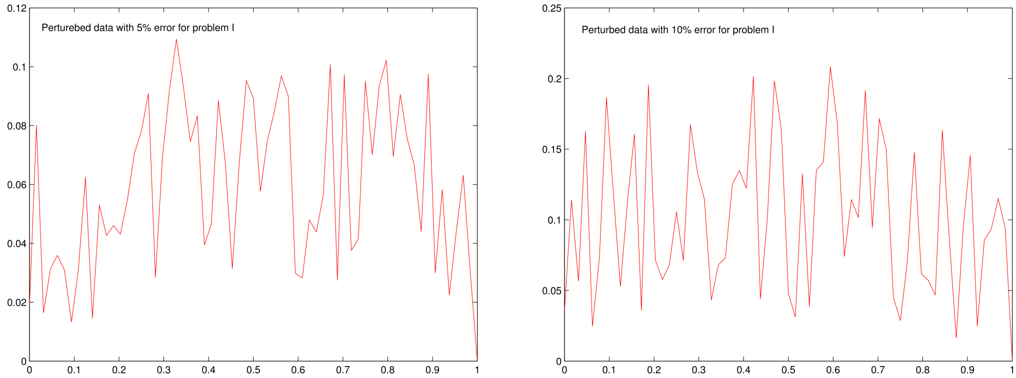


(a) Actual data (Example 3.1)



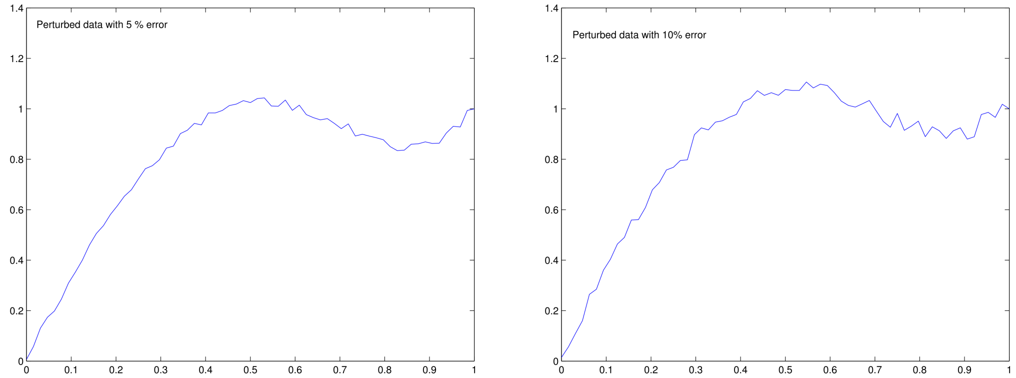
(b) Actual data (Example 3.2)

Fig. 3.1. Actual data for examples.



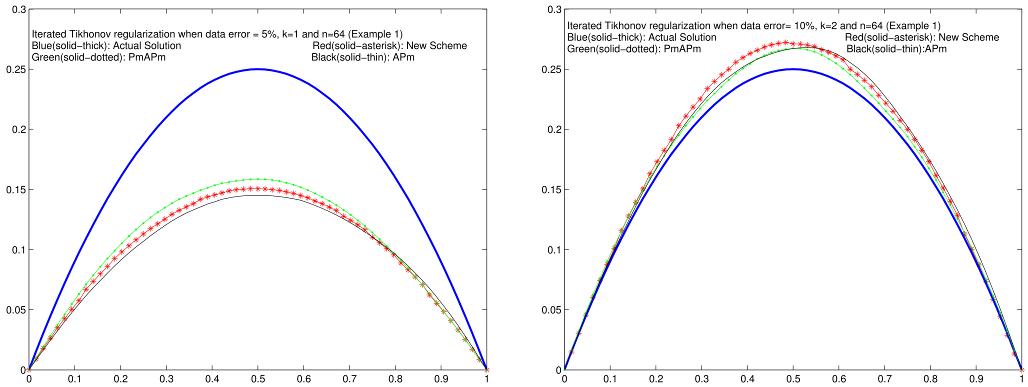
(a) Perturbed data with 5% errors (b) Perturbed data with 10% errors

Fig. 3.2. Perturbed data for Example 3.1.



(a) Perturbed data with 5% errors (b) Perturbed data with 10% errors

Fig. 3.3. Perturbed data for Example 3.2.



(a) Iterated Tikhonov solution, k=1 (b) Iterated Tikhonov solution, k=2

Fig. 3.4. Computed Solution for Example 3.1 when $\delta = 5\%$.

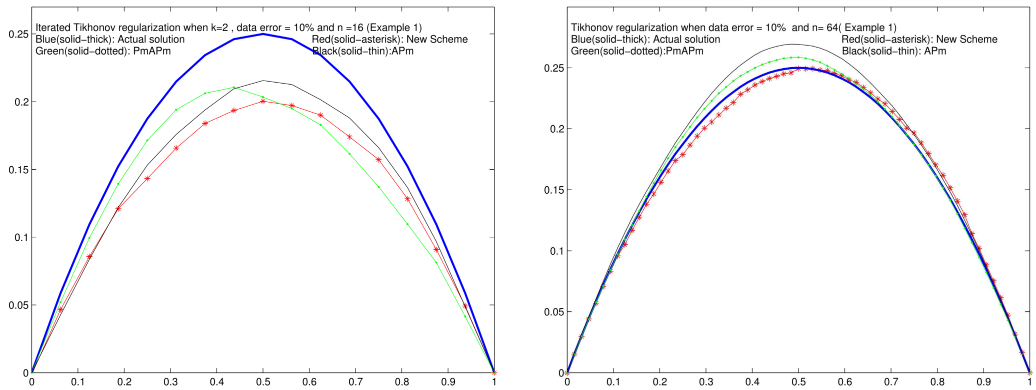
results indicate that we obtain a better result when $k = 2$ than $k = 1$ (cf. Fig. 3.4). When there is a data error of 10% in Example 3.1, the solution for ITR using different schemes with

Table 3.1: Computational results for Example 3.1, when $\nu = 0.5$.

δ	Dim	α	$\tilde{e}_{\alpha,m}$, k=1			$\tilde{e}_{\alpha,m}$, k=2		
			New	P_mAP_m	AP_m	New	P_mAP_m	AP_m
5%	4	0.235	0.160085	0.068701	0.148304	0.133506	0.037834	0.125462
	16	0.1175	0.135213	0.043836	0.140202	0.085102	0.008146	0.089224
	64	0.044062	0.073839	0.017777	0.068624	0.007418	0.112541	0.019276
10%	4	0.235	0.140675	0.056216	0.140808	0.134152	0.016067	0.098699
	16	0.1175	0.103375	0.008562	0.1045782	0.025654	0.057838	0.020692
	64	0.044062	0.001490	0.011082	0.010714	0.151194	0.242473	0.164910

Table 3.2: Computational results for Example 3.2, when $\nu = 1$.

δ	Dim	α	$\tilde{e}_{\alpha,m}$, k=1			$\tilde{e}_{\alpha,m}$, k=2		
			New	P_mAP_m	AP_m	New	P_mAP_m	AP_m
5%	4	1.106171	0.150711	0.148237	0.148304	0.080831	0.076263	0.075489
	16	0.695236	0.109478	0.108659	0.108414	0.0119959	0.009072	0.009433
	64	0.360357	0.010563	0.010195	0.009849	0.204137	0.204571	0.203984
10%	4	1.106171	0.146870	0.144352	0.143985	0.071879	0.067251	0.060570
	16	0.695236	0.102661	0.101498	0.101216	0.019621	0.018814	0.019165
	64	0.360357	0.011231	0.011082	0.010714	0.228818	0.229417	0.228833

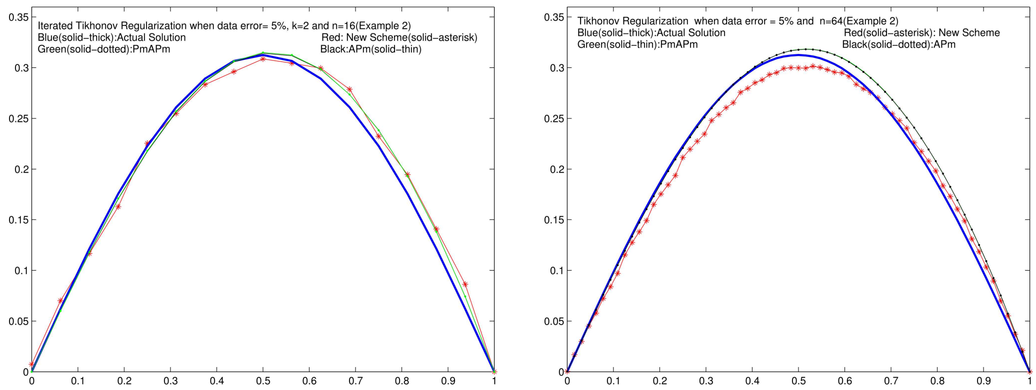


(a) Iterated Tikhonov solution, $\delta = 10\%$, k=2

(b) Tikhonov solution, $\delta = 10\%$

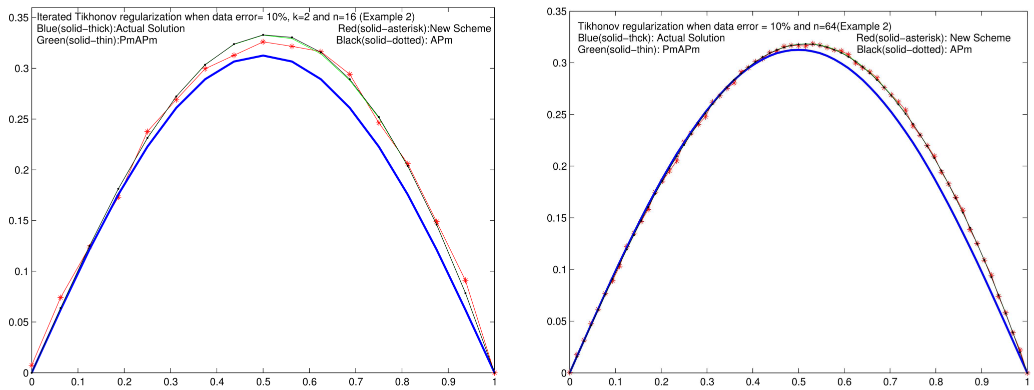
Fig. 3.5. Computed Solution for Example 3.1 when $\delta = 10\%$.

$k = 2$ and a system of size 16×16 is given in Fig. 3.5(a). For the same problem, the solution computed through TR for different schemes with system size 64×64 is given in Fig. 3.5(b). Similarly, when we have 5% data error in Example 3.2, the solution obtained through ITR with $k = 2$ and a system size of 16×16 is given in Fig. 3.6 whereas solution computed using TR with a system size of 64×64 is given in Fig. 3.6(b). The respective solutions with ITR and TR with different schemes and data error 10% are presented in Figs. 3.7(a) and 3.7(b).



(a) Iterated Tikhonov solution, $\delta = 5\%$, k=2 (b) Tikhonov solution, $\delta = 5\%$

Fig. 3.6. Computed Solution for Example 3.2 when $\delta = 5\%$.



(a) Iterated Tikhonov solution, $\delta = 10\%$, k=2 (b) Tikhonov solution, $\delta = 10\%$

Fig. 3.7. Computed Solution for Example 3.2 when $\delta = 10\%$.

We observed during the numerical experiments that ITR gives better results. In some cases, the ITR achieves the same result as that of Tikhonov regularization with smaller systems (cf. Figs. 3.6, 3.7). Further, we note that sometimes ITR explodes after a certain number of iterations, whereas Tikhonov regularization is a stable method, though it takes larger n to converge. It is obvious from theoretical analysis and the numerical experimentations that the proposed scheme is very economical in the sense that it requires only less amount of discrete information to solve the problem and achieves the same accuracy as compared with other traditional methods, emphasizing the computational efficiency of the proposed scheme. We have chosen α that depends on m in our examples namely, $\alpha \sim m2^{-m}$ in Example 3.1 and $\alpha \sim (m2^{-m})^{2/3}$ in Example 3.2. This is mainly because, in practice, one may not have complete information about the data error level, δ. In such situations, it is desirable to choose the parameter α that depends on m. We note that Theorem 2.2 enable us to choose α in such a manner.

Acknowledgements. We profoundly thank the unknown referee(s) for their careful reading of the manuscript and valuable suggestions that significantly improved the presentation of the paper as well.

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