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ANALYSIS OF ARBITRARY HIGH ORDER SPECTRAL VOLUME METHOD FOR HYPERBOLIC CONSERVATION LAWS OVER RECTANGULAR MESHES*

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Abstract

This paper investigates two spectral volume (SV) methods applied to 2D linear hyperbolic conservation laws on rectangular meshes. These methods utilize upwind fluxes and define control volumes using Gauss-Legendre (LSV) and right-Radau (RRSV) points within mesh elements. Within the framework of Petrov-Galerkin method, a unified proof is established to show that the proposed LSV and RRSV schemes are energy stable and have optimal error estimates in the L^2 norm. Additionally, we demonstrate superconvergence properties of the SV method at specific points and analyze the error in cell averages under appropriate initial and boundary discretizations. As a result, we show that the RRSV method coincides with the standard upwind discontinuous Galerkin method for hyperbolic problems with constant coefficients. Numerical experiments are conducted to validate all theoretical findings.

Mathematics subject classification: 65M15, 65M60, 65N30.
Key words: Spectral Volume methods, Energy stable, Superconvergence, Hyperbolic conservation laws.

1. Introduction

The spectral volume method is a significant high-order numerical technique for hyperbolic equations, extending the classical Godunov finite volume method [20]. Similar to other high-order methods such as the k-exact finite volume method [4, 19], the essentially nonoscillatory (ENO) method [11, 22], the weighted ENO (WENO) method [25, 27], and the discontinuous Galerkin (DG) method [15–17, 23], the SV method offers numerous advantages including high-order accuracy, compact stencils, and geometric flexibility (suitable for unstructured grids). Notably, the SV method preserves conservation laws on finer meshes, potentially offering higher resolution for discontinuities compared to other high-order methods (see [28]). Since its intro-

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duction in 2002 [36] and subsequent developments by Wang *et al.* [34, 37–39], the SV method has found widespread application in solving various PDEs such as the shallow water equation [14, 18], Navier-Stokes equation [21, 26, 29, 30], and electromagnetic field equations [26].

Despite the widespread application of SV methods, the exploration of their mathematical theory remains nascent, with limited available information on their convergence and superconvergence properties. To our knowledge, considerable attention has focused on the stability of lower-order SV schemes. For instance, Wang *et al.* [34, 38, 39] introduced the "Lebesgue constant" to assess SV methods in their early work. Van den Abeele *et al.* [31–33] investigated the stability of various low-order SV schemes: They found that 1D third and fourth-order SV schemes based on Gauss-Lobatto distributions are weakly unstable, whereas a 2D second-order SV scheme is stable, and a two-parameter family of third-order SV schemes exhibits weak instability. Zhang and Shu [35] employed Fourier-type analysis to study the stability of p-th order ($p \leq 3$) SV schemes over uniform meshes for 1D hyperbolic equations. In summary, there appears to be a lack of systematic studies on the mathematical theory encompassing stability, optimal-order convergence, and superconvergence.

A primary contribution of this study is to present a unified analysis of the stability and optimal-order convergence properties of SV schemes of arbitrary order for two-dimensional linear hyperbolic problems on rectangular meshes. To achieve this goal, we initially develop two classes of high-order SV schemes using control volumes defined by Gauss-Legendre points or right-Radau points within subintervals of the mesh. Ensuring stability, we adopt upwind numerical fluxes and reformulate the SV scheme in Petrov-Galerkin form. By introducing a specific mapping from the trial to the test space, we recast the SV method as a specialized Galerkin method. Leveraging the Galerkin framework and established numerical quadrature techniques, we establish the energy stability and optimal convergence rates of the LSV and RRSV schemes. As a notable outcome, we demonstrate that the proposed SV method can be interpreted as a discrete variant involving numerical quadratures of the well-known discontinuous Galerkin method. Particularly, the RRSV method exhibits identical behavior to the DG method when applied to hyperbolic equations with constant coefficients.

Another significant contribution of this work is the discovery and proof of notable superconvergence properties inherent in the SV method. Research into superconvergence behaviors of numerical methods has been extensive over the past decades (see, e.g. [1,3,5–9,12,13,24]). Among these studies, a well-known result is the $2k$-conjecture, which initially conjectured (based on substantial numerical evidence) that bi-k finite element approximations on rectangular meshes for the Poisson equation converge at a rate of h^{2k} at vertices of the mesh. This conjecture was proven by Chen *et al.* [12] after nearly 40 years. Subsequently, Cao *et al.* [9] demonstrated that the $2k$-conjecture holds for finite volume methods (FVM) and further extended it to the $(2k+1)$-conjecture for DG methods in [6] for 2D hyperbolic equations.

However, very little is known about the superconvergence properties of the SV method. To our knowledge, prior to our recent work on SV methods for 1D linear hyperbolic problems [10], no superconvergence analysis of the SV method had been reported. Motivated by the successful application of correction functions in DG and FVM methods (see, e.g. [6, 9]), we introduce a novel method to investigate the superconvergence properties of the proposed SV schemes. We prove that the $2k$-conjecture holds truw for LSV, while the $(2k+1)$-conjecture is valid for RRSV. Specifically, the bi-k LSV approximation and RRSV approximation over rectangular meshes converge at rates of h^{2k} and h^{2k+1}, respectively, at downwind nodal points. Additionally, we derive several other superconvergence results for the SV solution at specific points or lines:

(1) The error in the cell averages converges with orders of h^{2k} and h^{2k+1} for LSV and RRSV, respectively.

(2) For RRSV, the function value and derivative value exhibit separate superconvergence at right Radau points and left Radau lines, with orders of h^{k+2} and h^{k+1}, respectively.

(3) For LSV, the function value demonstrates $(k+2)$-th order superconvergence at Gauss points, while the derivative error achieves $(k+1)$-th order superconvergence at specific points.

The remainder of the paper is organized as follows: In Section 2, we introduce two classes of SV methods for linear conservation laws within the Petrov-Galerkin framework. Section 3 studies the bilinear form of these SV methods and contrasts them with DG methods. Section 4 rigorously establishes the energy inequality and L^2 stability for both SV methods. Sections 5 and 6 are dedicated to the convergence and superconvergence analyses of RRSV and LSV, respectively. In Section 7, numerical experiments are presented to validate all theoretical results. Section 8 contains concluding remarks. The appendix details the construction of correction functions.

2. Spectral Volume Method as a Special Petrov-Galerkin Method

We consider the two-dimensional linear hyperbolic conservation laws

$$u_t(x,y,t) + u_x(x,y,t) + u_y(x,y,t) = 0, \quad (x,y,t) \in [a,b] \times [c,d] \times (0,T]. \tag{2.1}$$

In addition, we suppose that the function u satisfies the initial condition

$$u(x,y,0) = u_0(x,y),$$

and the periodic boundary condition

$$\begin{aligned} u(a,y,t) &= u(b,y,t), \quad y \in [c,d], \\ u(x,c,t) &= u(x,d,t), \quad x \in [a,b], \end{aligned}$$

or the inflow boundary condition

$$\begin{aligned} u(a,y,t) &= \varphi_1(y,t), \quad y \in [c,d], \\ u(x,c,t) &= \varphi_2(x,t), \quad x \in [a,b], \end{aligned}$$

where all $u_0, \varphi_1, \varphi_2$ are smooth.

We begin the presentation of the SV method with some notation. Let

$$\begin{aligned} a &= x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{m+\frac{1}{2}} = b, \\ c &= y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{n+\frac{1}{2}} = d. \end{aligned}$$

Given positive integer r, define $\mathbb{Z}_r = \{1,2,\ldots,r\}, \mathbb{Z}_r^0 = \{0,1,\ldots,r\}$. For all $(p,q) \in \mathbb{Z}_m \times \mathbb{Z}_n$, let

$$\tau_p^x = \left[x_{p-\frac{1}{2}}, x_{p+\frac{1}{2}}\right], \quad \tau_q^y = \left[y_{q-\frac{1}{2}}, y_{q+\frac{1}{2}}\right], \quad \tau_{p,q} = \tau_p^x \times \tau_q^y$$

be the associated rectangle, and

$$h_\tau^x = x_{p+\frac{1}{2}} - x_{p-\frac{1}{2}}, \quad h_\tau^y = y_{q+\frac{1}{2}} - y_{q-\frac{1}{2}}$$

be the length along the x- and y- direction separately. We denote by $\mathcal{T}_h$ the mesh associated with the dividing points $(x_{p-1/2}, y_{q-1/2}) : (p,q) \in \mathbb{Z}_m \times \mathbb{Z}_n$, i.e.

$$\mathcal{T}_h = \{\tau_{p,q} : (p,q) \in \mathbb{Z}_m \times \mathbb{Z}_n\}.$$

Each rectangle $\tau \in \mathcal{T}_h$ is called as a spectral volume (SV). Let

$$h = \max_{\tau\in\mathcal{T}_h} \left(h_\tau^x, h_\tau^y\right), \quad h_{\min} = \min_{\tau\in\mathcal{T}_h} \left(h_\tau^x, h_\tau^y\right).$$

We assume that $\mathcal{T}_\langle$ is quasi-uniform. That is, there exists a positive constant c_1 satisfying $h \le c_1 h_{\min}$. Define

$$U_h = \left\{w \in L^2 : \; w|_\tau \in \mathbb{Q}_k = \mathbb{P}_k \times \mathbb{P}_k, \tau \in \mathcal{T}_h\right\},$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree at most k. Note that U_h is the standard broken finite element space which is widely used in the DG method.

To construct the SV scheme, we first partition each SV into control volumes by using some points and lines. Let

$$\begin{aligned} -1 &= s_0 < s_1 < \cdots < s_k < s_{k+1} = 1, \\ -1 &= \bar{s}_0 < \bar{s}_1 < \cdots < \bar{s}_k < \bar{s}_{k+1} = 1. \end{aligned}$$

Denote by F_τ the affine mapping from $[-1,1]^2$ to τ, and define

$$\mathcal{G}_\tau = \left\{g_{i,j}^\tau = \left(g_i^{\tau,x}, g_j^{\tau,y}\right) : g_{i,j}^\tau = F_\tau(s_i, \bar{s}_j), \; i \in \mathbb{Z}_{k+1}^0, \; j \in \mathbb{Z}_{k+1}^0\right\}$$

as the set of points in τ generated from $s_i, \bar{s}_j, (i,j) \in \mathbb{Z}_{k+1}^0 \times \mathbb{Z}_{k+1}^0$. With the above $(k+2)^2$ points, each τ of SV is partitioned into $(k+1)^2$ CVs with

$$V_{i,j}^\tau = \left[g_i^{\tau,x}, g_{i+1}^{\tau,x}\right] \times \left[g_j^{\tau,y}, g_{j+1}^{\tau,y}\right], \quad (i,j) \in \mathbb{Z}_k^0 \times \mathbb{Z}_k^0.$$

The dual mesh of $\mathcal{T}_h$ is defined as follows:

$$\mathcal{T}_h^* = \left\{V_{i,j}^\tau : \tau \in \mathcal{T}_h, \; (i,j) \in \mathbb{Z}_k^0 \times \mathbb{Z}_k^0\right\}.$$

The SV method for (2.1) is to find a function $u_h \in U_h$ such that

$$\int_{V_{i,j}^\tau} \partial_t u_h dxdy + \int_{\partial V_{i,j}^\tau} \hat{u}_h \boldsymbol{\nu} \cdot \mathbf{n} ds = 0, \tag{2.2}$$

where $\boldsymbol{\nu} = (1,1)$ and $\mathbf{n} = \mathbf{n}(x,y)$ is the unit outward normal vector at the point $(x,y) \in \partial V_{i,j}^\tau$, the boundary of $V_{i,j}^\tau$, and $\hat{u}_h$ is the numerical flux (induced by u_h) across the interface between two CVs, which is chosen as the upwind flux. Namely,

$$\hat{u}_h(x,y) = u_h^-(x,y), \quad (x,y) \in \partial V_{i,j}^\tau. \tag{2.3}$$

Here by u_h^-, u_h^+ we mean the left and right limits of u_h across each boundary of $V_{i,j}^\tau$, respectively. To complete the presentation of our SV method, we still need to define the numerical flux on the boundary $\partial\Omega$. If the problem (2.1) is associated with the periodic boundary condition, the numerical flux on $\partial\Omega$ are taken as

$$\begin{aligned} \hat{u}_h(a,y) &= u_h(b,y) = \hat{u}_h(b,y), \\ \hat{u}_h(x,c) &= u_h(x,d) = \hat{u}_h(x,d). \end{aligned}$$

If (2.1) is associated with the inflow boundary condition, there exist different methods to define the numerical flux on $(\partial\Omega)^-$, which can be chosen as the L^2 projection, some special interpolation function of the exact solution. For the purpose of superconvergence properties of u_h, we may take the following the numerical flux on $(\partial\Omega)^-$:

$$\hat{u}_h\big(x_{\frac{1}{2}}, y\big) = \tilde{u}_I(a, y), \quad \hat{u}_h(x, c) = \tilde{u}_I(x, c), \tag{2.4}$$

where $\tilde{u}_I$ denotes some particular projection of the exact solution u which will be specified later.

We now present the SV method under the framework of the Petrov-Galerkin method. To this end, we choose U_h as our trial space and define the test space as

$$V_h = \big\{w^* : w^*|_{V_{i,j}^\tau} \in \mathbb{P}_0\big(V_{i,j}^\tau\big),\ (i,j) \in \mathbb{Z}_k^0 \times \mathbb{Z}_k^0,\ \tau \in \mathcal{T}_h\big\}.$$

For any $w^* \in V_h$, note that

$$w^* = \sum_{\tau\in\mathcal{T}_h} \sum_{i,j=0}^{k} w_{i,j}^{\tau,*} \chi_{V_{i,j}^\tau}(x, y),$$

where $w_{i,j}^{\tau,*} = w_{i,j}^{\tau,*}(t)$ is a function of t, χ_A is the characteristic function satisfying $\chi_A = 1$ in A and $\chi_A = 0$ otherwise. Let

$$H_h = \big\{w : w|_\tau \in H^1,\ \forall\, \tau \in \mathcal{T}_h\big\}.$$

For all $(v, w^*) \in H_h \times V_h$, we define the elementwise bilinear form by

$$a_\tau(v, w^*) = \sum_{i,j=0}^{k} w_{i,j}^{\tau,*} \bigg(\int_{V_{i,j}^\tau} \partial_t v dxdy + \int_{\partial V_{i,j}^\tau} \hat{v} \boldsymbol{\nu} \cdot \mathbf{n} ds \bigg), \tag{2.5}$$

and the global bilinear form as

$$a(v, w^*) = \sum_{\tau\in\mathcal{T}_h} a_\tau(v, w^*). \tag{2.6}$$

Then the SV method for (2.1) is reformulated as : Find a $u_h \in U_h$ such that for all $w^* \in V_h$,

$$a(u_h, w^*) = 0. \tag{2.7}$$

We end this section with a discussion on the choice of different points $s_1, \ldots, s_k$ and $\bar{s}_1, \ldots, \bar{s}_k$, which leads to different SV schemes and affects the stability of the SV method. In this paper, we only consider the following two SV schemes:

(1) Legendre SV: Both $s_1, \ldots, s_k$ and $\bar{s}_1, \ldots, \bar{s}_k$ are taken as k Gauss points (i.e. the zeros of Legendre polynomial L_k).

(2) Right Radau SV: Both $s_1, \ldots, s_k$ and $\bar{s}_1, \ldots, \bar{s}_k$ are taken as k interior right Radau points (i.e. the zeros of $L_{k+1} - L_k$ except the point $s = 1$).

We would like to point out that the choice of Gauss and Radau points are not unique and may be extended to a broader class of subdivision points.

3. The SV Bilinear Form

To study the bilinear form of SV method, we first transform its Petrov-Galerkin form into the equivalent Galerkin form, and then discuss the associated properties under the framework of the Galerkin method.

We begin by relating the abscissae $s_j \in [-1,1]$, $j \in \mathbb{Z}_k$ to a quadrature

$$Q_k(f) = \sum_{j=1}^{k+1} A_j f(s_j), \quad f \in L^1([-1,1]),$$

which will be used to calculate the exact integral $I(f) = \int_{-1}^{1} f(s)ds$. Here $A_j, j \in \mathbb{Z}_{k+1}$ denote the weights associated with s_j. If $s_j, j \in \mathbb{Z}_k$ are taken as the Gauss points, then Q_k is the Gauss-Legendre quadrature with $A_{k+1} = 0$. If $s_j, j \in \mathbb{Z}_{k+1}$ are taken as the right Radau points, Q_k is referred as the right Radau quadrature. We define its residual by

$$R(f) = I(f) - Q_k(f).$$

We note that the k-point Gauss quadrature and the $(k+1)$-point Radau quadrature is exact for polynomials of degree $2k-1$ and $2k$, respectively.

The above definitions and notation can be easily generalized from the reference interval $[-1,1]$ to an arbitrary interval $[\alpha,\beta]$ by introducing an affine mapping $F = F_{[\alpha,\beta]}$ which maps a $s \in [-1,1]$ to

$$x = Fs = \frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2}s \in [\alpha,\beta].$$

Namely, for $f \in L^1([\alpha,\beta])$, we let

$$Q_k(f,[\alpha,\beta]) = \sum_{j=1}^{k+1} A_j^{[\alpha,\beta]} f(F_{[\alpha,\beta]}s_j),$$
$$R(f,[\alpha,\beta]) = I(f,[\alpha,\beta]) - Q_k(f,[\alpha,\beta])$$

with

$$A_j^{[\alpha,\beta]} = \frac{\beta-\alpha}{2}A_j, \quad j \in \mathbb{Z}_{k+1}.$$

They can also be generalized to an arbitrary rectangle $[\alpha,\beta] \times [\alpha_1,\beta_1]$ by using the mapping $F = (F_{[\alpha,\beta]}, F_{[\alpha_1,\beta_1]})$ which maps a $(\xi,\eta) \in [-1,1] \times [-1,1]$ to

$$(x,y) = F(\xi,\eta) = \big(F_{[\alpha,\beta]}(\xi), F_{[\alpha_1,\beta_1]}(\eta)\big) \in [\alpha,\beta] \times [\alpha_1,\beta_1].$$

Namely, for $f \in L^1([\alpha,\beta] \times [\alpha_1,\beta_1])$, we let

$$Q_k(f,[\alpha,\beta]\times[\alpha_1,\beta_1]) = \sum_{i,j=1}^{k+1} A_i^{[\alpha,\beta]} A_j^{[\alpha_1,\beta_1]} f(F_{[\alpha,\beta]}\xi_i, F_{[\alpha_1,\beta_1]}\eta_j),$$
$$R(f,[\alpha,\beta]\times[\alpha_1,\beta_1]) = I(f,[\alpha,\beta]\times[\alpha_1,\beta_1]) - Q_k(f,[\alpha,\beta]\times[\alpha_1,\beta_1]).$$

For all $\tau = \tau_p^x \times \tau_q^y \in \mathcal{T}_h$, we define three residuals for all $(x,y) \in \tau$ by

$$\big(E_\tau^x f\big)(y) = R\big(f(\cdot,y),\tau_p^x\big), \quad \big(E_\tau^y f\big)(x) = R\big(f(x,\cdot),\tau_q^y\big), \quad E_\tau f = R(f,\tau). \tag{3.1}$$

Next, we define a mapping from the trial space U_h to the test space V_h which will play an important role in transforming the SV method to a Galerkin method. In each $\tau \in \mathcal{T}_h$, suppose

$$w^*|_\tau = w^{\tau,*} := \sum_{i=0}^{k}\sum_{j=0}^{k} w_{i,j}^{\tau,*}(t)\chi_{V_{i,j}^\tau}(x,y). \tag{3.2}$$

We denote its jumps along x and y directions at $g_{i,j}^\tau = (g_i^{\tau,x}, g_j^{\tau,y})$ as

$$\left[w_{i,j}^{\tau,*}\right]^x = w_{i,j}^{\tau,*} - w_{i-1,j}^{\tau,*}, \quad \left[w_{i,j}^{\tau,*}\right]^y = w_{i,j}^{\tau,*} - w_{i,j-1}^{\tau,*},$$

and the double layer jump as

$$\lfloor w_{i,j}^{\tau,*} \rfloor = w_{i,j}^{\tau,*} + w_{i-1,j-1}^{\tau,*} - w_{i-1,j}^{\tau,*} - w_{i,j-1}^{\tau,*}, \quad (i,j) \in \mathbb{Z}_k \times \mathbb{Z}_k.$$

Define a transformation $\mathcal{F} : U_h \to V_h$ by letting $\mathcal{F}$ maps a $w \in U_h$ to $w^* = \mathcal{F}w$ whose coefficients $w_{i,j}^{\tau,*}, (i,j) \in \mathbb{Z}_k^0 \times \mathbb{Z}_k^0$ are determined by

$$w_{0,0}^{\tau,*} = w\big(g_0^{\tau,x}, g_0^{\tau,y}\big), \tag{3.3}$$

$$\left[w_{i,0}^{\tau,*}\right]^x = A_i^{\tau,x} w_x\big(g_i^{\tau,x}, g_0^{\tau,y}\big), \qquad i \in \mathbb{Z}_k, \tag{3.4}$$

$$\left[w_{0,j}^{\tau,*}\right]^y = A_j^{\tau,y} w_y\big(g_0^{\tau,x}, g_j^{\tau,y}\big), \qquad j \in \mathbb{Z}_k, \tag{3.5}$$

$$\lfloor w_{i,j}^{\tau,*} \rfloor = A_i^{\tau,x} A_j^{\tau,y} w_{xy}\big(g_i^{\tau,x}, g_j^{\tau,y}\big), \quad (i,j) \in \mathbb{Z}_k \times \mathbb{Z}_k, \tag{3.6}$$

where the weights $A_i^{\tau,x} = A_i^{\tau_p^x}, A_j^{\tau,y} = A_j^{\tau_q^y}$ with $\tau = \tau_p^x \times \tau_q^y$. We note that the definitions by (3.3)-(3.6) uniquely determine a mapping from U_h to V_h, since the $(k+1)^2$ unknown constants are calculated one by one. Precisely, we first obtain $w_{0,0}^{\tau,*}$ from (3.3), and then get $w_{i,0}^{\tau,*}, w_{j,0}^{\tau,*}$ from (3.4)-(3.5) and the value of $w_{0,0}^{\tau,*}$, and finally derive the constant $w_{i,j}^{\tau,*}$ by using the values of $w_{i-1,j-1}^{\tau,*}, w_{i-1,j}^{\tau,*}, w_{i,j-1}^{\tau,*}$ and the formula (3.6).

For any function $v \in H_h$, we introduce the notations

$$\partial_x^{-1} v|_{\tau_{p,q}} = \int_{x_{p-\frac{1}{2}}}^{x} v\,dx, \quad \partial_y^{-1} v|_{\tau_{p,q}} = \int_{y_{q-\frac{1}{2}}}^{y} v\,dy, \quad \partial_{xy}^{-2} v = \partial_x^{-1}\partial_y^{-1} v.$$

Lemma 3.1. *For all $v, w \in U_h$ and all $\tau_{p,q} \in \mathcal{T}_h$, there hold*

$$(v, w - w^*)_\tau = E_\tau\big(w_{xy}\partial_{xy}^{-2} v\big) - E_\tau^x\big(w_x \partial_{xy}^{-2} v\big)(y)\Big|_{y_{q-\frac{1}{2}}}^{y_{q+\frac{1}{2}}} - E_\tau^y\big(w_y \partial_{xy}^{-2} v\big)(x)\Big|_{x_{p-\frac{1}{2}}}^{x_{p+\frac{1}{2}}}, \tag{3.7}$$

$$\int_{(\partial\tau)^-} v(w - w^*)ds = E_\tau^y\big(\partial_y^{-1} v w_y\big)\big(g_0^{\tau,x}\big) + E_\tau^x\big(\partial_x^{-1} v w_x\big)\big(g_0^{\tau,y}\big). \tag{3.8}$$

Here,

$$F(\cdot, y)|_\alpha^\beta = F(\cdot,\beta) - F(\cdot,\alpha), \quad (\partial\tau)^- = \{(x,y) \in \partial\tau : \boldsymbol{\nu}\cdot\mathbf{n}(x,y) \le 0\}.$$

Here we omit the proof since the similar argument can be founded in [2].

We next present the SV method in its Galerkin form. Observing (2.5), we have for all $\tau \in \mathcal{T}_h$ that

$$\begin{aligned} a_\tau(v, w^*) &= (v_t + v_x + v_y, w^*)_\tau + \sum_{i,j=0}^{k} w_{i,j}^{\tau,*} \int_{\partial V_{i,j}^\tau} (\hat{v} - v)\hat{\mathbf{n}}\cdot\mathbf{n}ds \\ &= (v_t + v_x + v_y, w^*)_\tau + \sum_{i,j=0}^{k} w_{i,j}^{\tau,*} \int_{(\partial V_{i,j}^\tau)\cap(\partial\tau)^-} (\hat{v} - v)\hat{\mathbf{n}}\cdot\mathbf{n}ds \\ &= (v_t + v_x + v_y, w^*)_\tau + \int_{(\partial\tau)^-} [v] w^* ds, \end{aligned} \tag{3.9}$$

where $[v]$ denotes the jump of v with $[v] = v^+ - v^-$ each edge. Note that in the proof of (3.9), we have used the fact that $\hat{v} = v$ on $\partial\tau^+$ and the fact that v is continuous in the interior of τ.

Recall that the bilinear form of the upwind DG scheme, we have

$$\begin{aligned} a_\tau^{DG}(v,w) &= \int_\tau (v_t w - v w_x - v w_y)dx + \int_{\partial\tau} \hat{v} w \boldsymbol{\nu}\cdot\mathbf{n} ds \\ &= (v_t + v_x + v_y, w)_\tau + \int_{(\partial\tau)^-} [v] w ds. \end{aligned} \tag{3.10}$$

As a direct consequence of (3.7)-(3.10), we have the following relationship between the SV bilinear form and the DG bilinear form.

Theorem 3.1. *For all $\tau = \tau_{p,q} \in \mathcal{T}_h$, there holds*

$$a_\tau(v, w^*) = a_\tau^{DG}(v,w) + \sum_{i=1}^4 \mathcal{H}_\tau^i(v,w), \tag{3.11}$$

where

$$\begin{aligned} \mathcal{H}_\tau^1(v,w) &= -E_\tau\big(w_{xy}\partial_{xy}^{-2}v_t\big) + E_\tau^x\big(w_x\partial_{xy}^{-2}v_t\big)(y)\Big|_{y_{q-\frac12}}^{y_{q+\frac12}} + E_\tau^y\big(w_y\partial_{xy}^{-2}v_t\big)(x)\Big|_{x_{p-\frac12}}^{x_{p+\frac12}}, \\ \mathcal{H}_\tau^2(v,w) &= -E_\tau\big(w_{xy}\partial_y^{-1}v\big) + E_\tau^x\big(w_x\partial_y^{-1}v\big)(y)\Big|_{y_{q-\frac12}}^{y_{q+\frac12}} + E_\tau^y\big(w_y\partial_y^{-1}v\big)(x)\Big|_{x_{p-\frac12}}^{x_{p+\frac12}}, \\ \mathcal{H}_\tau^3(v,w) &= -E_\tau\big(w_{xy}\partial_x^{-1}v\big) + E_\tau^x\big(w_x\partial_x^{-1}v\big)(y)\Big|_{y_{q-\frac12}}^{y_{q+\frac12}} + E_\tau^y\big(w_y\partial_x^{-1}v\big)(x)\Big|_{x_{p-\frac12}}^{x_{p+\frac12}}, \\ \mathcal{H}_\tau^4(v,w) &= E_\tau^y\big(\big[\partial_y^{-1}v\big]w_y\big)\big(g_0^{\tau,x}\big) + E_\tau^x\big(\big[\partial_x^{-1}v\big]w_x\big)\big(g_0^{\tau,y}\big). \end{aligned}$$

We close this section with a discussion of the error terms $\mathcal{H}_\tau^i, 1 \le i \le 4$ given in (3.11). In the rest of this paper, standard notations for Sobolev spaces are adopted, such as $W^{r,p}(D)$ on subdomain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{r,p,D}$ and semi-norm $|\cdot|_{r,p,D}$. When $D = \Omega$, the index D is omitted. We set $W^{r,2}(D) = H^r(D)$, $\|\cdot\|_{r,2,D} = \|\cdot\|_{r,D}$, and $|\cdot|_{r,2,D} = |\cdot|_{r,D}$. The notation $A \lesssim B$ indicates $A \le cB$ with c a constant independent of the mesh size h.

For the RRSV method, since the Radau numerical quadrature exactly holds for polynomials of degree $2k$ and $\partial_{xy}^{-2}(v_t + v_x + v_y)w_{xy} \in \mathbb{Q}_{2k}$ for all $v, w \in U_h$, then

$$\mathcal{H}_\tau^i(v,w) = 0, \quad v, w \in U_h, \quad i = 1,2,3,4.$$

Consequently, for the RRSV method,

$$a(v, w^*) = a^{DG}(v,w), \quad \forall\, v, w \in U_h. \tag{3.12}$$

Namely, the RRSV method is identical to the DG method for the scalar constant-coefficient hyperbolic equation (2.1).

For the LSV method, the error terms $\mathcal{H}_\tau^i$ might be different from zero.

Lemma 3.2. *For all $\tau = \tau_{p,q}$, let $\mathcal{H}_\tau^i, i \le 4$ be defined in Theorem 3.1. Then for the LSV method, and for all $v \in U_h$,*

$$\mathcal{H}_\tau^1(v,v) = \frac{c_k}{2}\frac{d}{dt}\Big(\big(h_\tau^y\big)^{2k}\big\|\partial_y^k v\big\|_{0,\tau}^2 + c_k\big(h_\tau^x h_\tau^y\big)^{2k}\big\|\partial_y^k\partial_x^k v\big\|_{0,\tau}^2 + \big(h_\tau^x\big)^{2k}\big\|\partial_x^k v\big\|_{0,\tau}^2\Big), \tag{3.13}$$

$$\mathcal{H}_\tau^2(v,v) = \frac{c_k}{2}\big(h_\tau^y\big)^{2k+1}\Big(\big(\partial_y^k v\big)^2(x_{p+\frac12},\eta) - \big(\partial_y^k v\big)^2\big(x_{p-\frac12},\eta\big)\Big), \tag{3.14}$$

$$\mathcal{H}_\tau^3(v,v) = \frac{c_k}{2}\big(h_\tau^x\big)^{2k+1}\Big(\big(\partial_x^k v\big)^2\big(\xi, y_{q+\frac12}\big) - \big(\partial_x^k v\big)^2\big(\xi, y_{q-\frac12}\big)\Big), \tag{3.15}$$

$$\mathcal{H}_\tau^4(v,v) = c_k\Big(\big(h_\tau^y\big)^{2k+1}\big(\big[\partial_y^k v\big]\partial_y^k v\big)\big(x_{p-\frac12},\eta\big) + \big(h_\tau^x\big)^{2k+1}\big(\big[\partial_x^k v\big]\partial_x^k v\big)\big(\xi, y_{q-\frac12}\big)\Big), \tag{3.16}$$

where

$$c_k=\frac{(k!)^4}{(2k+1)[(2k)!]^3}\binom{2k}{k-1},\quad \xi:=\xi_p\in\big(x_{p-\frac12},x_{p+\frac12}\big),\quad \eta:=\eta_q\in\big(y_{q-\frac12},y_{q+\frac12}\big)$$

are arbitrary. Here by $v(x,y)$ on τ we mean the restriction of function v on τ, if no confusion is cased, i.e.

$$v\big(x_{p+\frac12},y\big)=v\big(x^-_{p+\frac12},y\big),\quad v\big(x_{p-\frac12},y\big)=v\big(x^+_{p-\frac12},y\big)$$

on each $\tau_{p,q}$.

Proof. First, we use the formula of Gauss quadrature error to derive that

$$\big(E^x_\tau f\big)(y)=\bar c_k\big(h^x_\tau\big)^{2k+1}\partial^{2k}_x f(\xi,y),\quad \big(E^y_\tau f\big)(x)=\bar c_k\big(h^y_\tau\big)^{2k+1}\partial^{2k}_y f(x,\eta)\tag{3.17}$$

with $\xi\in\tau^x_p,\eta\in\tau^y_q$ and

$$\bar c_k=\frac{(k!)^4}{(2k+1)[(2k)!]^3}.$$

Second, a direct calculation yields that

$$E_\tau f=\int_{\tau^x_p}\big(E^y_\tau f\big)(x)dx+\int_{\tau^y_p}\big(E^x_\tau f\big)(y)dy-E^x_\tau E^y_\tau f.$$

Substituting the above equation into the formulation of $\mathcal{H}^1_\tau(v,v)$ and using the integration by parts, we easily get

$$\mathcal{H}^1_\tau(v,v)=\int_{\tau^x_p}E^y_\tau\big(v_y\partial^{-1}_y v_t\big)(x)dx+\int_{\tau^y_p}E^x_\tau\big(v_x\partial^{-1}_x v_t\big)(y)dy+E^x_\tau E^y_\tau\big(v_{xy}\partial^{-2}_{xy}v_t\big).\tag{3.18}$$

By using (3.17) and the Leibnitz formula of derivatives, we have for all $v\in U_h$ that

$$E^x_\tau\big(v_x\partial^{-1}_x v_t\big)(y)=\bar c_k\binom{2k}{k-1}\big(h^x_\tau\big)^{2k+1}\big(\partial^k_x v\partial^k_x v_t\big)(\xi,y).$$

Noticing that $\partial^k_x v$ is a constant about the variable x for all $v\in U_h$, then $\xi\in(x_{p-1/2},x_{p+1/2})$ is arbitrary and thus

$$\int_{\tau^y_p}E^x_\tau\big(v_x\partial^{-1}_x v_t\big)(y)dy=c_k\big(h^x_\tau\big)^{2k}\int_\tau\partial^k_x v_t\partial^k_x v dxdy=\frac{c_k}{2}\big(h^x_\tau\big)^{2k}\frac{d}{dt}\big\|\partial^k_x v\big\|^2_{0,\tau}.$$

Similarly, there hold

$$\int_{\tau^x_p}E^y_\tau\big(v_y\partial^{-1}_y v_t\big)(x)dx=c_k\big(h^y_\tau\big)^{2k}\int_\tau\big(\partial^k_y v_t\partial^k_y v\big)(x,y)dxdy=\frac{c_k}{2}\big(h^y_\tau\big)^{2k}\frac{d}{dt}\big\|\partial^k_y v\big\|^2_{0,\tau},$$

$$E^y_\tau E^x_\tau\big(v_x\partial^{-1}_x v_t\big)=(c_k)^2\big(h^x_\tau h^y_\tau\big)^{2k+1}\big(\partial^k_y\partial^k_x v\partial^k_y\partial^k_x v_t\big)(\xi,\eta)=\frac{(c_k)^2}{2}\big(h^y_\tau h^x_\tau\big)^{2k}\frac{d}{dt}\big\|\partial^k_y\partial^k_x v\big\|^2_{0,\tau}.$$

Then (3.13) follows.

As for $\mathcal{H}^2_\tau$, by following the same argument as that in (3.18) and using the fact that Guass numerical quadrature is exact for $2k-1$ polynomial, we get

$$\begin{aligned}\mathcal{H}^2_\tau(v,v)&=\int_{\tau^x_p}E^y_\tau\big(v_y\partial^{-1}_y v_x\big)(x)dx+\int_{\tau^y_p}E^x_\tau(v_x v)(y)dy+E^x_\tau E^y_\tau\big(v_{xy}\partial^{-1}_y v\big)\\&=\int_{\tau^x_p}E^y_\tau\big(v_y\partial^{-1}_y v_x\big)(x)dx=\frac{c_k}{2}\big(h^y_\tau\big)^{2k+1}\int_{\tau^x_p}\partial_x\big(\partial^k_y v\big)^2(x,\eta)dx\end{aligned}$$

with $\eta\in(y_{q-1/2},y_{q+1/2})$ being arbitrary. Then (3.14) follows. The proofs of $\mathcal{H}^i_\tau(v,v),i=3,4$ are similar to that of $\mathcal{H}^2_\tau(v,v)$ and thus we omit it here. The proof is complete. □

4. L^2-norm Stability

We first introduce a so-called energy $\|\cdot\|_E$ defined by

$$\|v\|_E = (v, Fv)^{\frac{1}{2}} = (v, v^*)^{\frac{1}{2}}. \tag{4.1}$$

We next show that $\|\cdot\|_E$ is a norm equivalent to the standard L^2 norm in space U_h. Note that if $v \in U_h$, then for the RRSV, $(v, v^*) = (v, v)$ and thus $\|\cdot\|_E = \|\cdot\|_0$. As for the LSV, following the same argument as what we did in (3.18), we get for all $v, w \in U_h$,

$$\begin{aligned}
&(v, w^*)_\tau - (v, w)_\tau \\
=& \int_{\tau_p^x} E_\tau^y\big(w_y \partial_y^{-1} v\big)(x)dx + \int_{\tau_p^y} E_\tau^x\big(w_x \partial_x^{-1} v\big)(y)dy + E_\tau^x E_\tau^y\big(w_{xy}\partial_{xy}^{-2} v\big) \\
=& c_k\Big(\big(h_\tau^y\big)^{2k}\big(\partial_y^k v, \partial_y^k w\big)_\tau + c_k\big(h_\tau^x h_\tau^y\big)^{2k}\big(\partial_y^k\partial_x^k v, \partial_y^k \partial_x^k w\big)_\tau + \big(h_\tau^x\big)^{2k}\big(\partial_x^k v, \partial_x^k w\big)_\tau\Big).
\end{aligned} \tag{4.2}$$

Here in the second step, we have used the error of Gauss numerical quadrature and the fact that $\partial_x^k v, \partial_y^k v$ are constants for any $v \in U_h$. Consequently,

$$(v, v^*)_\tau = (v, v)_\tau + c_k\Big(\big(h_\tau^y\big)^{2k}\big\|\partial_y^k v\big\|_{0,\tau}^2 + c_k\big(h_\tau^x h_\tau^y\big)^{2k}\big\|\partial_y^k \partial_x^k v\big\|_{0,\tau}^2 + \big(h_\tau^x\big)^{2k}\big\|\partial_x^k v\big\|_{0,\tau}^2\Big). \tag{4.3}$$

Using the inverse inequality $\|v\|_{r,\tau} \lesssim h^{-r}\|v\|_{0,\tau}$ for all $v \in U_h$, we conclude that

$$\|v\|_0 \le \|v\|_E \lesssim \|v\|_0, \quad \forall\, v \in U_h. \tag{4.4}$$

On the other hand, we have from (4.2) and (4.3) that

$$\begin{aligned}
&(v, w^*) + (w, v^*) \\
=& 2(v, w) + 2c_k\Big(\big(h_\tau^y\big)^{2k}\big(\partial_y^k v, \partial_y^k w\big)_\tau + c_k\big(h_\tau^x h_\tau^y\big)^{2k}\big(\partial_y^k \partial_x^k v, \partial_y^k \partial_x^k w\big)_\tau + \big(h_\tau^x\big)^{2k}\big(\partial_x^k v, \partial_x^k w\big)_\tau\Big) \\
\le& 2(v, v^*)^{\frac{1}{2}}(w, w^*)^{\frac{1}{2}},
\end{aligned}$$

and thus

$$\|v + w\|_E \le \|v\|_E + \|w\|_E, \quad \forall\, v, w \in U_h,$$

which indicates (together with (4.4) and the fact that F a linear mapping) that $\|\cdot\|_E$ is a norm in U_h equivalent to the L^2 norm $\|\cdot\|_0$.

Now we are ready to present the L^2-norm stability for SV methods. In light of (3.9), we have

$$\begin{aligned}
a_\tau(v, v^*) &= (v_t + v_x + v_y, v^*)_\tau + \int_{(\partial\tau)^-} [v]v^* ds, \\
&= \frac{1}{2}\frac{d}{dt}\|v\|_{E,\tau}^2 + (v_x + v_y, v^*)_\tau + \int_{(\partial\tau)^-} [v]v^* ds.
\end{aligned} \tag{4.5}$$

Therefore, by (3.11)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|_{E,\tau}^2 &= a_\tau(v, v^*) - (v_x + v_y, v^*)_\tau - \int_{(\partial\tau)^-} [v]v^* ds \\
&= a_\tau(v, v^*) - (v_x + v_y, v)_\tau - \int_{(\partial\tau)^-} [v]v ds - \sum_{i=2}^{4} \mathcal{H}_\tau^i(v, v).
\end{aligned} \tag{4.6}$$

We have the following stability result.

Theorem 4.1. *Let $a(\cdot\,,\cdot)$ be the bilinear form induced by the SV method, then*

$$\frac{1}{2}\frac{d}{dt}\|v\|_E^2 \le a(v,v^*) - \int_{\partial\Omega} [\![v^2]\!] ds - H_{bd}, \quad \forall\, v \in U_h, \tag{4.7}$$

where

$$\int_{\partial\Omega} [\![v^2]\!] ds = \int_c^d \big(v^2(b,y) - v^2(a,y)\big)dy + \int_a^b \big(v^2(x,d) - v^2(x,c)\big)dx,$$

and for RRSV methods, the high order term on the boundary $H_{bd} := H_{bd}(v) = 0$; while for LSV methods,

$$\begin{aligned} H_{bd} := &\frac{c_k}{2}\sum_{q=1}^{n} \big(h_\tau^y\big)^{2k+1}\big(\big(\partial_y^k v\big)^2(b,\eta_q) - \big(\partial_y^k v\big)^2(a,\eta_q)\big) \\ &+ \frac{c_k}{2}\sum_{p=1}^{m} \big(h_\tau^x\big)^{2k+1}\big(\big(\partial_x^k v\big)^2(\xi_p,d) - \big(\partial_x^k v\big)^2(\xi_p,c)\big), \end{aligned}$$

where $\xi_p \in (x_{p-1/2}, x_{p+1/2}), \eta_q \in (y_{q-1/2}, y_{q+1/2})$ are arbitrary. Consequently, both LSV and RRSV are L^2 stable, i.e.

$$\frac{d}{dt}\|u_h\|_E^2 \lesssim C. \tag{4.8}$$

Here $C := C(t)$ with $C = 0$ and

$$C = \int_c^d |\varphi_1(y,t)|^2 dy + \int_a^b |\varphi_2(x,t)|^2 dx$$

for the periodic boundary condition and inflow boundary condition, respectively.

Proof. First, for all $\tau = \tau_{p,q} \in \mathcal{T}_h$, a direct calculation yields (also see [6])

$$\begin{aligned} &\sum_{\tau\in\mathcal{T}_h}\left((v_x+v_y,v)_\tau + \int_{(\partial\tau)^-}[v]v ds\right) \\ \ge\ &\frac{1}{2}\int_a^b \sum_{q\in\mathbb{Z}_n}\Big(v^2\big(x,y^-_{q+\frac12}\big) - v^2\big(x,y^-_{q-\frac12}\big)\Big)dx \\ &+\frac{1}{2}\int_c^d \sum_{p\in\mathbb{Z}_m}\Big(v^2\big(x^-_{p+\frac12},y\big) - v^2\big(x^-_{p-\frac12},y\big)\Big) \\ =\ &\int_{\partial\Omega}[\![v^2]\!]ds. \end{aligned}$$

Secondly, for RRSV method, it is easy to get that

$$\sum_{i=2}^{4}\mathcal{H}_\tau^i(v,v) = 0.$$

For the LSV method, we conclude from (3.14)-(3.16) that

$$\begin{aligned} &\sum_{\tau\in\mathcal{T}_h}\sum_{i=2}^{4}\mathcal{H}_\tau^i(v,v) \\ &= H_{bd} + \frac{c_k}{2}\sum_{p=1}^{m}\sum_{q=1}^{n}\Big(\big(h_\tau^y\big)^{2k+1}\big[\partial_y^k v\big]^2\big(x_{p-\frac12},\eta\big) + \big(h_\tau^x\big)^{2k+1}\big(\big[\partial_x^k v\big]^2\big)\big(\xi, y_{q-\frac12}\big)\Big) \ge H_{bd}. \end{aligned}$$

Consequently, we obtain the error inequality (4.7).

Now taking $v = u_h$ in (4.7), and noticing $a(u_h, u_h^*) = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_E^2 \le -\int_{\partial\Omega} [\![u_h^2]\!] ds - H_{bd}(u_h). \tag{4.9}$$

For the periodic boundary condition, there holds

$$\int_{\partial\Omega} [\![u_h^2]\!] ds = H_{bd}(u_h) = 0,$$

thus

$$\frac{d}{dt}\|u_h\|_E^2 \le 0 = C(t).$$

For the inflow boundary condition,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_h\|_E^2 &\le -\int_{\partial\Omega} [\![u_h^2]\!] ds - H_{bd}(u_h)\\
&\le \int_c^d u_h^2(a,y)dy + \int_a^b u_h^2(x,c)dx\\
&\quad + \frac{c_k}{2}\sum_{q=1}^{n}\left(h_\tau^y\right)^{2k+1}\left(\partial_y^k u_h\right)^2(a,\eta_q)\\
&\quad + \frac{c_k}{2}\sum_{p=1}^{m}\left(h_\tau^x\right)^{2k+1}\left(\partial_x^k u_h\right)^2(\xi_p,c)\\
&\lesssim \int_c^d \varphi_1^2(y,t)dy + \int_a^b \varphi_2^2(x,t)dx = C(t),
\end{aligned}$$

where in the last step, we have used the inverse inequality and the identities

$$\begin{aligned}
h_\tau^y\left(\partial_y^k u_h\right)^2(a,\eta_q) &= \int_{y_{q-\frac{1}{2}}}^{y_{q+\frac{1}{2}}} \left(\partial_y^k u_h\right)^2(a,y)dy,\\
h_\tau^x\left(\partial_x^k u_h\right)^2(\xi_p,c) &= \int_{x_{p-\frac{1}{2}}}^{x_{p+\frac{1}{2}}} \left(\partial_x^k u_h\right)^2(x,c)dx.
\end{aligned}$$

In summary, the stability (4.8) is valid for both the RRSV and LSV, and for both the periodic and inflow boundary conditions. □

5. Convergence with Optimal Orders

We begin with an estimate of $\|w^*\|_0$ for $w \in U_h$. In light of (3.3)-(3.6), it is easy to derive by using the inverse inequality

$$\left|w_{i,j}^{\tau,*}\right| \lesssim \|w\|_{0,\infty,\tau} + \left|w_{i,0}^{\tau,*}\right| \lesssim \|w\|_{0,\infty,\tau}, \quad \forall\,(i,j) \in \mathbb{Z}_0^k \times \mathbb{Z}_0^k.$$

Consequently,

$$(w^*, w^*)_\tau = \sum_{i,j=0}^{k} h_\tau^x h_\tau^y \left|w_{i,j}^{\tau,*}\right|^2 \lesssim (w,w)_\tau,$$

and thus

$$\|w^*\|_0 \lesssim \|w\|_0, \quad \forall\, w \in U_h. \tag{5.1}$$

We next introduce the Lagrange interpolation function. Given any function $v \in L^\infty$, we denote by $\mathcal{I}_h^x v \in \mathbb{P}_k(x)$ the one dimensional Lagrange interpolation function of v along the x direction satisfying

$$\mathcal{I}_h^x v\big(g_j^{\tau,x}, y\big) = v\big(g_j^{\tau,x}, y\big), \quad j \in \mathbb{Z}_{k+1}. \tag{5.2}$$

Similarly, we can define the interpolation function $\mathcal{I}_h^y v$ and the interpolation error $\mathcal{Q}^y v$ along the y-direction. Let $v_I \in U_h$ be the Lagrange interpolation of v satisfying

$$v_I\big(g_{i,j}^\tau\big) = v\big(g_{i,j}^\tau\big), \quad (i,j) \in \mathbb{Z}_{k+1} \times \mathbb{Z}_{k+1}, \quad \tau \in \mathcal{T}_h. \tag{5.3}$$

Apparently, we have $v_I = \mathcal{I}_h^x \mathcal{I}_h^y v$. Denote by $\mathcal{Q}^x v = v - \mathcal{I}_h^x v$ the interpolation error. Then there holds the error decomposition

$$v - v_I = \mathcal{Q}^x v + \mathcal{Q}^y v - \mathcal{Q}^x \mathcal{Q}^y v. \tag{5.4}$$

Furthermore, for smooth function v with $v \in H^j, j \le k+1$, we use the approximation theory to derive that

$$\big\|\partial_x^i \mathcal{Q}^x v\big\|_{0,\tau} \lesssim h^{j-i} \big\|\partial_x^j v\big\|_{0,\tau}, \quad i \le j. \tag{5.5}$$

We have the following error estimates for the interpolation function u_I.

Lemma 5.1. *Let $u \in H^{k+2}$ and u_I be the Lagrange interpolation function of u defined in* (5.3). *Then*

$$|a(u - u_I, w^*)| \lesssim h^{k+1} \|u\|_{k+2} \|w\|_0, \quad \forall\, w \in U_h. \tag{5.6}$$

Proof. By the error decomposition (5.4), we have

$$a_\tau(u - u_I, w^*) = a_\tau(\mathcal{Q}^x u, w^*) + a_\tau(\mathcal{Q}^y u, w^*) - a_\tau(\mathcal{Q}^x \mathcal{Q}^y u, w^*).$$

We next estimate $a_\tau(\mathcal{Q}^y u, w^*)$. By (3.9),

$$a_\tau(\mathcal{Q}^y u, w^*) = \big((\partial_t + \partial_x + \partial_y)\mathcal{Q}^y u, w^*\big)_\tau + \int_{(\partial\tau)^-} [\mathcal{Q}^y u] w^* ds.$$

Denote by $l_j, j \le k+1$ the Lagrange basis function associated with the interpolation points $g_j^{\tau,y}$. Then

$$\mathcal{Q}^y u = u - \mathcal{I}_h^y u = u(x,y,t) - \sum_{j=1}^{k+1} u\big(x, g_j^{\tau,y}, t\big) l_j(y).$$

Since $u \in H^{k+2}$ is smooth, then the interpolation function $\mathcal{I}_h^y u$ is smooth about the variable x, and thus $\mathcal{Q}^y u$ is continuous about x, which yields, together with the fact that $\mathcal{Q}^y u(x, g_j^{\tau,y}) = 0$ for all $j \in \mathbb{Z}_{k+1}$,

$$\big(\partial_y \mathcal{Q}^y u, w^*\big)_\tau + \int_{(\partial\tau)^-} [\mathcal{Q}^y u] w^* ds = 0.$$

Consequently,

$$a_\tau(\mathcal{Q}^y u, w^*) = \big((\partial_t + \partial_x)\mathcal{Q}^y u, w^*\big)_\tau = \big(\mathcal{Q}^y(u_t + u_x), w^*\big)_\tau. \tag{5.7}$$

Here in the second step, we have used the equation

$$(\partial_t + \partial_x)\mathcal{Q}^y u = (\partial_t + \partial_x) u(x,y,t) - \sum_{j=1}^{k+1} (\partial_t + \partial_x) u\big(x, g_j^{\tau,y}, t\big) l_j(y) = (\partial_t + \partial_x)\mathcal{Q}^y u.$$

By the Cauchy-Schwarz inequality, (5.1) and approximation property of $\mathcal{Q}^y$, we have

$$\big|a_\tau(\mathcal{Q}^y u, w^*)\big| \lesssim h^{k+1}\|u\|_{k+2,\tau}\|w\|_{0,\tau}.$$

Similarly, there holds

$$\big|a_\tau(\mathcal{Q}^x u, w^*)\big| = \big|\big(\mathcal{Q}^x(u_t+u_y), w^*\big)_\tau\big| \lesssim h^{k+1}\|u\|_{k+2,\tau}\|w\|_{0,\tau}.$$

Moreover, by (3.9) and the properties of $\mathcal{Q}^x$ and $\mathcal{Q}^y$, we get that

$$|a(\mathcal{Q}^x\mathcal{Q}^y u, w^*)| = \big|\big((\partial_t+\partial_x+\partial_y)\mathcal{Q}^x\mathcal{Q}^y u, w^*\big)_\tau\big| \lesssim h^{k+1}\|u\|_{k+2}\|w\|_0. \tag{5.8}$$

Then the desired result follows. □

Theorem 5.1. *Let $u\in H^{k+2}$ and u_I be the Gauss Lagrange interpolation function of the exact solution u defined by* (5.3)*. Let u_h be the solution of* (2.2) *with the initial value chosen as*

$$u_h(x,y,0) = u_I(x,y,0), \tag{5.9}$$

and for the inflow boundary condition

$$u_h(x,y) = u_I(x,y), \quad (x,y)\in(\partial\Omega)^-.$$

Then for both LSV and RRSV methods,

$$\|u(\cdot,t)-u_h(\cdot,t)\|_0 \lesssim h^{k+1}\|u\|_{k+2}. \tag{5.10}$$

Proof. Note that $(u_h-u_I)(x,y)=0$ on $(\partial\Omega)^-$ for the inflow boundary condition and

$$(u_h-u_I)\big(x^-_{m+\frac12},y\big) = (u_h-u_I)\big(x^-_{\frac12},y\big),$$
$$(u_h-u_I)\big(x,y^-_{n+\frac12}\big) = (u_h-u_I)\big(x,y^-_{\frac12}\big)$$

for the periodic boundary condition. Then for both the periodic and inflow boundary conditions, we take $v=u_h-u_I$ in (4.7) to derive that

$$\frac12\frac{d}{dt}\|u_h-u_I\|_E^2 \le a_h\big(u_h-u_I,(u_h-u_I)^*\big) = a\big(u-u_I,(u_h-u_I)^*\big).$$

By using (5.9), (4.4) and (5.6), we obtain that

$$\|(u_h-u_I)(\cdot,t)\|_0^2 \lesssim h^{k+1}\int_0^t \|u(\cdot,\tau)\|_{k+2}\|(u_h-u_I)(\cdot,\tau)\|d\tau.$$

The desired result follows from the Gronwall inequality and the triangle inequality. □

6. Superconvergence

6.1. Superconvergence of RRSV methods

As already proved in [10, Section 3.3], for the scalar linear hyperbolic equation with constant coefficients, the RRSV method is identical to the upwind DG method. and thus share all the same superconvergence results of the DG method given in [6]. That is, we have the following results for RRSV.

Theorem 6.1. *Let $u \in W^{2k+2,\infty}$, and $u_h \in U_h$ be the solution of the RRSV method with the initial value chosen as*

$$u_h(x, y, 0) = \tilde{u}_I(x, y, 0),$$

and for the inflow boundary condition

$$u_h(x, y) = \tilde{u}_I(x, y), \quad (x, y) \in (\partial\Omega)^-.$$

Here $\tilde{u}_I$ denotes the special projection of u defined in [6]. *Then for both the periodic and inflow boundary conditions,*

$$e_n := \left(\frac{1}{mn} \sum_{p=1}^{m} \sum_{q=1}^{n} \big(u - u_h\big)^2 \big(x^-_{p+\frac{1}{2}}, y^-_{q+\frac{1}{2}}, t\big) \right)^{\frac{1}{2}} \lesssim h^{2k+1} \|u\|_{2k+2,\infty}, \tag{6.1}$$

$$e_c := \left(\frac{1}{nm} \sum_{\tau \in \mathcal{T}_h} \left(\frac{1}{|\tau|} \int_\tau (u - u_h) \right)^2 dxdy \right)^{\frac{1}{2}} \lesssim h^{2k+1} \|u\|_{2k+2,\infty}, \tag{6.2}$$

$$e_r := \max_{P \in \mathcal{R}} |(u - u_h)(P, t)| \lesssim h^{k+2} \|u\|_{k+2,\infty}, \tag{6.3}$$

$$e_l := \max_{Q_0 \in \mathcal{L}^x} |\partial_x (u - u_h)(Q_0, t)| + \max_{Q_1 \in \mathcal{L}^y} |\partial_y (u - u_h)(Q_1, t)| \lesssim h^{k+1} \|u\|_{k+2,\infty}, \tag{6.4}$$

where $\mathcal{R}, \mathcal{L}^x, \mathcal{L}^y$ denotes the right Radau points, left Radau lines along x and y directions on the whole domain, respectively. That is,

$$\begin{aligned}
\mathcal{R} &= \big\{ r^\tau_{i,j} \,:\, r^\tau_{i,j} = F_\tau(r_i, r_j),\ i \in \mathbb{Z}_{k+1},\ j \in \mathbb{Z}_{k+1},\ \tau \in \mathcal{T}_h \big\}, \\
\mathcal{L}^x &= \big\{ l^{\tau,x}_j \,:\, l^{\tau,x}_j = F_\tau(l_j, y),\ i \in \mathbb{Z}_k,\ \tau \in \mathcal{T}_h \big\}, \\
\mathcal{L}^y &= \{ l^{\tau,y}_j \,:\, l^{\tau,y}_i = F_\tau(x, l_j),\ i \in \mathbb{Z}_k,\ \tau \in \mathcal{T}_h \}.
\end{aligned}$$

Here $r_i, i \in \mathbb{Z}_{k+1}$ and $l_j, j \in \mathbb{Z}_k$ denote the $k+1$ right Radau points and k interior left Radau points on $[-1, 1]$, respectively. Here by k interior left Radau points we mean zeros of $L_{k+1} + L_k$ except $s = -1$.

6.2. Superconvergence of LSV methods

To investigate the superconvergence property for LSV, we slightly modify the $u_I \in U_h$ of u with

$$\tilde{u}_I = u_I - \zeta_h, \tag{6.5}$$

where $\zeta_h \in U_h$ is called the correction function and is used to correct the error bound $a(u - u_I, w^*)$ such that $a(u - \tilde{u}_I, w^*) = a(u - u_I, w^*) + a(\zeta_h, w^*)$ for all $w \in U_h$ is of higher order.

Proposition 6.1. *If $u \in H^{2k+1}(\Omega)$, then there exists a function $\zeta_h \in U_h$ satisfying*

$$\begin{aligned}
&\|\zeta_h\|_{0,\tau} \lesssim h^{k+2} \|u\|_{k+2}, \\
&h\big|\zeta_h\big(x^-_{p-\frac{1}{2}}, y^-_{q-\frac{1}{2}}\big)\big| + |(\zeta_h, 1)_\tau| \lesssim h^{2k+1} \|u\|_{2k+1},
\end{aligned} \tag{6.6}$$

and for all $w \in U_h$,

$$|a(u - u_I + \zeta_h, w^*)| \lesssim h^{k+1+r} \|u\|_{k+2+r} \|w\|_0, \quad r \leq k - 1. \tag{6.7}$$

The detailed proof of Proposition 6.1 will be given in the Appendix A.

Define the following errors:

$$e_g := \left(\frac{1}{mn}\sum_{\tau\in\mathcal{T}_h}\sum_{i,j=1}^{k}(u-u_h)^2\big(g_{i,j}^{\tau},t\big)\right)^{\frac12},$$

$$e_d := \left(\frac{1}{mn}\sum_{\tau\in\mathcal{T}_h}\left(\sum_{i=1}^{k}\partial_x(u-u_h)^2\big(z_i^{\tau,x},y,t\big)+\sum_{j=1}^{k}\partial_y(u-u_h)^2\big(x,z_j^{\tau,y},t\big)\right)\right)^{\frac12},$$

where

$$\big(z_i^{\tau,x},z_j^{\tau,y}\big)=F_\tau(z_i,z_j),\quad z_i, i\in\mathbb{Z}_k$$

zeros of the polynomial $\partial_s((s-1)L_k(s))$ in $[-1,1]$. Thanks to the construction of the correction function ζ_h, we have the following superconvergence results for the LSV method.

Theorem 6.2. *Let $u\in H^{2k+1}$ be the solution of (2.1), and $u_h\in U_h$ be the solution of the LSV method with the initial value chosen as $u_h(x,y,0)=\tilde u_I(x,y,0)$ and $u_h=\tilde u_I, (x,y)\in(\partial\Omega)^-$ for the inflow boundary condition. Then*

$$\|u_h-u_I\|_0\lesssim h^{\min(k+2,2k)}\|u\|_{k+3}, \tag{6.8}$$

$$e_n\lesssim h^{2k}\|u\|_{2k+1},\qquad e_c\lesssim h^{2k}\|u\|_{2k+1}, \tag{6.9}$$

$$e_g\lesssim h^{\min(k+2,2k)}\|u\|_{k+3},\quad e_d\lesssim h^{\min(k+1,2k-1)}\|u\|_{k+3}. \tag{6.10}$$

Proof. Due to the special choice of the boundary discretization (2.3) and (4.7), we have for both the periodic and inflow boundary conditions that

$$\frac12\frac{d}{dt}\|u_h-\tilde u_I\|_E^2\le a\big(u_h-\tilde u_I,(u_h-\tilde u_I)^*\big)=\sum_{\tau\in\mathcal{T}_h}a_\tau\big(u-u_I+\zeta_h,(u_h-\tilde u_I)^*\big).$$

In light of (6.7), we have

$$\|(u_h-\tilde u_I)(\cdot,t)\|_E\lesssim\|(u_h-\tilde u_I)(\cdot,0)\|_E+h^{k+1+r}(1+t)\|u\|_{k+2+r}.$$

Using the special choice of the initial value and (4.4), we easily get

$$\|(u_h-\tilde u_I)(\cdot,t)\|_0\lesssim h^{k+1+r}(1+t)\|u\|_{k+2+r},\quad r\le k-1. \tag{6.11}$$

Taking $r=\min(1,k-1)$ in the above inequality and using the first inequality of (6.6) and the triangle inequality yields

$$\|(u_h-u_I)(\cdot,t)\|_0\lesssim\|(u_h-\tilde u_I)(\cdot,t)\|_0+\|\zeta_h\|_0\lesssim h^{\min(2k,k+2)}(1+t)\|u\|_{k+3}. \tag{6.12}$$

Then (6.8) follows. Noticing that

$$u_I\big(x^-_{p+\frac12},y^-_{q+\frac12},t\big)=u\big(x^-_{p+\frac12},y^-_{q+\frac12},t\big),$$

we have, from the second inequality of (6.6) and the inverse inequality that

$$\begin{aligned}e_n&\lesssim\left(\frac{1}{mn}\sum_{p=1}^{m}\sum_{q=1}^{n}(\tilde u_I-u_h)^2\big(x^-_{p+\frac12},y^-_{q+\frac12},t\big)+\zeta_h^2\big(x^-_{p+\frac12},y^-_{q+\frac12},t\big)\right)^{\frac12}\\&\lesssim\|(\tilde u_I-u_h)(\cdot,t)\|_0+h^{2k}\|u\|_{2k+1}.\end{aligned}$$

By taking $r = k-1$ in (6.11), we obtain the superconvergence property on the upwind points which is presented by the first inequality of (6.9).

We next estimate the cell average error e_c. Using the formula of Newton interpolating remainder, we have

$$\mathcal{Q}^y v(\cdot,y)|_{\tau_q^y} = \bar{v}(\cdot,y)\big(y-y_{q+\frac{1}{2}}\big)\prod_{j=1}^{k}(y-y_{q,j}) = \bar{v}(\cdot,y)\big(y-y_{q+\frac{1}{2}}\big)\tilde{L}_{q,k}(y), \tag{6.13}$$

where $\bar{v}(\cdot,y)$ denotes the Newton's difference quotient, $y_{q,j}, j\in\mathbb{Z}_k$ are k Gauss points on τ_q^y and $\tilde{L}_{q,k}(y)=\prod_{j=1}^{k}(y-y_{q,j})$. Since $\tilde{L}_{q,k}\perp\mathbb{P}_{k-1}(y)$, then for all $w\in\mathbb{P}_j(y), j\le k-2$,

$$\begin{aligned}\left|\int_{\tau_q^y}\mathcal{Q}^y vwdy\right| &= \left|\int_{\tau_q^y}\tilde{L}_{q,k}\big(y-y_{q+\frac{1}{2}}\big)\big(\bar{v}-\mathcal{I}_{k-2-j}^y\bar{v}\big)wdy\right| \\ &\lesssim h^{k+1}\big\|\bar{v}-\mathcal{I}_{k-2-j}^y\bar{v}\big\|_{0,\tau_q^y}\|w\|_{0,\tau_q^y} \\ &\lesssim h^{k+1+r}\|v\|_{k+1+r,\tau_q^y}\|w\|_{0,\tau_q^y}, \quad r\le k-j-1.\end{aligned} \tag{6.14}$$

Here $\mathcal{I}_{k-2-m}^y v\in\mathbb{P}_{k-2-m}(y)$ denotes the interpolation function of v, and in the last step, we have used the formula of Newton's difference quotient $\|\bar{v}\|_{0,\tau_q^y}\lesssim\|v\|_{k+1,\tau_q^y}$. Especially, we choose $r=k-1$ and $w=1$ in (6.14) and then obtain

$$|(\mathcal{Q}^y u,1)_\tau|\lesssim h^{2k+1}\|u\|_{2k+1,\tau}.$$

Similar results also hold true for $\mathcal{Q}^x u, \mathcal{Q}^x\mathcal{Q}^y u$. Then

$$|(u-u_I,1)_\tau| = \big|(\mathcal{Q}^x u,1)_\tau+(\mathcal{Q}^y u,1)_\tau-(\mathcal{Q}^x\mathcal{Q}^y u,1)_\tau\big|\lesssim h^{2k+1}\|u\|_{2k+1,\tau}.$$

Consequently,

$$\begin{aligned}|(u-u_h,1)_\tau| &= |(u-u_I,1)_\tau+(\zeta_h,1)_\tau+(\tilde{u}_I-u_h,1)_\tau| \\ &\lesssim h^{2k+1}\|u\|_{2k+1,\tau}+|(\tilde{u}_I-u_h,1)_\tau|.\end{aligned}$$

Since $\mathcal{T}_h$ is quasi-uniform, we have $|\tau|^{-1}/nm\lesssim 1$ and thus

$$e_c\lesssim\|(\tilde{u}_I-u_h)(\cdot,t)\|_0+h^{2k}\|u\|_{2k+1}\lesssim h^{2k}\|u\|_{2k+1}.$$

Then the second inequality of (6.9) is obtained.

Next we prove (6.10). First, by the definition of u_I and the triangle university,

$$e_g=\left(\frac{1}{mn}\sum_{\tau\in\mathcal{T}_h}\sum_{i,j=1}^{k}(u_I-u_h)^2\big(g_{i,j}^\tau,t\big)\right)^{\frac{1}{2}}\lesssim\|u_I-u_h\|_0.$$

Then the first inequality of (6.10) follows from (6.8).

On the other hand, by (6.13), we have

$$\begin{aligned}&\big|\partial_y\mathcal{Q}^y u\big(x,z_i^{\tau,y},t\big)\big|\lesssim h^{k+1}\big|\partial_y\bar{v}\big(x,z_i^{\tau,y},t\big)\big|\lesssim h^{k+1}\|v\|_{k+2,\tau}, \\ &\big|\partial_x\mathcal{Q}^y u\big(z_i^{\tau,x},y,t\big)\big|\lesssim h^{k+1}\|v\|_{k+2,\tau}.\end{aligned}$$

Consequently,

$$e_d\le\left(\frac{1}{mn}\sum_{\tau\in\mathcal{T}_h}\|u_I-u_h\|_{1,\infty,\tau}^2\right)^{\frac{1}{2}}+h^{k+1}\|u\|_{k+2}$$

$$\lesssim h^{-1}\|(u_I - u_h)(\cdot,t)\|_0 + h^{k+1}\|u\|_{k+2}.$$

The second inequality of (6.10) follows from (6.8). The proof is complete. □

Remark 6.1. The assumption $u \in H^{2k+1}$ in Theorem 6.2 implies that we also require $u_0 \in H^{2k+1}$. In addition, the process of initial and boundary discretizations for LSV are provided in the Appendix A. For RRSV, we refer to [6] for the detailed information about the initial and boundary discretizations.

Remark 6.2. We would like to point out that the stability and error analysis of SV methods on 2D rectangular grid is not a direct generalization from 1D results in [10] . On the one hand, the main challenge in the SV stability analysis is to define a special mapping from the trial space to the test space. Due to the extra variable and the interplay between the two variables x, y, the mapping is much more sophisticated than that for 1D. As defined in (3.3)-(3.6), how to construct the $(k+1)^2$ piecewise constants in 2D is very technical, which is not a simple tensor product of that in 1D. On the other hand, in the error analysis, especially the superconvergence analysis, again the correction function is not a simple tensor product, which should be carefully designed to eliminate the impact of the interplay between the two variables x, y.

7. Numerical Experiments

Some numerical experiments are presented in this section to justify the results in Theorems 5.1, 6.1 and 6.2. We shall test various errors including the standard L^2 error $\|e\|_0 := \|u-u_h\|_0$, the cell average error e_c, the function value error at Gauss points e_g, the function value error at right Radau points e_r, and the derivative error e_l and e_d at superconvergent lines. In our numerical experiments, we take the correction initial discretization $u_h(x,y,0) = \tilde{u}_I(x,y,0)$ and choose (2.4) as the boundary discretization for the Dirichlet boundary condition. The fourth order Runge-Kutta method is used for time discretization with $\triangle t = 0.05h^2_{\min}$.

Example 7.1. We consider

$$\begin{aligned}
&u_t + u_x + u_y = 0, && (x,y,t) \in [0,2\pi]\times[0,2\pi]\times(0,1],\\
&u(x,y,0) = \sin(x+y), && (x,y) \in [0,2\pi]\times[0,2\pi],\\
&u(0,y,t) = u(2\pi,y,t), && \forall\, y \in [0,2\pi],\\
&u(x,0,t) = u(x,2\pi,t), && \forall\, x \in [0,2\pi],
\end{aligned} \tag{7.1}$$

which admits the exact solution $u(x,y,t) = \sin(x+y-2t)$.

Non-uniform meshes of $N \times N$ rectangles are obtained by equally dividing $[0,\pi/2]$ and $[\pi/2, 2\pi]$ into $N/2$ elements in each x and y directions. The problem is solved by using LSV and RRSV methods with $\mathbb{Q}_k, k = 1,2,3$ polynomials. We compute the numerical solution at $t = 1$.

Listed in Tables 7.1 and 7.2 are errors and their corresponding convergence rates for LSV and RRSV methods, respectively. From these two tables, we first observe optimal orders (i.e. $(k+1)$-th order) for the L^2 error $\|e\|_0$, for both the LSV and RRSV methods, which confirm Theorem 5.1. Moreover, from Table 7.1, we observe superconvergence rates of $(2k)$-th order for the numerical cell averages and numerical approximation at the downwind points (i.e. e_c and e_n), $(k+2)$-th order for the numerical solution at the Gauss points (e_g) and

Table 7.1: Errors and convergence rates of the LSV for Example 7.1.

k	N	$\|e\|_0$	Rate	e_c	Rate	e_g	Rate	e_n	Rate	e_d	Rate
1	8	2.4e-0	–	1.5e-1	–	2.7e-1	–	6.5e-2	–	4.3e-1	–
	16	7.0e-1	1.8	4.7e-2	1.7	7.1e-2	2.0	1.4e-2	2.2	1.4e-1	1.7
	32	1.9e-1	1.9	1.5e-2	1.7	2.1e-2	1.7	7.8e-3	0.9	3.8e-2	1.8
	64	4.9e-2	2.0	4.2e-3	1.8	6.1e-3	1.8	2.7e-3	1.5	1.0e-2	1.9
	128	1.3e-2	2.0	1.1e-3	1.9	1.6e-3	1.9	7.7e-4	1.8	2.6e-3	2.0
	256	3.1e-3	2.0	2.9e-4	2.0	4.2e-4	2.0	2.1e-4	1.9	6.5e-4	2.0
2	4	3.0e-0	–	5.6e-2	–	6.7e-1	–	2.3e-2	–	1.3e-0	–
	8	5.0e-1	2.6	4.9e-3	3.5	1.1e-1	2.7	2.3e-2	–	2.8e-1	2.3
	16	6.7e-2	2.9	1.4e-3	1.8	7.9e-3	3.7	2.7e-3	3.1	3.4e-2	3.0
	32	8.5e-3	3.0	1.2e-4	3.6	5.3e-4	3.9	1.7e-4	3.9	4.2e-3	3.0
	64	1.1e-3	3.0	8.6e-6	3.8	3.4e-5	4.0	1.1e-5	3.9	5.2e-4	3.0
	128	1.3e-4	3.0	5.6e-7	3.9	2.2e-6	4.0	7.3e-7	4.0	6.4e-5	3.0
3	4	1.1e-0	–	1.3e-2	–	2.9e-1	–	1.8e-2	–	7.9e-1	–
	8	7.9e-2	3.8	3.4e-4	5.2	1.0e-2	4.8	1.5e-3	3.5	5.7e-2	3.8
	16	5.1e-3	4.0	2.3e-5	3.9	3.8e-4	4.8	3.8e-5	5.4	4.0e-3	3.8
	32	3.2e-4	4.0	4.0e-7	5.9	1.2e-5	5.0	6.3e-7	5.9	2.5e-4	4.0
	64	2.0e-5	4.0	6.7e-9	6.0	3.7e-7	5.0	8.7e-9	6.2	1.6e-5	4.0
	128	1.2e-6	4.0	1.1e-10	6.0	1.2e-8	5.0	1.4e-10	6.0	9.9e-7	4.0

Table 7.2: Errors and convergence rates of the RRSV for Example 7.1.

k	N	$\|e\|_0$	Rate	e_c	Rate	e_r	Rate	e_n	Rate	e_l	Rate
1	8	1.5e-0	–	8.4e-2	–	2.9e-1	–	1.3e-1	–	4.2e-1	–
	16	3.7e-1	2.1	1.7e-2	2.3	4.8e-2	2.6	2.5e-2	2.4	1.2e-1	1.9
	32	9.2e-2	2.0	2.7e-3	2.7	6.7e-3	2.8	3.7e-3	2.8	2.8e-2	2.0
	64	2.3e-2	2.0	3.7e-4	2.9	8.8e-4	2.9	4.9e-4	2.9	6.8e-3	2.1
	128	5.7e-3	2.0	4.8e-5	3.0	1.1e-4	3.0	6.3e-5	3.0	1.6e-3	2.0
	256	1.4e-3	2.0	6.1e-6	3.0	1.4e-5	3.0	7.9e-6	3.0	4.0e-4	2.0
2	4	3.3e-0	–	1.6e-1	–	1.4e-0	–	4.2e-1	–	1.9e-0	–
	8	3.7e-1	3.2	2.0e-2	3.0	1.0e-1	3.8	2.1e-2	4.3	2.2e-1	3.1
	16	4.4e-2	3.0	6.7e-4	4.9	4.7e-3	4.4	1.7e-3	3.7	3.0e-2	2.9
	32	5.4e-3	3.0	1.9e-5	5.2	2.2e-4	4.4	2.4e-5	6.1	3.0e-3	3.3
	64	6.8e-4	3.0	4.5e-7	5.4	1.3e-5	4.0	5.5e-7	5.5	3.7e-4	3.0
	128	8.5e-5	3.0	1.2e-8	5.2	8.2e-7	4.0	1.6e-8	5.1	4.6e-5	3.0
3	4	7.2e-1	–	1.1e-2	–	2.4e-1	–	2.8e-2	–	6.0e-1	–
	8	4.9e-2	4.0	1.5e-4	6.3	6.6e-3	5.2	4.6e-4	5.9	3.9e-2	4.0
	16	3.1e-3	4.0	3.0e-6	5.6	2.2e-4	4.9	3.4e-6	7.1	2.6e-3	3.9
	32	2.0e-4	4.0	2.5e-8	6.9	7.1e-6	5.0	3.5e-8	6.6	1.8e-4	4.0
	64	1.2e-5	4.0	2.0e-10	7.0	2.2e-7	5.0	3.0e-10	6.9	1.1e-5	4.0
	128	7.7e-7	4.0	1.6e-12	7.0	6.9e-9	5.0	2.0e-12	7.2	6.6e-7	4.0

$(k+1)$-th order for the partial derivatives error (e_d) when $k \geq 2$, which confirm our theoretical results of the LSV method in Theorem 6.2. We note that for $k=1$, no superconvergence phenomenon exists for the function value approximation while superconvergence for the derivative approximation is still observed. Similarly, Table 7.2 demonstrates an order of $2k+1$ for $e_c, e_n, k+2$ for the numerical solution at the right Radau points e_r, and $k+1$ for the partial derivatives of the approximation at the interior left Radau lines e_l, respectively. These results are consistent with the theoretical results in Theorem 6.1 for RRSV methods.

Example 7.2. We solve the following problem:

$$\begin{aligned} &u_t + u_x + u_y = 0, \quad (x,y,t) \in [0,1]\times[0,1]\times(0,1],\\ &u(x,y,0) = e^{x+y}, \quad (x,y) \in [0,1]\times[0,1],\\ &u(0,y,t) = e^{y-2t}, \quad \forall\, y \in [0,1],\\ &u(x,0,t) = e^{x-2t}, \quad \forall\, x \in [0,1], \end{aligned} \tag{7.2}$$

which admits the exact solution $u(x,y,t) = e^{x+y-2t}$.

We obtain our meshes by equally dividing the unit square into $N \times N$ rectangles. We test this problem by LSV and RRSV methods using $\mathbb{Q}_k$ polynomials with $k = 1, 2, 3$. Listed in Tables 7.3 and 7.4 are the corresponding errors and convergence rates. We note that the errors e_n, e_c converge to zero and achieve the Matlab machine epsilon rapidly for $k=3$, and thus only the numerical results at some coarse meshes, i.e. $N \leq 16$, are provided.

From Tables 7.3 and 7.4, we observe that, the L^2 error converges to zero with an optimal convergence rate $k+1$ for both LSV and RRSV methods, the convergence rates of e_r and e_g

Table 7.3: Errors and convergence rates of the LSV for Example 7.2.

k	N	$\|e\|_0$	Rate	e_c	Rate	e_g	Rate	e_n	Rate	e_d	Rate
1	2	1.9e-2	–	1.6e-2	–	1.7e-2	–	9.6e-3	–	2.5e-2	
	4	4.6e-3	2.0	3.8e-3	2.1	3.6e-3	2.3	1.9e-3	2.3	5.8e-3	2.1
	8	1.1e-3	2.0	9.1e-4	2.1	7.7e-4	2.2	4.0e-4	2.3	1.4e-3	2.0
	16	2.8e-4	2.0	2.2e-4	2.0	1.8e-4	2.1	9.0e-5	2.2	3.5e-4	2.0
	32	7.0e-5	2.0	5.5e-5	2.0	4.2e-5	2.1	2.1e-5	2.1	8.6e-5	2.0
	64	1.7e-5	2.0	1.4e-5	2.0	1.0e-5	2.0	5.2e-6	2.0	2.1e-5	2.0
2	2	6.3e-4	–	3.4e-5	–	1.9e-4	–	4.3e-5	–	2.5e-3	
	4	7.6e-5	3.0	2.5e-6	3.7	1.1e-5	4.1	1.4e-6	4.9	2.8e-4	3.1
	8	9.3e-6	3.0	1.7e-7	3.9	7.0e-7	4.0	6.7e-8	4.4	3.4e-5	3.1
	16	1.2e-6	3.0	1.1e-8	4.0	4.4e-8	4.0	3.7e-9	4.2	4.1e-6	3.0
	32	1.4e-7	3.0	6.9e-10	4.0	2.8e-9	4.0	2.2e-10	4.0	5.1e-7	3.0
	64	1.8e-8	3.0	4.4e-11	4.0	1.7e-10	4.0	1.4e-11	4.0	6.3e-8	3.0
3	4	1.2e-6	–	1.1e-9	–	1.5e-7	–	5.0e-9	–	7.5e-6	–
	6	2.3e-7	4.0	1.0e-10	5.8	2.0e-8	5.1	3.6e-10	6.5	1.5e-6	4.0
	8	7.3e-8	4.0	1.9e-11	5.8	4.7e-9	5.0	5.9e-11	6.3	4.6e-7	4.0
	10	3.0e-8	4.0	5.1e-12	5.9	1.5e-9	5.0	1.5e-11	6.3	1.9e-7	4.0
	12	1.4e-8	4.0	1.8e-12	5.8	6.1e-10	5.0	4.6e-12	6.3	9.0e-8	4.0
	14	7.8e-9	4.0	6.9e-13	6.2	2.8e-10	5.0	1.8e-12	6.2	4.8e-8	4.0

Table 7.4: Errors and convergence rates of the RRSV for Example 7.2.

k	N	$\|e\|_0$	Rate	e_c	Rate	e_r	Rate	e_n	Rate	e_l	Rate
1	2	8.5e-3	–	1.4e-3	–	3.3e-3	–	3.6e-4	–	1.9e-2	–
	4	2.0e-3	2.1	1.9e-4	2.9	4.2e-4	3.0	4.4e-5	3.0	4.3e-3	2.2
	8	4.8e-4	2.1	2.4e-5	3.0	5.3e-5	3.0	6.0e-6	2.9	1.0e-3	2.1
	16	1.2e-4	2.0	3.0e-6	3.0	6.6e-6	3.0	7.6e-7	3.0	2.5e-4	2.0
	32	2.9e-5	2.0	3.8e-7	3.0	8.3e-7	3.0	9.4e-8	3.0	6.3e-5	2.0
	64	7.2e-6	2.0	4.8e-8	3.0	1.0e-7	3.0	1.2e-8	3.0	1.6e-5	2.0
2	2	3.9e-4	–	3.2e-6	–	1.0e-4	–	8.0e-6	–	1.6e-3	–
	4	4.8e-5	3.0	1.1e-7	4.8	6.1e-6	4.1	1.8e-7	5.5	2.0e-4	3.0
	8	5.9e-6	3.0	3.7e-9	4.9	3.7e-7	4.0	4.5e-9	5.3	2.4e-5	3.0
	16	7.3e-7	3.0	1.3e-10	5.0	2.3e-8	4.0	1.2e-10	5.2	3.0e-6	3.0
	32	9.1e-8	3.0	3.8e-12	5.0	1.4e-9	4.0	3.3e-12	5.2	3.7e-7	3.0
3	4	7.3e-7	–	5.8e-11	–	8.9e-8	–	2.7e-11	–	4.9e-6	–
	5	3.0e-7	4.0	1.2e-11	7.0	2.9e-8	5.0	5.6e-12	7.1	2.0e-6	4.0
	6	1.4e-7	4.0	3.4e-12	7.0	1.2e-8	5.0	1.5e-12	7.1	9.6e-7	4.0
	7	7.7e-8	4.0	1.1e-12	7.2	5.4e-9	5.0	5.2e-13	7.1	5.2e-7	4.0
	8	4.5e-8	4.0	4.5e-13	6.9	2.8e-9	5.0	–	–	3.0e-7	4.0

are both $k+2$, and the order of e_l and e_d are $k+1$, the convergence rate for e_c and e_n can reach $2k$ for LSV methods, while $2k+1$ for RRSV methods. Again these results confirm our theoretical findings in Theorems 5.1, 6.1 and 6.2, which indicates that the results are also valid for the Dirichlet boundary condition.

8. Concluding Remarks

In this work, we have presented a unified analysis of the stability and convergence properties of two arbitrary-order semi-discrete SV methods for 2D linear hyperbolic equations on non-uniform rectangular meshes. Our approach involves reformulating the SV method as a specialized (nonsymmetric) Galerkin method, facilitated by a carefully crafted one-to-one mapping from the trial space to the test space. This methodological departure from traditional matrix-based approaches used for stability analysis over uniform meshes distinguishes our work.

Additionally, we have uncovered and substantiated novel superconvergence properties not previously documented in the literature. Notably, we demonstrate that the RRSV method achieves convergence rates of $(2k+1)$ at downwind points, while the LSV method achieves $(2k)$-th order convergence. Considering that the $2k$-conjecture has only recently been established for other numerical methods such as finite element methods, finite volume methods, and discontinuous Galerkin (DG) methods, our findings represent a significant advancement for SV methods.

While our analysis provides insights into the mathematical underpinnings of the SV method, it also highlights the ongoing challenges and opportunities for further exploration. The theoretical framework developed here may potentially extend to linear problems with variable coefficients or linear hyperbolic systems. However, extending our findings to nonlinear problems, fully discrete SV methods, and SV methods on unstructured meshes poses substantial challenges and forms the basis of our current and future research endeavors.

Appendix A. Proof of Proposition 6.1

In this appendix, we construct a special $\zeta_h \in U_h$ which satisfies the properties in Proposition 6.1. To this end, we first note the decomposition of the term

$$a(u-u_I, w^*) = a(\mathcal{Q}^x u, w^*) + a(\mathcal{Q}^y u, w^*) - a(\mathcal{Q}^x\mathcal{Q}^y u, w^*), \quad \forall\, w \in U_h.$$

In light of the approximation property of $\mathcal{Q}^x$ and $\mathcal{Q}^y$, the term $\mathcal{Q}^x\mathcal{Q}^y u$ is of higher order, comparing to other two terms $\mathcal{Q}^x u, \mathcal{Q}^y u$. Therefore, to achieve our superconvergence goal, we only need to construct correction functions $\zeta_h^x, \zeta_h^y \in U_h$ to correct the first two terms $a(\mathcal{Q}^x u, w^*)$ and $a(\mathcal{Q}^y u, w^*)$ respectively such that

$$\left|a\big(\mathcal{Q}^x u + \zeta_h^x, w^*\big)\right| + \left|a\big(\mathcal{Q}^y u + \zeta_h^y, w^*\big)\right| \lesssim h^{k+1+r}\|w\|_0$$

for some positive r. Then by defining $\zeta_h = \zeta_h^x + \zeta_h^y$, we have

$$|a(u-\tilde{u}_I, w^*)| = \left|a\big(\mathcal{Q}^x u + \zeta_h^x, w^*\big) + a\big(\mathcal{Q}^y u + \zeta_h^y, w^*\big) - a(\mathcal{Q}^x\mathcal{Q}^y u, w^*)\right| \lesssim h^{k+1+r}\|w\|_0.$$

This is the basic idea on how to construct the correction functions.

We next demonstrate how to construct ζ_h^x, ζ_h^y, respectively.

1. Correction of the term $a(\mathcal{Q}^y u, w^*)$.

Let $\zeta_0 = \mathcal{Q}^y u$ and for $1 \le i \le k-1$, we define

$$\zeta_i = \mathcal{I}_h^y \partial_y^{-1}(\partial_t + \partial_x)\zeta_{i-1} = \big(\mathcal{I}_h^y \partial_y^{-1}\big)^i (\partial_t + \partial_x)^i \zeta_0, \tag{A.1}$$

where $\mathcal{I}_h^y$ is defined in (5.2). Apparently, $\zeta_i \in \mathbb{P}_k(y)$. In addition,

$$\zeta_i\big(x, g_1^{\tau,y}\big) = \int_{g_0^{\tau,y}}^{g_1^{\tau,y}} (\partial_t + \partial_x)\zeta_{i-1}\, dy,$$

$$\zeta_i\big(x, g_{j+1}^{\tau,y}\big) - \zeta_i\big(x, g_j^{\tau,y}\big) = \int_{g_j^{\tau,y}}^{g_{j+1}^{\tau,y}} (\partial_t + \partial_x)\zeta_{i-1}\, dy.$$

For all $\tau_{p,q} \in \mathcal{T}_h$, multiplying both sides of the above equation by $w_{l,j}^{\tau,*}$ and summing up all l, j yields

$$(\partial_y \zeta_i, w^*)_\tau + \int_{\tau_p^x} \zeta_i\big(x, y_{q-\frac12}^+\big) w^* dx = \big((\partial_t + \partial_x)\zeta_{i-1}, w^*\big)_\tau. \tag{A.2}$$

For any $v, w \in H_h$, we define

$$\langle w, v\rangle_{\tau_q^y} = \int_{\tau_q^y} (vw)\, dy, \quad \langle w, v\rangle_{\tau_p^x} = \int_{\tau_p^x} (vw)\, dx, \quad (p,q) \in \mathbb{Z}_m \times \mathbb{Z}_n.$$

We have the following estimates for $\zeta_i, i \le k-1$.

Lemma A.1. *For all $\tau = \tau_{p,q} \in \mathcal{T}_h$ and $i \le k-1$, there hold*

$$\left\|(\partial_t + \partial_x)^l \zeta_i\right\|_{0,\tau} \lesssim h^{i+r}\left\|(\partial_t + \partial_x)^l u\right\|_{r+i,\tau}, \quad |(\zeta_i, 1)_\tau| \lesssim h^{2k+1}\|u\|_{2k+1,\tau}, \tag{A.3}$$

$$\left|\zeta_i\big(x, y_{q+\frac12}^-\big)\right| + \left(\int_{\tau_p^x} \left|\zeta_i\big(x, y_{q+\frac12}^-\big)\right|^2 dx\right)^{\frac12} \lesssim h^{k+r-\frac12}\|u\|_{k+r-1,\tau}, \quad r \le k+1. \tag{A.4}$$

Proof. First, by the boundedness of $\mathcal{I}_h^y$,

$$\big\|(\partial_t+\partial_x)^l\zeta_i\big\|_{0,\tau}\lesssim\big\|\partial_y^{-1}(\partial_t+\partial_x)^{l+1}\zeta_{i-1}\big\|_{0,\tau}\lesssim h\big\|(\partial_t+\partial_x)^{l+1}\zeta_{i-1}\big\|_{0,\tau}.$$

By using the method of recurrence and the approximation property of $\mathcal{Q}^y$,

$$\big\|(\partial_t+\partial_x)^l\zeta_i\big\|_{0,\tau}\lesssim h^i\big\|(\partial_t+\partial_x)^{l+i}\zeta_0\big\|_{0,\tau}\lesssim h^{r+i}\big\|(\partial_t+\partial_x)^l u\big\|_{r+i,\tau},\quad\forall\, r\le k+1.$$

This finishes the proof of the first inequality of (A.3).

Second, for any function $v\in H_h$, note that

$$\big\langle\big(\mathcal{I}_h^y\partial_y^{-1}\big)^i v,1\big\rangle_{\tau_q^y}=\sum_{j=1}^k\big(\mathcal{I}_h^y\partial_y^{-1}\big(\mathcal{I}_h^y\partial_y^{-1}\big)^{i-1}v\big)\big(g_j^{\tau,y}\big)=\sum_{j=1}^k\big(\partial_y^{-1}\big(\mathcal{I}_h^y\partial_y^{-1}\big)^{i-1}v\big)\big(g_j^{\tau,y}\big).$$

Using the fact that the k-point Gauss numerical quadrature is exact for polynomial of degree not more than $2k$, we have

$$\big\langle\big(\mathcal{I}_h^y\partial_y^{-1}\big)^i v,1\big\rangle_{\tau_q^y}=\big\langle\partial_y^{-1}\big(\mathcal{I}_h^y\partial_y^{-1}\big)^{i-1}v,1\big\rangle_{\tau_q^y}=-\big\langle\big(\mathcal{I}_h^y\partial_y^{-1}\big)^{i-1}v,y-y_{q+\frac12}\big\rangle_{\tau_q^y},$$

where in the last step, we have used the integration by parts. Using the method of recurrence again, we obtain

$$\big\langle\big(\mathcal{I}_h^y\partial_y^{-1}\big)^i v,1\big\rangle_{\tau_q^y}=\frac{(-1)^{i-1}}{(i-1)!}\big\langle\mathcal{I}_h^y\partial_y^{-1}v,\big(y-y_{q+\frac12}\big)^{i-1}\big\rangle_{\tau_q^y}.$$

Choosing $v=(\partial_t+\partial_x)^i\zeta_0$ and using the equation $\mathcal{I}_h^y=I-\mathcal{Q}^y$ in the above equation, we have

$$\langle\zeta_i,1\rangle_{\tau_q^y}=\frac{(-1)^{i-1}}{(i-1)!}\big\langle(I-\mathcal{Q}^y)\big(\partial_y^{-1}(\partial_t+\partial_x)^i\zeta_0\big),\big(y-y_{q+\frac12}\big)^{i-1}\big\rangle_{\tau_q^y}.$$

In light of (6.14), we have for all $r\le k-i$ that

$$\begin{aligned}&\big|\big\langle\mathcal{Q}^y\partial_y^{-1}(\partial_t+\partial_x)^i\zeta_0,\big(y-y_{q+\frac12}\big)^{i-1}\big\rangle_{\tau_q^y}\big|\\ \lesssim{}& h^{k+1+r}\big\|\partial_y^{-1}(\partial_t+\partial_x)^i\zeta_0\big\|_{k+1+r,\tau_q^y}\big\|\big(y-y_{q+\frac12}\big)^{i-1}\big\|_{0,\tau_q^y}\\ \lesssim{}& h^{k+i+r+\frac12}\|u\|_{k+i+r,\tau_q^y}.\end{aligned}$$

Similarly, by using the integration by parts,

$$\begin{aligned}&\big\langle\partial_y^{-1}(\partial_t+\partial_x)^i\zeta_0,\big(y-y_{q+\frac12}\big)^{i-1}\big\rangle_{\tau_q^y}\\ ={}&\big\langle\partial_y^{-1}\mathcal{Q}^y(\partial_t+\partial_x)^i u,\big(y-y_{q+\frac12}\big)^{i-1}\big\rangle_{\tau_q^y}\\ ={}&-\frac{1}{i}\big\langle\mathcal{Q}^y(\partial_t+\partial_x)^i u,\big(y-y_{q+\frac12}\big)^{i}\big\rangle_{\tau_q^y},\end{aligned}$$

which yields, together with the approximation property of $\mathcal{Q}^y$ that

$$\big|\big\langle\partial_y^{-1}(\partial_t+\partial_x)^i\zeta_0,\big(y-y_{q+\frac12}\big)^{i-1}\big\rangle_{\tau_q^y}\big|\lesssim h^{k+i+r+\frac12}\|u\|_{k+i+r,\tau_q^y},\quad r\le k-i.$$

Consequently,

$$|\langle\zeta_i,1\rangle_{\tau_q^y}|\lesssim h^{k+i+r+\frac12}\|u\|_{k+i+r,\tau_q^y},\quad r\le k-i.\tag{A.5}$$

Then by the Cauchy-Schwarz inequality,

$$|(\zeta_i,1)_\tau| \lesssim h^{k+i+r+1}\|u\|_{k+i+r,\tau}, \quad r\le k-i.$$

The second inequality of (A.3) then follows by taking $r=k-i$.

To prove (A.4), we note that

$$\begin{aligned}\zeta_i\big(x,y^-_{q+\frac12}\big) &= \sum_{j=1}^{k}\big(\zeta_i\big(x,g^{\tau,y}_{j+1}\big)-\zeta\big(x,g^{\tau,y}_j\big)\big)+\zeta\big(x,g^{\tau,y}_1\big)\\ &=\sum_{j=1}^{k}\int_{g^{\tau,y}_j}^{g^{\tau,y}_{j+1}}\zeta_{i-1}(x,y)dy+\int_{g^{\tau,y}_0}^{g^{\tau,y}_1}\zeta_{i-1}(x,y)dy=\langle\zeta_{i-1},1\rangle_{\tau^y_q}. \end{aligned}\tag{A.6}$$

Then (A.4) follows directly from (6.14), (A.5) and the Cauchy-Schwarz inequality. □

Define

$$\zeta^y(x,y)=\sum_{i=1}^{k-1}(-1)^i\zeta_i(x,y), \quad \zeta^y_h(x,y)=\mathcal{I}^x_k\zeta^y(x,y). \tag{A.7}$$

Here $\mathcal{I}^x_k\zeta\in\mathbb{P}_k(x)$ denotes the Gauss-Lobatto interpolation function of ζ along the x direction. That is, in each element τ^x_p,

$$\mathcal{I}^x_k\zeta(x_{p,j},y)=\zeta(x_{p,j},y), \quad j\in\mathbb{Z}_{k+1},$$

where $x_{p,j}$ are zeros of the Lobatto polynomial in τ^x_p. Note that ζ_i are continuous about x, and thus $\zeta^y_h\in U_h$ is also continuous about the variable x.

Lemma A.2. *Let ζ^y_h be defined in* (A.7)*. Then for all $\tau=\tau_{p,q}\in\mathcal{T}_h$,*

$$\begin{aligned}&\big\|\zeta^y_h\big\|_{0,\tau}\lesssim h^{k+2}\|u\|_{k+2,\tau},\\ &h\big|\zeta^y_h\big(x^-_{p+\frac12},y^-_{q+\frac12}\big)\big|+\big|\big(\zeta^y_h,1\big)_\tau\big|\lesssim h^{2k+1}\|u\|_{2k+1,\tau}.\end{aligned}\tag{A.8}$$

Moreover, there holds for all $w\in U_h$,

$$\big|a\big(\mathcal{Q}^y u+\zeta^y_h,w^*\big)\big|\lesssim h^{k+1+r}\|u\|_{k+r+2}\|w\|_0, \quad \forall\, r\le k-1. \tag{A.9}$$

Proof. The first inequality of (A.8) follows directly from (A.7) and by taking $r=k+2-i$ in (A.3). By using the first inequality of (A.4) and the trace inequality,

$$\big|\zeta^y_h\big(x^-_{p+\frac12},y^-_{q+\frac12}\big)\big|\lesssim h^{2k}(\|u\|_{2k,\tau}+h\|u\|_{2k+1,\tau})\lesssim h^{2k}\|u\|_{2k+1,\tau}.$$

We next estimate the cell average error. In light of the second inequality of (A.3), we get

$$\big|(\zeta^y,1)_\tau\big|\lesssim\sum_{i=1}^{k-1}|(\zeta_i,1)_\tau|\lesssim h^{2k+1}\|u\|_{2k+1,\tau}.$$

Consequently,

$$\big|\big(\zeta^y_h,1\big)_\tau\big|\lesssim\big|\big(\zeta^y-\zeta^y_h,1\big)_\tau\big|+\big|(\zeta^y,1)_\tau\big|\lesssim h^{k+2}\big\|\partial^{k+1}_x\zeta^y\big\|_{0,\tau}+h^{2k+1}\|u\|_{2k+1,\tau}.$$

Then the second inequality of (A.8) follows directly by taking $i+r=k-1, k\geq 2$ in the first inequality of (A.3).

To show (A.9), we first divide the term in the right hand side into two parts, i.e.

$$a_\tau\big(\mathcal{Q}^y u+\zeta_h^y, w^*\big)=a_\tau(\mathcal{Q}^y u+\zeta^y, w^*)-a_\tau\big(\zeta^y+\zeta_h^y, w^*\big). \tag{A.10}$$

We next estimate the two parts respectively. By (5.7), we have

$$a_\tau(\mathcal{Q}^y u, w^*)=\big((\partial_t+\partial_x)\zeta_0, w^*\big)_\tau.$$

Noticing that ζ^y is continuous about x, we have from (3.9),

$$a_\tau(\zeta^y, w^*)=\big((\partial_t+\partial_x+\partial_y)\zeta^y, w^*\big)_\tau+\int_{\tau_p^x}[\zeta^y]\big(x, y_{q-\frac12}\big)w^* dx.$$

Combining the last two equations and (A.2), we get

$$a_\tau(\mathcal{Q}^y u+\zeta^y, w^*)=(-1)^k\big((\partial_t+\partial_x)\zeta_{k-1}, w^*\big)_\tau-\sum_{i=1}^{k-1}\int_{\tau_p^x}\zeta_i\big(x, y^-_{q-\frac12}\big)w^* dx,$$

which yields, together with the first inequality of (A.3) and (A.4),

$$\begin{aligned}|a_\tau(\mathcal{Q}^y u+\zeta^y, w^*)| &\lesssim h^{k-1+r}\|u\|_{k+r,\tau}\|w^*\|_{0,\tau}+h^{k+r-\frac12}\|u\|_{k+r-1,\tau}\|w^*\|_{0,\tau_q^y}\\ &\lesssim h^{k+r-1}\|u\|_{k+r,\tau}\|w\|_{0,\tau}, \quad r\leq k+1,\end{aligned} \tag{A.11}$$

where in the last step, we have used (5.1) and the inverse inequality.

On the other hand, we conclude from (3.9) that

$$a_\tau\big(\zeta^y-\mathcal{I}_k^x\zeta^y, w^*\big)=\big((\partial_t+\partial_x+\partial_y)\big(\zeta^y-\mathcal{I}_k^x\zeta^y\big), w^*\big)_\tau+\int_{\tau_p^x}\big[\zeta^y-\mathcal{I}_k^x\zeta^y\big]\big(x, y_{q-\frac12}\big)w^* dx.$$

Summing up all τ and using the trace inequality and (5.1),

$$\begin{aligned}\big|a\big(\zeta^y-\mathcal{I}_k^x\zeta^y, w^*\big)\big| &\lesssim \big(h^{-1}\big\|\zeta^y-\mathcal{I}_k^x\zeta^y\big\|_0+\big\|\zeta^y-\mathcal{I}_k^x\zeta^y\big\|_1+\big\|\partial_t\big(\zeta^y-\mathcal{I}_k^x\zeta^y\big)\big\|_0\big)\|w\|_0\\ &\lesssim h^{k+r}\|u\|_{k+r+1}\|w\|_0, \quad \forall\, r\leq k.\end{aligned} \tag{A.12}$$

Substituting (A.11) and (A.12) into (A.10), we get (A.9) directly. □

2. Correction of $a(\mathcal{Q}^x u, w^*)$.

Similar to that for $a(\mathcal{Q}^y u, w^*)$, we correct $a(\mathcal{Q}^x u, w^*)$ by defining $\bar\zeta_0=\mathcal{Q}^x u$ and defining for all $1\leq i\leq k-1$,

$$\bar\zeta_i=\mathcal{I}_h^x\partial_x^{-1}(\partial_t+\partial_y)\bar\zeta_{i-1}=\big(\mathcal{I}_h^x\partial_x^{-1}\big)^i(\partial_t+\partial_y)^i\bar\zeta_0. \tag{A.13}$$

We further denote

$$\zeta^x=\sum_{i=1}^k(-1)^i\bar\zeta_i, \quad \zeta_h^x=\sum_{i=1}^k\mathcal{I}_k^y\bar\zeta_i. \tag{A.14}$$

Following same arguments in Theorem A.2, we can show that

$$\begin{aligned}&\big\|\zeta_h^x\big\|_{0,\tau}\lesssim h^{k+2}\|u\|_{k+2,\tau},\\ &h\big|\zeta_h^x\big(x^-_{p+\frac12}, y^-_{q+\frac12}\big)\big|+\big|\big(\zeta_h^x, 1\big)_\tau\big|\lesssim h^{2k+1}\|u\|_{2k+1,\tau},\end{aligned} \tag{A.15}$$

$$\big|a\big(\mathcal{Q}^x u+\zeta_h^x, w^*\big)\big|\lesssim h^{k+r+1}\|u\|_{k+2+r}\|w\|_0, \quad \forall\, r\leq k-1, \quad w\in U_h. \tag{A.16}$$

3. Proof of Proposition 6.1.

Let

$$\zeta_h = \zeta_h^y + \zeta_h^x,$$

where ζ_h^y, ζ_h^x are given in (A.7) and (A.14), respectively. Next we show the above ζ_h satisfies all the properties given in Proposition 6.1.

Proof. First, (6.6) follows directly from (A.8) and (A.15). Secondly, we use the estimates in (5.5) to derive

$$\|\mathcal{Q}^x\mathcal{Q}^y u\|_{0,\tau} \lesssim h^{k+1+r}\|u\|_{k+1+r,\tau}, \quad r \le k+1.$$

In light of (5.8), we have

$$|a(\mathcal{Q}^x\mathcal{Q}^y u, w^*)| \lesssim h^{k+l}\|u\|_{k+l+1}\|w\|_0, \quad l \le k,$$

which yields, together with (A.9) and (A.16),

$$\begin{aligned}\big|a\big(u-u_I+\zeta_h, w^*\big)\big| &= \big|a\big(\mathcal{Q}^x+\zeta_h^x, w^*\big) + a\big(\mathcal{Q}^y u+\zeta_h^y, w^*\big) - a\big(\mathcal{Q}^x\mathcal{Q}^y u, w^*\big)\big| \\ &\lesssim h^{k+1+r}\|u\|_{k+2+r}\|w\|_0, \quad r \le k-1.\end{aligned}$$

The proof of (6.7) is complete. □

We end this section by discussing how to implement the initial and boundary discretizations. Note that $u_t + u_x + u_y = 0$, we have

$$(\partial_t+\partial_x)^i u(x,y,0) = (-1)^i\partial_y^i u_0, \quad (\partial_t+\partial_y)^i u(x,y,0) = (-1)^i\partial_x^i u_0.$$

Then the process of the initial discretization can be divided into the following steps:

1. Compute the interpolation functions $\mathcal{I}_h^y((-1)^i\partial_y^i u_0)$ to $\mathcal{I}_h^x((-1)^i\partial_x^i u_0)$ obtain $(\partial_t+\partial_x)^i\zeta_0$ and $(\partial_t+\partial_y)^i\bar{\zeta}_0$.
2. Obtain ζ_i and $\bar{\zeta}_i$ from (A.1) and (A.13).
3. Calculating the Gauss-Lobatto interpolation function of ζ_i and $\bar{\zeta}_i$, and then derive ζ_h^y, ζ_h^x from (A.7) and (A.14).
4. Figure out $u_h(x,y,0) = (u_0)_I - \zeta_h^y - \zeta_h^x$.

The process of the boundary discretization follows the same way, with u_0 replaced by φ_1, φ_2, and thus we omit it here.

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