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A NOVEL SECOND ORDER SCHEME WITH ONE STEP FOR FORWARD BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we design a novel explicit second order scheme with one step for forward backward stochastic differential equations, and the Crank-Nicolson scheme is a specific case of our proposed framework. We first establish a rigorous stability result, and then we derive precise error estimates. Moreover, we confirm that the proposed novel scheme is second order convergent. The theoretical results for the proposed methods are supported by numerical experiments.

Mathematics subject classification: 60H35, 60H10, 65C05.
Key words: Forward backward stochastic differential equations, Second order one step scheme, Numerical analysis.

1. Introduction

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ represents the natural filtration generated by a standard q-dimensional Brownian motion. Within this probability space, this paper does not prioritize the regularity of the solutions to the forward backward stochastic differential equations (FBSDEs) but rather focuses on probabilistic numerical methods for solving them

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, & (SDE) \\ Y_t = \Phi(X_T) + \displaystyle\int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, & (BSDE) \end{cases} \tag{1.1}$$

where $T > 0$ denotes a fixed terminal time; $X_0 \in \mathbb{R}^q$ provides the initial condition for the SDE; $\Phi(X_T) \in \mathbb{R}^n$ specifies the terminal condition for the BSDE; $b : [0, T] \times \mathbb{R}^q \to \mathbb{R}^q$ is the drift coefficient; $\sigma : [0, T] \times \mathbb{R}^q \to \mathbb{R}^{q \times q}$ represents the diffusion matrix; $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times q} \to \mathbb{R}^n$ is the generator function; and $\Phi : \mathbb{R}^q \to \mathbb{R}^n$ is a real-valued function. To facilitate analysis, we assume that the functions b, σ, f, and Φ adhere to the following conditions:

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(i) The functions b and $\sigma \in C_b^1$, where C_b^k denotes the space of continuous functions with uniformly bounded derivatives up to order k. In particular, we assume that

$$\sup_{0\le t\le T}\{|b(t,0)| + |\sigma(t,0)|\} \le L$$

with the non-negative constant L representing the Lipschitz constants for all relevant functions.

(ii) We assume that σ satisfies

$$\sigma(t,x)\sigma^\top(t,x) \ge \frac{I_q}{L}, \quad \forall\,(t,x)\in[0,T]\times\mathbb{R}^q.$$

(iii) b, σ, f, and $\Phi \in \mathcal{C}_L$, where $\mathcal{C}_L$ denotes the set of uniformly Lipschitz continuous functions with respect to the spatial variables. Additionally, we assume

$$\sup_{0\le t\le T}\{|f(t,0,0)| + |\Phi(0)|\} \le L.$$

Under these conditions, the FBSDEs (1.1) is well-posed. Additionally, by taking the conditional expectation on both sides of the backward component, the integrands are shown to be continuous with respect to time.

The well-posedness of the FBSDEs (1.1), including both existence and uniqueness of the solutions, is established in [20]. Moreover, the papers [21, 22] show that the solution (Y_t, Z_t) to the FBSDEs (1.1) can be expressed as

$$Y_t = u(t,X_t), \quad Z_t = \nabla u(t,X_t)\sigma(t,X_t), \quad \forall\, t\in[0,T], \tag{1.2}$$

where ∇u denotes the gradient of $u(t,x)$ with respect to x; $u(t,x) \in C_b^{1,2}([0,T]\times\mathbb{R}^q)$ is the solution of the following nonlinear parabolic partial differential equation:

$$\mathcal{L}^{(0)}u(t,x) + f\big(u(t,x), \nabla u(t,x)\sigma(t,x)\big) = 0 \tag{1.3}$$

with the terminal condition $u(T,x) = \Phi(x)$; and

$$\mathcal{L}^{(0)} = \frac{\partial}{\partial t} + \sum_{i=1}^{q} b_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{q}(\sigma\sigma^\top)_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

The interest in FBSDEs has grown among researchers due to their extensive applications across disciplines, including fields such as mathematical finance [6, 8], stochastic optimal control problems [28, 30], risk measures [17, 25], and more within the social and natural sciences. However, finding analytical solutions to FBSDEs is rarely feasible because the majority of these equations are complex. Thus, numerical algorithms have to be constructed to approximate their solutions. Consequently, significant work has been devoted to the numerical solutions of FBSDEs. Specifically, the Euler schemes, which have been shown to achieve a convergence order of 1/2, are discussed in [4, 9, 16, 23, 27]. For higher orders of convergence, the multistep schemes, which generalize the Euler schemes by utilizing more previously known information, have been developed and analyzed in [1–3, 5, 10–15, 24]. However, since the multistep schemes require multiple previous solution points to calculate the next one, they necessitate several initial values, increasing computational complexity.

To address the limitations of the multistep schemes, which require multiple initial values, and to maintain higher orders of convergence, this paper introduces a novel one step second order numerical scheme that generalizes the Euler schemes. The proposed scheme, developed by a predictor-corrector approach, is entirely explicit for Y and Z. When solving FBSDEs, the explicit schemes compute the next time step directly from known values, bypassing the need to solve the nonlinear equations required by implicit methods, thereby enabling faster computations. Additionally, the explicit schemes avoid the complex matrix operations and iterative processes inherent in implicit methods, significantly reducing computational complexity, particularly in high-dimensional problems. Moreover, a stability result will be established to provide precise error estimates and confirm that our method achieves second order convergence. The key distinctions between this paper and the references [7,29] are that our one step second order scheme is fully explicit for both Y and Z, whereas the schemes in [7,29] are explicit for Y while being implicit for Z. Furthermore, when $\alpha = 1$, our approach simplifies to the Crank-Nicolson method discussed in [7,29], highlighting the Crank-Nicolson method as a particular case within our framework.

The structure of this paper is as follows: Section 2 outlines the explicit one step method used to solve the FBSDEs (1.1). In Section 3, we conduct a comprehensive stability analysis of the proposed method. Section 4 focuses on evaluating the local truncation errors for Y and Z, and discusses the second order convergence of the numerical approach. In Section 5, numerical experiments are presented to verify the theoretical findings. Lastly, Section 6 concludes the paper.

2. Novel One Step Discretization of the BSDE

This section presents the time-discretization of the BSDE in (1.1) using a novel one-step scheme. To achieve this, a uniform discrete mesh $\pi = \{t_0, t_1, \cdots, t_N\}$ is defined over the time interval $[0, T]$, with the step size $h_t = T/N$, where $N \in \mathbb{N}^+$. The BSDE in (1.1) is thus expressed at the mesh points t_i as follows:

$$Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f_s ds - \int_{t_i}^{t_{i+1}} Z_s dW_s, \tag{2.1}$$

where $f_s = f(s, Y_s, Z_s)$. Applying conditional expectations to both sides of (2.1), we get

$$Y_{t_i} = \mathbb{E}_{t_i}^x[Y_{t_{i+1}}] + \int_{t_i}^{t_{i+1}} \mathbb{E}_{t_i}^x[f_s] ds, \tag{2.2}$$

where

$$\mathbb{E}_{t_i}^x[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_{t_i}, X_{t_i} = x].$$

As explained in Section 1, the integrand $\mathbb{E}_{t_i}^x[f_s]$ in the equation above depends deterministically on s. Thus, the one step method can be used to approximate it, namely

$$\int_{t_i}^{t_{i+1}} \mathbb{E}_{t_i}^x[f_s] ds = h_t \mathbb{E}_{t_i}^x \left[\frac{1}{2\alpha} f_{t_{i+1-\alpha}} + \left(1 - \frac{1}{2\alpha}\right) f_{t_{i+1}} \right] + R_{yf}^i, \tag{2.3}$$

where

$$f_{t_{i+1-\alpha}} = f(t_{i+1-\alpha}, Y_{t_{i+1-\alpha}}, Z_{t_{i+1-\alpha}}), \quad \alpha \in (0, 1],$$

and

$$R^i_{yf} = \int_{t_i}^{t_{i+1}} \mathbb{E}^x_{t_i}[f_s]ds - h_t\mathbb{E}^x_{t_i}\left[\frac{1}{2\alpha}f_{t_{i+1-\alpha}} + \left(1-\frac{1}{2\alpha}\right)f_{t_{i+1}}\right].$$

Inserting (2.3) into (2.2), we obtain

$$Y_{t_i} = \mathbb{E}^x_{t_i}\left[Y_{t_{i+1}} + \frac{h_t}{2\alpha}f_{t_{i+1-\alpha}} + h_t\left(1-\frac{1}{2\alpha}\right)f_{t_{i+1}}\right] + R^i_{yf}. \tag{2.4}$$

The values of $(Y_{t_{i+1-\alpha}}, Z_{t_{i+1-\alpha}})$ are at non-grid points and unknown for $i = 0, 1, \ldots, N-1$. Calculating Y_{t_i} with an explicit numerical scheme requires approximating $Y_{t_{i+1-\alpha}}$ and $Z_{t_{i+1-\alpha}}$ in $f_{t_{i+1-\alpha}}$. Firstly, we approximate the value of $Y_{t_{i+1-\alpha}}$ in $f_{t_{i+1-\alpha}}$ and then the computation expression with respect to $Z_{t_{i+1-\alpha}}$ will be given soon. We approximate the value of $Y_{t_{i+1-\alpha}}$ by the explicit Euler scheme, namely

$$Y_{t_{i+1-\alpha}} = \mathbb{E}^x_{t_{i+1-\alpha}}[Y_{t_{i+1}} + \alpha h_t f_{t_{i+1}}] + \widetilde{R}^i_{yf}, \tag{2.5}$$

where

$$\begin{aligned}
&\mathbb{E}^x_{t_{i+1-\alpha}}[\cdot] = \mathbb{E}[\,\cdot\,|\,\mathcal{F}_{t_{i+1-\alpha}}, X_{t_{i+1-\alpha}} = x],\\
&f_{t_i} = f(t_i, Y_{t_i}, Z_{t_i}), \quad i = 0, 1, \ldots, N-1,\\
&\widetilde{R}^i_{yf} = \int_{t_{i+1-\alpha}}^{t_{i+1}} \mathbb{E}^x_{t_{i+1-\alpha}}[f_s]ds - \alpha h_t\mathbb{E}^x_{t_{i+1-\alpha}}[f_{t_{i+1}}].
\end{aligned}$$

Thus, the computation expression with respect to $Y_{t_{i+1-\alpha}}$ is

$$Y^\pi_{i+1-\alpha} = \mathbb{E}^x_{t_{i+1-\alpha}}\left[Y^\pi_{i+1} + \alpha h_t f^\pi_{i+1}\right], \tag{2.6}$$

where $f^\pi_i = f(t_i, Y^\pi_i, Z^\pi_i)$.

Then, we derive the expression for Z. By multiplying (2.1) with

$$\Delta W^\top_{i,i+1} := (W_{t_{i+1}} - W_{t_i})^\top$$

and applying the conditional expectation, we obtain

$$0 = \mathbb{E}^x_{t_i}\left[Y_{t_{i+1}}\Delta W^\top_{i,i+1}\right] + \int_{t_i}^{t_{i+1}} \mathbb{E}^x_{t_i}\left[f_s\Delta W^\top_{t_i,s}\right]ds - \int_{t_i}^{t_{i+1}} \mathbb{E}^x_{t_i}[Z_s]ds, \tag{2.7}$$

where $\Delta W_{t_i,s} = W_s - W_{t_i}$. Similarly, we approximate the integral terms $\mathbb{E}^x_{t_i}[f_s\Delta W^\top_{t_i,s}]$ and $\mathbb{E}^x_{t_i}[Z_s]$ on the right-hand side of (2.7) using the same method as for calculating Y_{t_i}, namely

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} \mathbb{E}^x_{t_i}\left[f(s, Y_s, Z_s)\Delta W^\top_{t_i,s}\right]ds\\
&= h_t\mathbb{E}^x_{t_i}\left[\frac{1}{2\alpha}f_{t_{i+1-\alpha}}\Delta W^\top_{i,i+1-\alpha} + \left(1-\frac{1}{2\alpha}\right)f_{t_{i+1}}\Delta W^\top_{i,i+1}\right] + R^i_{z,1}, \qquad (2.8)\\
&\int_{t_i}^{t_{i+1}} \mathbb{E}^x_{t_i}[Z_s]ds = \frac{h_t}{2}\mathbb{E}^x_{t_i}[Z_{t_i} + Z_{t_{i+1}}] + R^i_{z,2}, \qquad (2.9)
\end{aligned}$$

where

$$\begin{aligned}
&\Delta W_{i,i+1-\alpha} = W_{t_{i+1-\alpha}} - W_{t_i},\\
&R^i_{z,1} = \int_{t_i}^{t_{i+1}} \mathbb{E}^x_{t_i}\left[f(s, Y_s, Z_s)\Delta W^\top_{t_i,s}\right]ds - h_t\mathbb{E}^x_{t_i}\left[\frac{1}{2\alpha}f_{t_{i+1-\alpha}}\Delta W^\top_{i,i+1-\alpha} + \left(1-\frac{1}{2\alpha}\right)f_{t_{i+1}}\Delta W^\top_{i,i+1}\right],\\
&R^i_{z,2} = \int_{t_i}^{t_{i+1}} \mathbb{E}^x_{t_i}[Z_s]ds - \frac{h_t}{2}\mathbb{E}^x_{t_i}[Z_{t_i} + Z_{t_{i+1}}].
\end{aligned}$$

Plugging (2.8) and (2.9) into (2.7), we deduce

$$
\begin{aligned}
0 = \mathbb{E}_{t_i}^x \left[Y_{t_{i+1}} \Delta W_{i,i+1}^\top \right] + h_t \mathbb{E}_{t_i}^x \left[\frac{1}{2\alpha} f_{t_{i+1-\alpha}} \Delta W_{i,i+1-\alpha}^\top + \left(1 - \frac{1}{2\alpha}\right) f_{t_{i+1}} \Delta W_{i,i+1}^\top \right] \\
- \frac{h_t}{2} \mathbb{E}_{t_i}^x [Z_{t_i} + Z_{t_{i+1}}] + R_z^i,
\end{aligned}
$$

where $R_z^i = R_{z,1}^i - R_{z,2}^i$. Furthermore,

$$
\begin{aligned}
Z_{t_i} = \mathbb{E}_{t_i}^x \left[\frac{2}{h_t} Y_{t_{i+1}} \Delta W_{i,i+1}^\top \right] + 2\mathbb{E}_{t_i}^x \left[\frac{1}{2\alpha} f_{t_{i+1-\alpha}} \Delta W_{i,i+1-\alpha}^\top + \left(1 - \frac{1}{2\alpha}\right) f_{t_{i+1}} \Delta W_{i,i+1}^\top \right] \\
- \mathbb{E}_{t_i}^x [Z_{t_{i+1}}] + \frac{2}{h_t} R_z^i.
\end{aligned} \tag{2.10}
$$

Analogously, we approximate the value of $Z_{t_{i+1-\alpha}}$ in $f_{t_{i+1-\alpha}}$ by the Euler scheme as below

$$
\alpha h_t Z_{t_{i+1-\alpha}} = \mathbb{E}_{t_{i+1-\alpha}}^x \left[Y_{t_{i+1}} \Delta W_{i+1-\alpha,i+1}^\top + \alpha h_t f_{t_{i+1}} \Delta W_{i+1-\alpha,i+1}^\top \right] + R_{z,3}^i, \tag{2.11}
$$

where

$$
\begin{aligned}
&\Delta W_{i+1-\alpha,i+1} = W_{t_{i+1}} - W_{t_{i+1-\alpha}}, \\
&\Delta W_{i+1-\alpha,s} = W_s - W_{t_{i+1-\alpha}}, \\
&R_{z,3}^i = \mathbb{E}_{t_{i+1-\alpha}}^x \left[R_{z,31}^i - R_{z,32}^i \right], \\
&R_{z,31}^i = \int_{t_{i+1-\alpha}}^{t_{i+1}} f_s \Delta W_{i+1-\alpha,s}^\top ds - \alpha h_t f_{t_{i+1}} \Delta W_{i+1-\alpha,i+1}^\top, \\
&R_{z,32}^i = \int_{t_{i+1-\alpha}}^{t_{i+1}} Z_s ds - \alpha h_t Z_{t_{i+1-\alpha}}.
\end{aligned}
$$

Thus, the computation expression with respect to $Z_{t_{i+1-\alpha}}$ is

$$
Z_{i+1-\alpha}^\pi = \mathbb{E}_{t_{i+1-\alpha}}^x \left[\frac{1}{\alpha h_t} Y_{i+1}^\pi \Delta W_{i+1-\alpha,i+1}^\top + f_{i+1}^\pi \Delta W_{i+1-\alpha,i+1}^\top \right]. \tag{2.12}
$$

Combining (2.4) with (2.5), (2.11), we have

$$
Y_{t_i} = \mathbb{E}_{t_i}^x \left[Y_{t_{i+1}} + \frac{h_t}{2\alpha} \widetilde{f}_{t_{i+1-\alpha}} + h_t \left(1 - \frac{1}{2\alpha}\right) f_{t_{i+1}} \right] + R_y^i, \tag{2.13}
$$

where

$$
\begin{aligned}
&\widetilde{f}_{t_{i+1-\alpha}} = f\left(t_{i+1-\alpha}, \widetilde{Y}_{t_{i+1-\alpha}}, \widetilde{Z}_{t_{i+1-\alpha}}\right), \quad i = 0, 1, \ldots, N-1, \\
&R_y^i = R_{yf}^i + \widehat{R}_{yf}^i, \widehat{R}_{yf}^i = h_t \mathbb{E}_{t_i}^x [f_{t_{i+1-\alpha}} - \widetilde{f}_{t_{i+1-\alpha}}], \\
&\widetilde{Y}_{t_{i+1-\alpha}} = \mathbb{E}_{t_{i+1-\alpha}}^x [Y_{t_{i+1}} + \alpha h_t f_{t_{i+1}}], \\
&\widetilde{Z}_{t_{i+1-\alpha}} = \mathbb{E}_{t_{i+1-\alpha}}^x \left[\frac{1}{\alpha h_t} Y_{t_{i+1}} \Delta W_{i+1-\alpha,i+1}^\top + f_{t_{i+1}} \Delta W_{i+1-\alpha,i+1}^\top \right].
\end{aligned}
$$

Thus, the time-discretization of Y_{t_i} is

$$
Y_i^\pi = \mathbb{E}_{t_i}^x \left[Y_{i+1}^\pi + \frac{h_t}{2\alpha} \widetilde{f}_{i+1-\alpha}^\pi + h_t \left(1 - \frac{1}{2\alpha}\right) f_{i+1}^\pi \right], \tag{2.14}
$$

where

$$\widetilde{f}^{\pi}_{i+1-\alpha} = f\big(t_{i+1-\alpha}, Y^{\pi}_{i+1-\alpha}, Z^{\pi}_{i+1-\alpha}\big), \quad i = 0, 1, \ldots, N-1.$$

From (2.10), we have the time-discretization of Z_{t_i},

$$Z^{\pi}_i = \mathbb{E}^x_{t_i}\left[\frac{2}{h_t}Y^{\pi}_{i+1}\Delta W^{\top}_{i,i+1} + \frac{1}{\alpha}\widetilde{f}^{\pi}_{i+1-\alpha}\Delta W^{\top}_{i,i+1-\alpha} + \frac{2\alpha-1}{\alpha}f^{\pi}_{i+1}\Delta W^{\top}_{i,i+1} - Z^{\pi}_{i+1}\right]. \tag{2.15}$$

Thus, we deduce the discrete-time approximation (Y^{π}, Z^{π}) for (Y, Z) at t_i for $i = N, N-1, \ldots, 1, 0$:

1. The terminal condition is $(Y^{\pi}_N, Z^{\pi}_N) = (\Phi(X_T), \nabla u(T, X_T)\sigma(T, X_T))$.
2. For $0 \le i < N$, the step from $i+1$ to i follows the transition rule described as

$$\begin{cases} Y^{\pi}_{i+1-\alpha} = \mathbb{E}^x_{t_{i+1-\alpha}}\big[Y^{\pi}_{i+1} + \alpha h_t f^{\pi}_{i+1}\big], \\ Z^{\pi}_{i+1-\alpha} = \mathbb{E}^x_{t_{i+1-\alpha}}\left[\dfrac{1}{\alpha h_t}Y^{\pi}_{i+1}\Delta W^{\top}_{i+1-\alpha,i+1} + f^{\pi}_{i+1}\Delta W^{\top}_{i+1-\alpha,i+1}\right], \\ Y^{\pi}_i = \mathbb{E}^x_{t_i}\left[Y^{\pi}_{i+1} + \dfrac{h_t}{2\alpha}\widetilde{f}^{\pi}_{i+1-\alpha} + h_t\left(1 - \dfrac{1}{2\alpha}\right)f^{\pi}_{i+1}\right], \\ Z^{\pi}_i = \mathbb{E}^x_{t_i}\left[\dfrac{2}{h_t}Y^{\pi}_{i+1}\Delta W^{\top}_{i,i+1} + \dfrac{1}{\alpha}\widetilde{f}^{\pi}_{i+1-\alpha}\Delta W^{\top}_{i,i+1-\alpha} + \dfrac{2\alpha-1}{\alpha}f^{\pi}_{i+1}\Delta W^{\top}_{i,i+1} - Z^{\pi}_{i+1}\right]. \end{cases} \tag{2.16}$$

3. Stability Analysis

In this section, we thoroughly analyze the stability for the numerical scheme. To simplify notation, we first focus on a one-dimensional scenario. This approach can be easily generalized to higher dimensions. We denote the perturbations affecting the generator f and the terminal values Y^{π}_N and Z^{π}_N by $\varepsilon_f, \varepsilon^{\pi}_{y,N}$, and $\varepsilon^{\pi}_{z,N}$, respectively. Note that ε_f is an $\mathcal{F}_t$-adapted process. At any point $(s, Y, Z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, we define $Y^{\pi}_{\varepsilon,N}, Z^{\pi}_{\varepsilon,N}$, and f_{ε} as follows:

$$\begin{cases} Y^{\pi}_{\varepsilon,N} = Y^{\pi}_N + \varepsilon^{\pi}_{y,N}, \\ Z^{\pi}_{\varepsilon,N} = Z^{\pi}_N + \varepsilon^{\pi}_{z,N}, \\ f_{\varepsilon}(s, Y_s, Z_s) = f(s, Y_s, Z_s) + \varepsilon_f. \end{cases} \tag{3.1}$$

Furthermore, we define the quantities $f^{\varepsilon,\pi}_i, \widetilde{f}^{\varepsilon,\pi}_{i+1-\alpha}, \varepsilon^{i,\pi}_f, \widetilde{\varepsilon}^{i+1-\alpha,\pi}_f, f^{\pi}_{\varepsilon,i}$, and $\widetilde{f}^{\pi}_{\varepsilon,i}$ as follows:

$$\begin{aligned} & f^{\varepsilon,\pi}_i = f\big(t_i, Y^{\pi}_{\varepsilon,i}, Z^{\pi}_{\varepsilon,i}\big), \\ & \widetilde{f}^{\varepsilon,\pi}_{i+1-\alpha} = f\big(t_i, Y^{\pi}_{\varepsilon,i+1-\alpha}, Z^{\pi}_{\varepsilon,i+1-\alpha}\big), \\ & \varepsilon^{i,\pi}_f = \varepsilon_f\big(t_i, Y^{\pi}_{\varepsilon,i}, Z^{\pi}_{\varepsilon,i}\big), \\ & \widetilde{\varepsilon}^{i+1-\alpha,\pi}_f = \varepsilon_f\big(t_{i+1-\alpha}, Y^{\pi}_{\varepsilon,i+1-\alpha}, Z^{\pi}_{\varepsilon,i+1-\alpha}\big), \\ & f^{\pi}_{\varepsilon,i} = f^{\varepsilon,\pi}_i + \varepsilon^{i,\pi}_f, \\ & \widetilde{f}^{\pi}_{\varepsilon,i} = \widetilde{f}^{\varepsilon,\pi}_{i+1-\alpha} + \widetilde{\varepsilon}^{i+1-\alpha,\pi}_f. \end{aligned}$$

Here $Y^{\pi}_{\varepsilon,i}$, $Z^{\pi}_{\varepsilon,i}$, $Y^{\pi}_{\varepsilon,i+1-\alpha}$, and $Z^{\pi}_{\varepsilon,i+1-\alpha}$ are obtained by the Scheme 3.1.

Scheme 3.1. Given

$$\big(Y^{\pi}_{\varepsilon,N}, Z^{\pi}_{\varepsilon,N}\big) = \big(Y^{\pi}_{N} + \varepsilon^{\pi}_{y,N}, Z^{\pi}_{N} + \varepsilon^{\pi}_{z,N}\big), \quad i = N-1, \ldots, 0,$$

solve $(Y^{\pi}_{\varepsilon,i}, Z^{\pi}_{\varepsilon,i})$ by

$$\begin{cases} Y^{\pi}_{\varepsilon,i+1-\alpha} = \mathbb{E}^{x}_{t_{i+1-\alpha}}\left[Y^{\pi}_{\varepsilon,i+1} + \alpha h_t f^{\pi}_{\varepsilon,i+1}\right], \\ Z^{\pi}_{\varepsilon,i+1-\alpha} = \mathbb{E}^{x}_{t_{i+1-\alpha}}\left[\dfrac{1}{\alpha h_t} Y^{\pi}_{\varepsilon,i+1}\Delta W^{\top}_{i+1-\alpha,i+1} + f^{\pi}_{\varepsilon,i+1}\Delta W^{\top}_{i+1-\alpha,i+1}\right], \\ Y^{\pi}_{\varepsilon,i} = \mathbb{E}^{x}_{t_i}\left[Y^{\pi}_{\varepsilon,i+1} + \dfrac{h_t}{2\alpha}\widetilde{f}^{\pi}_{\varepsilon,i+1-\alpha} + h_t\left(1-\dfrac{1}{2\alpha}\right) f^{\pi}_{\varepsilon,i+1}\right], \\ Z^{\pi}_{\varepsilon,i} = \mathbb{E}^{x}_{t_i}\left[\dfrac{2}{h_t} Y^{\pi}_{\varepsilon,i+1}\Delta W^{\top}_{i,i+1} + \dfrac{1}{\alpha}\widetilde{f}^{\pi}_{\varepsilon,i+1-\alpha}\Delta W^{\top}_{i,i+1-\alpha} + \dfrac{2\alpha-1}{\alpha} f^{\pi}_{\varepsilon,i+1}\Delta W^{\top}_{i,i+1} - Z^{\pi}_{\varepsilon,i+1}\right]. \end{cases} \tag{3.2}$$

To enable further analysis, the above scheme can be equivalently written as

$$\begin{cases} Y^{\pi}_{\varepsilon,i+1-\alpha} = \mathbb{E}^{x}_{t_{i+1-\alpha}}\left[Y^{\pi}_{\varepsilon,i+1} + \alpha h_t f^{\varepsilon,\pi}_{i+1}\right] + \widetilde{R}^{\pi}_{\varepsilon y,i}, \\ Z^{\pi}_{\varepsilon,i+1-\alpha} = \mathbb{E}^{x}_{t_{i+1-\alpha}}\left[\dfrac{1}{\alpha h_t} Y^{\pi}_{\varepsilon,i+1}\Delta W^{\top}_{i+1-\alpha,i+1} + f^{\varepsilon,\pi}_{i+1}\Delta W^{\top}_{i+1-\alpha,i+1}\right] + \widetilde{R}^{\pi}_{\varepsilon z,i}, \\ Y^{\pi}_{\varepsilon,i} = \mathbb{E}^{x}_{t_i}\left[Y^{\pi}_{\varepsilon,i+1} + \dfrac{h_t}{2\alpha}\widetilde{f}^{\varepsilon,\pi}_{i+1-\alpha} + h_t\left(1-\dfrac{1}{2\alpha}\right) f^{\varepsilon,\pi}_{i+1}\right] + R^{\pi}_{\varepsilon y,i}, \\ Z^{\pi}_{\varepsilon,i} = \mathbb{E}^{x}_{t_i}\left[\dfrac{2}{h_t} Y^{\pi}_{\varepsilon,i+1}\Delta W^{\top}_{i,i+1} + \dfrac{1}{\alpha}\widetilde{f}^{\varepsilon,\pi}_{i+1-\alpha}\Delta W^{\top}_{i,i+1-\alpha} + \dfrac{2\alpha-}{\alpha} f^{\varepsilon,\pi}_{i+1}\Delta W^{\top}_{i,i+1} - Z^{\pi}_{\varepsilon,i+1}\right] + R^{\pi}_{\varepsilon z,i}, \end{cases} \tag{3.3}$$

where

$$\begin{cases} \widetilde{R}^{\pi}_{\varepsilon y,i} = \mathbb{E}^{x}_{t_{i+1-\alpha}}\left[\alpha h_t \varepsilon^{i+1,\pi}_{f}\right], \\ \widetilde{R}^{\pi}_{\varepsilon z,i} = \mathbb{E}^{x}_{t_{i+1-\alpha}}\left[\varepsilon^{i+1,\pi}_{f}\Delta W^{\top}_{i+1-\alpha,i+1}\right], \\ R^{\pi}_{\varepsilon y,i} = \mathbb{E}^{x}_{t_i}\left[\dfrac{h_t}{2\alpha}\widetilde{\varepsilon}^{i+1-\alpha,\pi}_{f} + h_t\left(1-\dfrac{1}{2\alpha}\right)\varepsilon^{i+1,\pi}_{f}\right], \\ R^{\pi}_{\varepsilon z,i} = \mathbb{E}^{x}_{t_i}\left[\dfrac{1}{\alpha}\widetilde{\varepsilon}^{i+1-\alpha,\pi}_{f}\Delta W^{\top}_{i,i+1-\alpha} + \dfrac{2\alpha-1}{\alpha}\varepsilon^{i+1,\pi}_{f}\Delta W^{\top}_{i,i+1}\right]. \end{cases}$$

To obtain the error analysis expression, we let

$$\varepsilon^{\pi}_{y,N} = Y^{\pi}_{\varepsilon,N} - Y^{\pi}_{N}, \quad \varepsilon^{\pi}_{z,N} = Z^{\pi}_{\varepsilon,N} - Z^{\pi}_{N}.$$

Then subtracting (2.16) from (3.3), we obtain

$$\begin{cases} \varepsilon^{\pi}_{y,i+1-\alpha} = \mathbb{E}^{x}_{t_{i+1-\alpha}}\left[\varepsilon^{\pi}_{y,i+1} + \alpha h_t\big(f^{\varepsilon,\pi}_{i+1} - f^{\pi}_{i+1}\big)\right] + \widetilde{R}^{\pi}_{\varepsilon y,i}, \\ \varepsilon^{\pi}_{z,i+1-\alpha} = \mathbb{E}^{x}_{t_{i+1-\alpha}}\left[\dfrac{1}{\alpha h_t}\varepsilon^{\pi}_{y,i+1}\Delta W^{\top}_{i+1-\alpha,i+1} + \big(f^{\varepsilon,\pi}_{i+1} - f^{\pi}_{i+1}\big)\Delta W^{\top}_{i+1-\alpha,i+1}\right] + \widetilde{R}^{\pi}_{\varepsilon z,i}, \\ \varepsilon^{\pi}_{y,i} = \mathbb{E}^{x}_{t_i}\left[\varepsilon^{\pi}_{y,i+1} + \dfrac{h_t}{2\alpha}\big(\widetilde{f}^{\varepsilon,\pi}_{i+1-\alpha} - \widetilde{f}^{\pi}_{i+1-\alpha}\big) + h_t\left(1-\dfrac{1}{2\alpha}\right)\big(f^{\varepsilon,\pi}_{i+1} - f^{\pi}_{i+1}\big)\right] + R^{\pi}_{\varepsilon y,i}, \\ \varepsilon^{\pi}_{z,i} = \mathbb{E}^{x}_{t_i}\left[\dfrac{2}{h_t}\varepsilon^{\pi}_{y,i+1}\Delta W^{\top}_{i,i+1} + \dfrac{1}{\alpha}\big(\widetilde{f}^{\varepsilon,\pi}_{i+1-\alpha} - \widetilde{f}^{\pi}_{i+1-\alpha}\big)\Delta W^{\top}_{i,i+1-\alpha}\right. \\ \qquad\qquad \left. + \dfrac{2\alpha-1}{\alpha}\big(f^{\varepsilon,\pi}_{i+1} - f^{\pi}_{i+1}\big)\Delta W^{\top}_{i,i+1}\right] + R^{\pi}_{\varepsilon z,i}. \end{cases} \tag{3.4}$$

We refer to (3.4) as the permutation error equations for scheme (2.16). To prove the stability of scheme (2.16), stability is defined as follows.

Definition 3.1. *The scheme* (2.16) *is defined as stable. If, for any $\varepsilon > 0$ and $0 \le i \le N-1$, there exists a $\delta > 0$ such that*

$$\mathbb{E}\left[\left|\varepsilon^{\pi}_{y,i}\right|^2 + h_t \sum_{\ell=i}^{N-1} \left|\varepsilon^{\pi}_{z,\ell}\right|^2\right] < \varepsilon,$$

provided that

$$\mathbb{E}\left[\left|\varepsilon^{i,\pi}_{f}\right|^2 + \left|\widetilde{\varepsilon}^{i+1-\alpha,\pi}_{f}\right|^2\right] < \delta, \quad \mathbb{E}\left[\left|\varepsilon^{\pi}_{y,N}\right|^2 + \left|\varepsilon^{\pi}_{z,N}\right|^2\right] < \delta.$$

Theorem 3.1. *Suppose the assumptions* (i)-(iii) *hold and $(t, Y_t, Z_t) \mapsto f(t, Y_t, Z_t) \in C^3_b$. Then for a sufficiently small step size h_t, we have*

$$\begin{aligned}
&\mathbb{E}\left[\left|\varepsilon^{\pi}_{y,i}\right|^2\right] + \mathbb{E}\left[h_t \sum_{\ell=i}^{N-1} |\varepsilon^{\pi}_{z,\ell}|^2\right] \\
&\le C\Big(\mathbb{E}\left[\left|\varepsilon^{\pi}_{y,N}\right|^2\right] + h_t\mathbb{E}\left[\left|\varepsilon^{\pi}_{z,N}\right|^2\right]\Big) \\
&\quad + \sum_{\ell=i}^{N-1} C\mathbb{E}\left[\left|R^{\pi}_{\varepsilon y,\ell}\right|^2 + \left(h_t + h_t^2\right)\left(\left|\widetilde{R}^{\pi}_{\varepsilon y,\ell}\right|^2 + \left|\widetilde{R}^{\pi}_{\varepsilon z,\ell}\right|^2\right) + h_t\left|R^{\pi}_{\varepsilon z,\ell}\right|^2\right],
\end{aligned}$$

where C denotes a positive constant.

Proof. From (3.4) and the Lipschitz-condition assumption, we have

$$\begin{aligned}
\left|\varepsilon^{\pi}_{z,i}\right| \le{}& \left|\mathbb{E}^x_{t_i}\left[\varepsilon^{\pi}_{y,i+1}\frac{2\Delta W^{\top}_{i,i+1}}{h_t}\right]\right| + \frac{L}{\alpha}\mathbb{E}^x_{t_i}\left[\left(\left|\varepsilon^{\pi}_{y,i+1-\alpha}\right| + \left|\varepsilon^{\pi}_{z,i+1-\alpha}\right|\right)\left|\Delta W^{\top}_{i,i+1-\alpha}\right|\right] \\
&+ \left|\frac{2\alpha-1}{\alpha}\right| L\mathbb{E}^x_{t_i}\left[\left(\left|\varepsilon^{\pi}_{y,i+1}\right| + \left|\varepsilon^{\pi}_{z,i+1}\right|\right)\left|\Delta W^{\top}_{i,i+1-\alpha}\right|\right] + \mathbb{E}^x_{t_i}\left[\left|\varepsilon^{\pi}_{z,i+1}\right|\right] + \left|R^{\pi}_{\varepsilon z,i}\right|. \qquad (3.5)
\end{aligned}$$

From the Cauchy-Schwarz inequality, the inequality (3.5) is represented as

$$\begin{aligned}
\left|\varepsilon^{\pi}_{z,i}\right| \le{}& \left|\frac{2}{h_t}\sqrt{h_t\mathbb{E}^x_{t_i}\left[\left|\varepsilon^{\pi}_{y,i+1}\right|^2\right]}\right| \\
&+ \frac{L}{\alpha}\left(\sqrt{(1-\alpha)h_t\mathbb{E}^x_{t_i}\left[\left|\varepsilon^{\pi}_{y,i+1-\alpha}\right|^2\right]} + \sqrt{(1-\alpha)h_t\mathbb{E}^x_{t_i}\left[\left|\varepsilon^{\pi}_{z,i+1-\alpha}\right|^2\right]}\right) \\
&+ \left|\frac{2\alpha-1}{\alpha}\right| L\left(\sqrt{h_t\mathbb{E}^x_{t_i}\left[\left|\varepsilon^{\pi}_{y,i+1}\right|^2\right]} + \sqrt{h_t\mathbb{E}^x_{t_i}\left[\left|\varepsilon^{\pi}_{z,i+1}\right|^2\right]}\right) \\
&+ \mathbb{E}^x_{t_i}\left[\left|\varepsilon^{\pi}_{z,i+1}\right|\right] + \left|R^{\pi}_{\varepsilon z,i}\right|. \qquad (3.6)
\end{aligned}$$

From (3.4), we have

$$\begin{aligned}
\left|\varepsilon^{\pi}_{y,i+1-\alpha}\right|^2 &= \left|\mathbb{E}^x_{t_{i+1-\alpha}}\left[\varepsilon^{\pi}_{y,i+1} + \alpha h_t\left(f^{\varepsilon,\pi}_{i+1} - f^{\pi}_{i+1}\right)\right] + \widetilde{R}^{\pi}_{\varepsilon y,i}\right|^2 \\
&\le \left|\mathbb{E}^x_{t_{i+1-\alpha}}\left[\varepsilon^{\pi}_{y,i+1} + \alpha h_t L\left(\left|\varepsilon^{\pi}_{y,i+1}\right| + \left|\varepsilon^{\pi}_{z,i+1}\right|\right)\right] + \widetilde{R}^{\pi}_{\varepsilon y,i}\right|^2, \qquad (3.7) \\
\left|\varepsilon^{\pi}_{z,i+1-\alpha}\right|^2 &= \left|\mathbb{E}^x_{t_{i+1-\alpha}}\left[\frac{1}{\alpha h_t}\left|\varepsilon^{\pi}_{y,i+1}\right|\Delta W^{\top}_{i+1-\alpha,i+1} + \left(f^{\varepsilon,\pi}_{i+1} - f^{\pi}_{i+1}\right)\Delta W^{\top}_{i+1-\alpha,i+1}\right] + \widetilde{R}^{\pi}_{\varepsilon z,i}\right|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \left|\mathbb{E}_{t_{i+1-\alpha}}^{x}\left[\frac{1}{\alpha h_t}\left|\varepsilon_{y,i+1}^{\pi}\right|\Delta W_{i+1-\alpha,i+1}^{\top}+L\left(\left|\varepsilon_{y,i+1}^{\pi}\right|+\left|\varepsilon_{z,i+1}^{\pi}\right|\right)\Delta W_{i+1-\alpha,i+1}^{\top}\right]+\widetilde{R}_{\varepsilon z,i}^{\pi}\right|^{2}\\
&\le 2\mathbb{E}_{t_{i+1-\alpha}}^{x}\left[\left(\frac{2}{\alpha h_t}+2\alpha h_t L^2+4L\right)\left|\varepsilon_{y,i+1}^{\pi}\right|^{2}+2\alpha h_t L^2\left|\varepsilon_{z,i+1}^{\pi}\right|^{2}\right]+2\left|\widetilde{R}_{\varepsilon z,i}^{\pi}\right|^{2}. \qquad (3.8)
\end{aligned}
$$

Plugging (3.7) and (3.8) into (3.6), we obtain

$$
\begin{aligned}
\left|\varepsilon_{z,i}^{\pi}\right| \le& \left|\frac{2}{h_t}\sqrt{h_t\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{y,i+1}^{\pi}\right|^{2}\right]}\right|\\
&+\frac{L}{\alpha}\left(\sqrt{2(1-\alpha)h_t\mathbb{E}_{t_i}^{x}\left[\left((1+\alpha h_t L)\left|\varepsilon_{y,i+1}^{\pi}\right|+\alpha h_t L\left|\varepsilon_{z,i+1}^{\pi}\right|\right)^{2}+\left(\widetilde{R}_{\varepsilon y,i}^{\pi}\right)^{2}\right]}\right)\\
&+\frac{L}{\alpha}\sqrt{2(1-\alpha)h_t\mathbb{E}_{t_i}^{x}\left[\left(\frac{2}{\alpha h_t}+2\alpha h_t L^2+4L\right)\left|\varepsilon_{y,i+1}^{\pi}\right|^{2}+2\alpha h_t L^2\left|\varepsilon_{z,i+1}^{\pi}\right|^{2}+\left|\widetilde{R}_{\varepsilon z,i}^{\pi}\right|^{2}\right]}\\
&+\left|\frac{2\alpha-1}{\alpha}\right|L\left(\sqrt{h_t\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{y,i+1}^{\pi}\right|^{2}\right]}+\sqrt{h_t\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{z,i+1}^{\pi}\right|^{2}\right]}\right)+\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{z,i+1}^{\pi}\right|\right]+\left|R_{\varepsilon z,i}^{\pi}\right|. \qquad (3.9)
\end{aligned}
$$

Squaring (3.9), we deduce

$$
\begin{aligned}
\left|\varepsilon_{z,i}^{\pi}\right|^{2} \le& \left(\frac{20}{h_t}+\frac{10(1-\alpha)h_t L^2}{\alpha^2}\left(4+4\alpha^2 h_t L^2+8\alpha h_t L+\frac{2}{\alpha h_t}+2\alpha h_t L^2+4L\right)\right.\\
&\left.+\frac{10(2\alpha-1)^2 h_t L^2}{\alpha^2}\right)\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{y,i+1}^{\pi}\right|^{2}\right]\\
&+\left(\frac{10(1-\alpha)h_t L^2}{\alpha^2}\left(4\alpha^2 h_t^2 L^2+2\alpha h_t L^2\right)+\frac{10(2\alpha-1)^2 h_t L^2}{\alpha^2}+5\right)\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{z,i+1}^{\pi}\right|^{2}\right]\\
&+\frac{20(1-\alpha)h_t L^2}{\alpha^2}\left(\left|\widetilde{R}_{\varepsilon z,i}^{\pi}\right|^{2}+\left|\widetilde{R}_{\varepsilon y,i}^{\pi}\right|^{2}\right)+5\left|R_{\varepsilon z,i}^{\pi}\right|^{2}. \qquad (3.10)
\end{aligned}
$$

From (3.4), we preliminarily obtain

$$
\begin{aligned}
\left|\varepsilon_{y,i}^{\pi}\right| \le& \left|\mathbb{E}_{t_i}^{x}\left[\varepsilon_{y,i+1}^{\pi}\right]\right|+\left|\frac{h_t}{2\alpha}\mathbb{E}_{t_i}^{x}\left[\widetilde{f}_{i+1-\alpha}^{\varepsilon,\pi}-\widetilde{f}_{i+1-\alpha}^{\pi}\right]\right|\\
&+\left|\frac{h_t(2\alpha-1)}{2\alpha}\mathbb{E}_{t_i}^{x}\left[f_{i+1}^{\varepsilon,\pi}-f_{i+1}^{\pi}\right]\right|+\left|R_{\varepsilon y,i}^{\pi}\right|\\
\le& \left|\mathbb{E}_{t_i}^{x}\left[\varepsilon_{y,i+1}^{\pi}\right]\right|+\frac{h_t}{2\alpha}L\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{y,i+1-\alpha}^{\pi}\right|+\left|\varepsilon_{z,i+1-\alpha}^{\pi}\right|\right]\\
&+\left|\frac{2\alpha-1}{2\alpha}\right|h_t L\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{y,i+1}^{\pi}\right|+\left|\varepsilon_{z,i+1}^{\pi}\right|\right]+\left|R_{\varepsilon y,i}^{\pi}\right|. \qquad (3.11)
\end{aligned}
$$

Combining (3.4) and (3.11), we can further derive

$$
\begin{aligned}
\left|\varepsilon_{y,i}^{\pi}\right| \le& \left(1+\frac{h_t L}{2\alpha}(1+\alpha h_t L)+\left|\frac{2\alpha-1}{\alpha}\right|h_t L\right)\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{y,i+1}^{\pi}\right|\right]\\
&+\left(\frac{h_t^2 L^2}{\alpha^2}+\left|\frac{2\alpha-1}{\alpha}\right|h_t L\right)\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{z,i+1}^{\pi}\right|\right]+\left(\frac{L}{2\alpha^2}+\frac{h_t L^2}{2\alpha}\right)\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{y,i+1}^{\pi}\Delta W_{i+1-\alpha,i+1}^{\top}\right|\right]\\
&+\frac{h_t L^2}{2\alpha}\mathbb{E}_{t_i}^{x}\left[\left|\varepsilon_{z,i+1}^{\pi}\Delta W_{i+1-\alpha,i+1}^{\top}\right|\right]+\left|R_{\varepsilon y,i}^{\pi}\right|+\frac{h_t L}{2\alpha}\left|\widetilde{R}_{\varepsilon y,i}^{\pi}\right|+\frac{h_t L}{2\alpha}\left|\widetilde{R}_{\varepsilon z,i}^{\pi}\right|. \qquad (3.12)
\end{aligned}
$$

Squaring (3.12), we deduce

$$\begin{aligned}\left|\varepsilon^{\pi}_{y,i}\right|^2 \le{}& 5\bigg(4+\frac{h_t^2L^2}{\alpha^2}+h_t^4L^4+\frac{4(2\alpha-1)^2}{\alpha^2}h_t^2L^2+\frac{h_tL^2}{2\alpha^3}+\frac{h_t^3L^4}{2\alpha}\bigg)\mathbb{E}^x_{t_i}\Big[\left|\varepsilon^{\pi}_{y,i+1}\right|^2\Big]\\&+5\bigg(\frac{h_t^4L^4}{2}+\frac{2(2\alpha-1)^2}{\alpha^2}h_t^2L^2+\frac{h_t^3L^4}{4\alpha}\bigg)\mathbb{E}^x_{t_i}\Big[\left|\varepsilon^{\pi}_{z,i+1}\right|^2\Big]\\&+15\left|R^{\pi}_{\varepsilon y,i}\right|^2+\frac{15h_tL}{2\alpha}\left|\widetilde{R}^{\pi}_{\varepsilon y,i}\right|^2+\frac{15h_tL}{2\alpha}\left|\widetilde{R}^{\pi}_{\varepsilon z,i}\right|^2. \end{aligned}\tag{3.13}$$

Adding (3.10) to (3.13), we deduce

$$\begin{aligned}&\left|\varepsilon^{\pi}_{y,i}\right|^2+h_t\left|\varepsilon^{\pi}_{z,i}\right|^2\\&\le 5\bigg(8+\frac{4(1-\alpha)}{\alpha^3}L^2h_t+\frac{6(2\alpha-1)^2+1+8(1-\alpha)+8L(1-\alpha)}{\alpha^2}L^2h_t^2\\&\quad+\frac{L+32(1-\alpha)+8L(1-\alpha)}{2\alpha}L^3h_t^3+\big(1+8(1-\alpha)\big)L^4h_t^4\bigg)\mathbb{E}^x_{t_i}\Big[\left|\varepsilon^{\pi}_{y,i+1}\right|^2\Big]\\&\quad+5\bigg(h_t+\frac{4(2\alpha-1)^2}{\alpha^2}L^2h_t^2+\frac{16(1-\alpha)+1}{4\alpha}L^4h_t^3+\frac{16(1-\alpha)+1}{2}L^4h_t^4\bigg)\mathbb{E}^x_{t_i}\Big[\left|\varepsilon^{\pi}_{z,i+1}\right|^2\Big]\\&\quad+15\left|R^{\pi}_{\varepsilon y,i}\right|^2+\frac{15h_tL}{2\alpha}\left|\widetilde{R}^{\pi}_{\varepsilon y,i}\right|^2+\frac{15h_tL}{2\alpha}\left|\widetilde{R}^{\pi}_{\varepsilon z,i}\right|^2\\&\quad+\frac{20(1-\alpha)h_t^2L^2}{\alpha^2}\Big(\left|\widetilde{R}^{\pi}_{\varepsilon z,i}\right|^2+\left|\widetilde{R}^{\pi}_{\varepsilon y,i}\right|^2\Big)+5h_t\left|R^{\pi}_{\varepsilon z,i}\right|^2\\&=40(1+C_1h_t)\mathbb{E}^x_{t_i}\Big[\left|\varepsilon^{\pi}_{y,i+1}\right|^2\Big]+C_2h_t\mathbb{E}^x_{t_i}\Big[\left|\varepsilon^{\pi}_{z,i+1}\right|^2\Big]\\&\quad+15\left|R^{\pi}_{\varepsilon y,i}\right|^2+\bigg(\frac{15h_tL}{2\alpha}+\frac{20(1-\alpha)h_t^2L^2}{\alpha^2}\bigg)\Big(\left|\widetilde{R}^{\pi}_{\varepsilon z,i}\right|^2+\left|\widetilde{R}^{\pi}_{\varepsilon y,i}\right|^2\Big)+5h_t\left|R^{\pi}_{\varepsilon z,i}\right|^2,\end{aligned}\tag{3.14}$$

where

$$\begin{aligned}C_1&=\frac{1}{8}\bigg(\frac{4(1-\alpha)}{\alpha^3}L^2+\frac{6(2\alpha-1)^2+1+8(1-\alpha)+8L(1-\alpha)}{\alpha^2}L^2h_t\\&\qquad+\frac{L+32(1-\alpha)+8L(1-\alpha)}{2\alpha}L^3{h_t}^2+\big(1+8(1-\alpha)\big)L^4{h_t}^3\bigg),\\C_2&=5\bigg(1+\frac{4(2\alpha-1)^2}{\alpha^2}L^2h_t+\frac{16(1-\alpha)+1}{4\alpha}L^4{h_t}^2+\frac{16(1-\alpha)+1}{2}L^4{h_t}^3\bigg).\end{aligned}$$

Two positive constants, denoted as C_0 and C_3, exist such that

$$\begin{aligned}&C_0\left|\varepsilon^{\pi}_{y,i}\right|^2+h_t\left|\varepsilon^{\pi}_{z,i}\right|^2\\&\le(1+C_3h_t)\Big(C_0\mathbb{E}^x_{t_i}\Big[\left|\varepsilon^{\pi}_{y,i+1}\right|^2\Big]+C_2h_t\mathbb{E}^x_{t_i}\Big[\left|\varepsilon^{\pi}_{z,i+1}\right|^2\Big]\Big)\\&\quad+C_3\left|R^{\pi}_{\varepsilon y,i}\right|^2+\big(C_3h_t+C_3h_t^2\big)\Big(\left|\widetilde{R}^{\pi}_{\varepsilon z,i}\right|^2+\left|\widetilde{R}^{\pi}_{\varepsilon y,i}\right|^2\Big)+C_3h_t\left|R^{\pi}_{\varepsilon z,i}\right|^2.\end{aligned}\tag{3.15}$$

By applying the tower property of conditional expectations to (3.15), it follows that

$$\begin{aligned}&C_0\mathbb{E}\Big[\left|\varepsilon^{\pi}_{y,i}\right|^2\Big]+(1-C_2)h_t\sum_{\ell=i}^{N-1}(1+C_3h_t)^{\ell-i}\mathbb{E}\Big[\left|\varepsilon^{\pi}_{z,\ell}\right|^2\Big]\\&\le(1+C_3h_t)^{N-i}\Big(C_0\mathbb{E}\Big[\left|\varepsilon^{\pi}_{y,N}\right|^2\Big]+C_2h_t\mathbb{E}\Big[\left|\varepsilon^{\pi}_{z,N}\right|^2\Big]\Big)\end{aligned}$$

$$+\sum_{\ell=i}^{N-1}(1+C_3h_t)^{\ell-i}\mathbb{E}\Big[C_3\big|R^{\pi}_{\varepsilon y,\ell}\big|^2+\big(C_3h_t+C_3h_t^2\big)\Big(\big|\widetilde{R}^{\pi}_{\varepsilon z,\ell}\big|^2+\big|\widetilde{R}^{\pi}_{\varepsilon y,\ell}\big|^2\Big)$$
$$+C_3h_t\big|R^{\pi}_{\varepsilon z,\ell}\big|^2\Big]. \tag{3.16}$$

Furthermore, we have

$$\mathbb{E}\big[\big|\varepsilon^{\pi}_{y,i}\big|^2\big]+\mathbb{E}\left[h_t\sum_{\ell=i}^{N-1}\big|\varepsilon^{\pi}_{z,\ell}\big|^2\right]$$
$$\le C\Big(\mathbb{E}\Big[\big|\varepsilon^{\pi}_{y,N}\big|^2\Big]+h_t\mathbb{E}\Big[\big|\varepsilon^{\pi}_{z,N}\big|^2\Big]\Big)$$
$$+\sum_{\ell=i}^{N-1}C\mathbb{E}\Big[\big|R^{\pi}_{\varepsilon y,\ell}\big|^2+\big(h_t+h_t^2\big)\Big(\big|\widetilde{R}^{\pi}_{\varepsilon y,\ell}\big|^2+\big|\widetilde{R}^{\pi}_{\varepsilon z,\ell}\big|^2\Big)+h_t\big|R^{\pi}_{\varepsilon z,\ell}\big|^2\Big]. \tag{3.17}$$

The proof is complete. □

Remark 3.1. In stability theory, the impact of small perturbations to the terminal conditions and the generator of BSDE is analyzed to assess the stability of the numerical scheme under these disturbances. Specifically, perturbations $(\varepsilon^{\pi}_{y,N},\varepsilon^{\pi}_{z,N})$ are applied to the terminal values (Y^{π}_N,Z^{π}_N), and a perturbation ε_f is introduced to the generator $f(s,Y_s,Z_s)$. This stability analysis confirms the robustness of the scheme described in (2.16) for practical applications.

4. Error Estimates

In this section, prior to presenting the error estimates for the scheme (2.16), we first introduce a key proposition and three lemmas. To facilitate understanding, we define certain symbols before discussing the Itô-Taylor expansions as outlined in [18, Theorem 5.5.1]. Let $\gamma:=(\gamma_1,\gamma_2,\dots,\gamma_p)$ denote a multi-index of finite length, where $p\in\mathbb{N}^+$; each component $\gamma_k\in\{0,1,2,\dots,q\}$, $k=1,2,\dots,p$, corresponds to a differential operator defined as

- For $\gamma_k=0$,
$$\mathcal{L}^{(0)}=\frac{\partial}{\partial t}+\sum_{i=1}^{q}b_i(t,x)\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{q}(\sigma\sigma^\top)_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j},$$
- For $\gamma_k\in\{1,2,\dots,q\}$,
$$\mathcal{L}^{(\gamma_k)}=\sum_{i=1}^{q}\sigma_{i\gamma_k}(t,x)\frac{\partial}{\partial x_i},$$

define $p=\ell(\gamma)$ as the length of the multi-index γ. The composite operator $\mathcal{L}^\gamma$ is recursively defined by $\mathcal{L}^\gamma=\mathcal{L}^{(\gamma_1)}\circ\cdots\circ\mathcal{L}^{(\gamma_p)}$. The space $\mathcal{A}^\gamma$ consists of functions $v:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d_1}$ for which $\mathcal{L}^\gamma v$ is continuous and well-defined. The subspace $\mathcal{A}^\gamma_b$ further requires $\mathcal{L}^\gamma v$ to be bounded. For a positive integer m, $\mathcal{A}^m_b$ denotes the set of functions v such that $v^\gamma\in\mathcal{A}^\gamma_b$ for all $\gamma\in\{\gamma\mid\ell(\gamma)\le m\}\setminus\{\emptyset\}$.

Proposition 4.1 ([5, 18, 26]). *Let $m\ge 0$. Then for a function $v\in\mathcal{A}^{m+1}_b$,*

$$\mathbb{E}^x_t[v(t+h_t,X_{t+h_t})]$$

$$= v_t + h_t v_t^{(0)} + \frac{h_t^2}{2} v_t^{(0,0)} + \cdots + \frac{h_t^m}{m!} v_t^{(0)_m} + \mathcal{O}\big(h_t^{m+1}\big), \tag{4.1}$$

$$\mathbb{E}_t^x[(W_{t+h_t} - W_t) v(t+h_t, X_{t+h_t})]$$
$$= h_t v_t^{(1)} + h_t^2 v_t^{(1,0)} + \frac{h_t^3}{2} v_t^{(1)*(0,0)} + \cdots + \frac{h_t^{m+1}}{m!} v_t^{(1)*(0)_m} + \mathcal{O}\big(h_t^{m+2}\big), \tag{4.2}$$

where

$$v_t = v(t, X_t), \quad v_t^{(0)} = \mathcal{L}^{(0)} v(t, X_t),$$
$$v_t^{(0,0)} = \mathcal{L}^{(0)} \circ \mathcal{L}^{(0)} v(t, X_t), \quad \cdots, \quad v_t^{(0)_m} = \underbrace{\mathcal{L}^{(0)} \circ \cdots \circ \mathcal{L}^{(0)}}_{m} v(t, X_t),$$
$$v_t^{(1)} = \mathcal{L}^{(1)} v(t, X_t),$$
$$v_t^{(1,0)} = \mathcal{L}^{(1)} \circ \mathcal{L}^{(0)} v(t, X_t), \quad \cdots, \quad v_t^{(1)*(0)_m} = \mathcal{L}^{(1)} \circ \underbrace{\mathcal{L}^{(0)} \circ \cdots \circ \mathcal{L}^{(0)}}_{m} v(t, X_t).$$

Lemma 4.1. *Suppose the assumptions* (i)-(iii) *hold. Furthermore, let*

$$(t, Y_t, Z_t) \ \mapsto \ f(t, Y_t, Z_t) \in C_b^3.$$

Then, for $i = 0, 1, \ldots, N-1$*,*

$$R_{yf}^i = \mathcal{O}\big(h_t^3\big), \quad \widetilde{R}_{yf}^i = \mathcal{O}\big(h_t^2\big).$$

Proof. By the assumptions (i)-(iii), the integrand $\mathbb{E}_t^x[f_s]$, for $s > t$, remains continuous in s. Consequently, we arrive at the ordinary differential equation

$$\frac{d\mathbb{E}_t^x[Y_s]}{ds} = -\mathbb{E}_t^x[f_s], \quad s \in [t, T]. \tag{4.3}$$

From the definition of R_{yf}^i and (2.4), we know

$$R_{yf}^i = \int_{t_i}^{t_{i+1}} \mathbb{E}_{t_i}^x[f_s]\, ds - h_t \mathbb{E}_{t_i}^x\left[\frac{1}{2\alpha} f_{t_{i+1-\alpha}} + \left(1 - \frac{1}{2\alpha}\right) f_{t_{i+1}}\right]$$
$$= Y_{t_i} - \mathbb{E}_{t_i}^x\left[Y_{t_{i+1}} + \frac{h_t}{2\alpha} f_{t_{i+1-\alpha}} + h_t\left(1 - \frac{1}{2\alpha}\right) f_{t_{i+1}}\right]. \tag{4.4}$$

Substituting (4.3) into (4.4), then applying (1.3) to the derived equation, we have

$$R_{yf}^i = \mathbb{E}_{t_i}^x\left[u_{t_i} - u_{t_{i+1}} + \frac{h_t}{2\alpha} u_{t_{i+1-\alpha}}^{(0)} + h_t\left(1 - \frac{1}{2\alpha}\right) u_{t_{i+1}}^{(0)}\right], \tag{4.5}$$

where $u_{t_i} = u(t_i, X_{t_i})$. Applying Itô-Taylor expansion (4.1) at (t_i, X_{t_i}), we have

$$\begin{aligned}
\mathbb{E}_{t_i}^x[u_{t_{i+1}}] &= u_{t_i} + h_t u_{t_i}^{(0)} + \frac{h_t^2}{2} u_{t_i}^{(0,0)} + \frac{h_t^3}{3!} u_{t_i}^{(0,0,0)} + \mathcal{O}\big(h_t^4\big),\\
\mathbb{E}_{t_i}^x\big[u_{t_{i+1-\alpha}}^{(0)}\big] &= u_{t_i}^{(0)} + (1-\alpha) h_t u_{t_i}^{(0,0)} + \frac{(1-\alpha)^2 h_t^2}{2} u_{t_i}^{(0,0,0)} + \mathcal{O}\big(h_t^3\big),\\
\mathbb{E}_{t_i}^x\big[u_{t_{i+1}}^{(0)}\big] &= u_{t_i}^{(0)} + h_t u_{t_i}^{(0,0)} + \frac{h_t^2}{2} u_{t_i}^{(0,0,0)} + \mathcal{O}\big(h_t^3\big).
\end{aligned} \tag{4.6}$$

Plugging (4.6) into (4.5), we get

$$R_{yf}^i = \mathbb{E}_{t_i}^x\left[\frac{(3\alpha - 2) h_t^3}{12} u_{t_i}^{(0,0,0)} + \mathcal{O}\big(h_t^4\big)\right]. \tag{4.7}$$

Eq. (4.7) shows that the result $R^i_{yf} = \mathcal{O}(h_t^3)$ is obvious. Analogously, by the definition of $\widetilde{R}^i_{yf}$ and (2.5), we obtain

$$\begin{aligned}\widetilde{R}^i_{yf} &= \int_{t_{i+1-\alpha}}^{t_{i+1}} \mathbb{E}^x_{t_{i+1-\alpha}}[f(s,Y_s,Z_s)]ds - \alpha h_t \mathbb{E}^x_{t_{i+1-\alpha}}[f_{t_{i+1}}] \\ &= Y_{t_{i+1-\alpha}} - \mathbb{E}^x_{t_{i+1-\alpha}}[Y_{t_{i+1}} + \alpha h_t f_{t_{i+1}}]. \end{aligned} \tag{4.8}$$

Substituting (4.3) into (4.8) and then using Itô-Taylor expansion (4.1) at $(t_{i+1-\alpha}, X_{t_{i+1-\alpha}})$ on the derived equation, we have

$$\begin{aligned}\widetilde{R}^i_{yf} &= \mathbb{E}^x_{t_{i+1-\alpha}}\big[u_{t_{i+1-\alpha}} - u_{t_{i+1}} + \alpha h_t u^{(0)}_{t_{i+1}}\big] \\ &= \mathbb{E}^x_{t_{i+1-\alpha}}\Bigg[u_{t_{i+1-\alpha}} - \Big(u_{t_{i+1-\alpha}} + \alpha h_t u^{(0)}_{t_{i+1-\alpha}} + \frac{\alpha^2 h_t^2}{2} u^{(0,0)}_{t_{i+1-\alpha}} + \mathcal{O}\big(h_t^3\big)\Big) \\ &\qquad + \alpha h_t \big(u^{(0)}_{t_{i+1-\alpha}} + \alpha h_t u^{(0,0)}_{t_{i+1-\alpha}} + \mathcal{O}\big(h_t^2\big)\big)\Bigg] \\ &= \mathbb{E}^x_{t_{i+1-\alpha}}\left[\frac{\alpha^2 h_t^2}{2} u^{(0,0)}_{t_{i+1-\alpha}} + \mathcal{O}\big(h_t^3\big)\right]. \end{aligned} \tag{4.9}$$

Notice that the above result implies the result $\widetilde{R}^i_{yf} = \mathcal{O}(h_t^2)$. □

Lemma 4.2. *Suppose the assumptions* (i)-(iii) *hold. Furthermore, let*

$$(t, Y_t, Z_t) \ \mapsto \ f(t, Y_t, Z_t) \in C_b^3.$$

Then, for $i = 0, 1, \ldots, N-1$,

$$R^i_z = \mathcal{O}\big(h_t^3\big), \quad R^i_{z,3} = \mathcal{O}\big(h_t^3\big).$$

Proof. By Itô-Taylor expansions (4.1) and (4.2), we have

$$\begin{aligned}\mathbb{E}^x_{t_i}\big[Y_{t_{i+1}} \Delta W^\top_{i,i+1}\big] &= h_t u^{(1)}_{t_i} + h_t^2 u^{(1,0)}_{t_i} + \frac{h_t^3}{2} u^{(1,0,0)}_{t_i} + \mathcal{O}\big(h_t^4\big), \\ \mathbb{E}^x_{t_i}[Z_{t_{i+1}}] &= u^{(1)}_{t_i} + h_t u^{(1,0)}_{t_i} + \frac{h_t^2}{2} u^{(1,0,0)}_{t_i} + \mathcal{O}\big(h_t^3\big). \end{aligned} \tag{4.10}$$

By (1.2), (1.3) and Itô-Taylor expansion (4.2), we get

$$\begin{aligned}\mathbb{E}^x_{t_i}\big[f_{t_{i+1-\alpha}} \Delta W^\top_{i,i+1-\alpha}\big] &= -\mathbb{E}^x_{t_i}\big[u^{(0)}_{t_{i+1-\alpha}} \Delta W^\top_{i,i+1-\alpha}\big] \\ &= -(1-\alpha) h_t u^{(1,0)}_{t_i} - (1-\alpha)^2 h_t^2 u^{(1,0,0)}_{t_i} - \mathcal{O}\big(h_t^3\big), \\ \mathbb{E}^x_{t_i}\big[f_{t_{i+1}} \Delta W^\top_{i,i+1}\big] &= -\mathbb{E}^x_{t_i}\big[u^{(0)}_{t_{i+1}} \Delta W^\top_{i,i+1}\big] = -h_t u^{(1,0)}_{t_i} - h_t^2 u^{(1,0,0)}_{t_i} - \mathcal{O}\big(h_t^3\big). \end{aligned} \tag{4.11}$$

Substituting (4.10), (4.11) into (2.10), we obtain

$$\frac{2}{h_t} R^i_z = \left(\alpha - \frac{1}{2}\right) h_t^2 u^{(1,0,0)}_{t_i} + \mathcal{O}\big(h_t^3\big). \tag{4.12}$$

From (4.12), we obtain the result $R^i_z = \mathcal{O}(h_t^3)$. Similarly, from (2.11), Itô-Taylor expansions (4.1) and (4.2), we have

$$\begin{aligned}R^i_{z,3} &= \alpha h_t Z_{t_{i+1-\alpha}} - \mathbb{E}^x_{t_{i+1-\alpha}}\big[Y_{t_{i+1}} \Delta W^\top_{i+1-\alpha,i+1} + \alpha h_t f_{t_{i+1}} \Delta W^\top_{i+1-\alpha,i+1}\big] \\ &= \mathbb{E}^x_{t_{i+1-\alpha}}\left[\frac{\alpha^3 h_t^3}{2} u^{(1,0,0)}_{t_{i+1-\alpha}} + \mathcal{O}\big(h_t^4\big)\right]. \end{aligned} \tag{4.13}$$

From (4.13), we deduce the result $R^i_{z,3} = \mathcal{O}(h_t^3)$. □

Lemma 4.3. *Suppose the assumptions* (i)-(iii) *hold. Furthermore, let*

$$(t, Y_t, Z_t) \ \mapsto \ f(t, Y_t, Z_t) \in C_b^3.$$

Then, for $i = 0, 1, \ldots, N-1$, $R_y^i = \mathcal{O}(h_t^3)$.

Proof. From the definition of $\widehat{R}_{yf}^i$, we know

$$\widehat{R}_{yf}^i = h_t \mathbb{E}_{t_i}^x [f_{t_{i+1-\alpha}} - \widetilde{f}_{t_{i+1-\alpha}}] \leq h_t \big|\mathbb{E}_{t_i}^x [f_{t_{i+1-\alpha}} - \widetilde{f}_{t_{i+1-\alpha}}]\big|. \tag{4.14}$$

By the Lipschitz condition and the definitions of $\widetilde{Y}_{t_{i+1-\alpha}}$ and $\widetilde{Z}_{t_{i+1-\alpha}}$, we can rewrite (4.14) as

$$\begin{aligned}\widehat{R}_{yf}^i &\leq h_t L \mathbb{E}_{t_i}^x \left[|Y_{t_{i+1-\alpha}} - \widetilde{Y}_{t_{i+1-\alpha}}| + |Z_{t_{i+1-\alpha}} - \widetilde{Z}_{t_{i+1-\alpha}}|\right] \\ &= h_t L \mathbb{E}_{t_i}^x \Bigg[\big|Y_{t_{i+1-\alpha}} - \mathbb{E}_{t_{i+1-\alpha}}^x [Y_{t_{i+1}} + \alpha h_t f_{t_{i+1}}]\big| \\ &\qquad + \left|Z_{t_{i+1-\alpha}} - \mathbb{E}_{t_{i+1-\alpha}}^x \left[\frac{1}{\alpha h_t} Y_{t_{i+1}} \Delta W_{i+1-\alpha,i+1}^\top + f_{t_{i+1}} \Delta W_{i+1-\alpha,i+1}^\top\right]\right|\Bigg] \\ &= h_t L \mathbb{E}_{t_i}^x \left[\big|\widetilde{R}_{yf}^i\big| + \left|\frac{1}{\alpha h_t} R_{z,3}^i\right|\right].\end{aligned}$$

From Lemmas 4.1 and 4.2, we have

$$\widehat{R}_{yf}^i = \mathcal{O}\big(h_t^3\big). \tag{4.15}$$

By the definition of R_y^i, (4.15) and Lemma 4.1, we obtain

$$R_y^i = R_{yf}^i + \widehat{R}_{yf}^i = \mathcal{O}\big(h_t^3\big).$$

The proof is complete. □

In what follows, we focus on analyzing the convergence property of the scheme (2.16).

Theorem 4.1. *Assuming that conditions* (i)-(iii) *and* $(t, Y_t, Z_t) \mapsto f(t, Y_t, Z_t) \in C_b^3$ *are met, we further require that the initial values satisfy*

$$\max\big\{\mathbb{E}\big[\big|Y_{t_N} - Y_N^\pi\big|\big], \mathbb{E}\big[\big|Z_{t_N} - Z_N^\pi\big|\big]\big\} = \mathcal{O}\big(h_t^2\big).$$

Then we have, for sufficiently small time step h_t,

$$\mathbb{E}\Big[\sup_{0 \leq i \leq N} |\Delta Y_i|^2\Big] + \sum_{\ell=i}^{N-1} \mathbb{E}\big[h_t |\Delta Z_\ell|^2\big] \leq C h_t^4, \tag{4.16}$$

where C *denotes a positive constant.*

Proof. Let

$$\begin{aligned}\Delta Y_i &= Y_{t_i} - Y_i^\pi, & \Delta Z_i &= Z_{t_i} - Z_i^\pi, \\ \Delta Y_{i+1-\alpha} &= Y_{t_{i+1-\alpha}} - Y_{i+1-\alpha}^\pi, & \Delta Z_{i+1-\alpha} &= Z_{t_{i+1-\alpha}} - Z_{i+1-\alpha}^\pi.\end{aligned}$$

From (2.10) and (2.15), we have

$$\begin{aligned}|\Delta Z_i| = \Bigg|\mathbb{E}_{t_i}^x \Bigg[&\Delta Y_{i+1} \frac{2\Delta W_{i,i+1}^\top}{h_t} + \frac{1}{\alpha}\big(f_{i+1-\alpha} - \widetilde{f}_{i+1-\alpha}^\pi\big)\Delta W_{i,i+1-\alpha}^\top \\ &+ \frac{2\alpha - 1}{\alpha}\big(f_{t_{i+1}} - f_{i+1}^\pi\big)\Delta W_{i,i+1}^\top - \Delta Z_{i+1}\Bigg] + \frac{2}{h_t} R_z^i\Bigg|.\end{aligned} \tag{4.17}$$

The expression in (4.17) can be reformulated by applying the Lipschitz condition, as follows:

$$
\begin{aligned}
|\Delta Z_i| \leq & \left|\mathbb{E}^x_{t_i}\left[\Delta Y_{i+1}\frac{2\Delta W^\top_{i,i+1}}{h_t}\right]\right| + \frac{L}{\alpha}\mathbb{E}^x_{t_i}\left[(|\Delta Y_{i+1-\alpha}| + |\Delta Z_{i+1-\alpha}|)\left|\Delta W^\top_{i,i+1-\alpha}\right|\right] \\
& + \left|\frac{2\alpha-1}{\alpha}\right| L\mathbb{E}^x_{t_i}\left[(|\Delta Y_{i+1}| + |\Delta Z_{i+1}|)\left|\Delta W^\top_{i,i+1-\alpha}\right|\right] + \mathbb{E}^x_{t_i}\left[|\Delta Z_{i+1}|\right] + \left|\frac{2}{h_t}R^i_z\right|. \quad (4.18)
\end{aligned}
$$

From the Cauchy-Schwarz inequality, the inequality (4.18) is represented as

$$
\begin{aligned}
|\Delta Z_i| \leq & \left|\frac{2}{h_t}\sqrt{h_t\mathbb{E}^x_{t_i}\left[|\Delta Y_{i+1}|^2\right]}\right| \\
& + \frac{L}{\alpha}\Big(\sqrt{(1-\alpha)h_t\mathbb{E}^x_{t_i}\left[|\Delta Y_{i+1-\alpha}|^2\right]} + \sqrt{(1-\alpha)h_t\mathbb{E}^x_{t_i}\left[|\Delta Z_{i+1-\alpha}|^2\right]}\Big) \\
& + \left|\frac{2\alpha-1}{\alpha}\right| L\Big(\sqrt{h_t\mathbb{E}^x_{t_i}\left[|\Delta Y_{i+1}|^2\right]} + \sqrt{h_t\mathbb{E}^x_{t_i}\left[|\Delta Z_{i+1}|^2\right]}\Big) \\
& + \mathbb{E}^x_{t_i}\left[|\Delta Z_{i+1}|\right] + \left|\frac{2}{h_t}R^i_z\right|. \quad (4.19)
\end{aligned}
$$

From (2.5) and (2.6), we have

$$
\begin{aligned}
|\Delta Y_{i+1-\alpha}|^2 &= \left|\mathbb{E}^x_{t_{i+1-\alpha}}\left[\Delta Y_{i+1} + \alpha h_t(f_{t_{i+1}} - f^\pi_{i+1})\right] + \widetilde{R}^i_{yf}\right|^2 \\
&\leq \left|\mathbb{E}^x_{t_{i+1-\alpha}}\left[\Delta Y_{i+1} + \alpha h_t L(|\Delta Y_{i+1}| + |\Delta Z_{i+1}|)\right] + \widetilde{R}^i_{yf}\right|^2. \quad (4.20)
\end{aligned}
$$

From (2.11) and (2.12), we have

$$
\begin{aligned}
|\Delta Z_{i+1-\alpha}| &= \left|\mathbb{E}^x_{t_{i+1-\alpha}}\left[\frac{1}{\alpha h_t}\Delta Y_{i+1}\Delta W^\top_{i+1-\alpha,i+1} + \left(f_{t_{i+1}} - f^\pi_{i+1}\right)\Delta W^\top_{i+1-\alpha,i+1}\right] + R^i_{z,3}\right|^2 \\
&\leq \left|\mathbb{E}^x_{t_{i+1-\alpha}}\left[\frac{1}{\alpha h_t}\Delta Y_{i+1}\Delta W^\top_{i+1-\alpha,i+1} + L(|\Delta Y_{i+1}| + |\Delta Z_{i+1}|)\Delta W^\top_{i+1-\alpha,i+1}\right] + R^i_{z,3}\right|^2 \\
&\leq 2\mathbb{E}^x_{t_{i+1-\alpha}}\left[\left(\frac{2}{\alpha h_t} + 2\alpha h_t L^2 + 4L\right)\left|\Delta Y_{i+1}\right|^2 + 2\alpha h_t L^2\left|\Delta Z_{i+1}\right|^2\right] + 2\left|R^i_{z,3}\right|^2. \quad (4.21)
\end{aligned}
$$

Plugging (4.20) and (4.21) into (4.19), we obtain

$$
\begin{aligned}
|\Delta Z_i| \leq & \left|\frac{2}{h_t}\sqrt{h_t\mathbb{E}^x_{t_i}\left[|\Delta Y_{i+1}|^2\right]}\right| \\
& + \frac{L}{\alpha}\sqrt{2(1-\alpha)h_t\mathbb{E}^x_{t_i}\left[\left((1+\alpha h_t L)|\Delta Y_{i+1}| + \alpha h_t L|\Delta Z_{i+1}|\right)^2 + \left(\widetilde{R}^i_{yf}\right)^2\right]} \\
& + \frac{L}{\alpha}\sqrt{2(1-\alpha)h_t\mathbb{E}^x_{t_i}\left[\left(\frac{2}{\alpha h_t} + 2\alpha h_t L^2 + 4L\right)|\Delta Y_{i+1}|^2 + 2\alpha h_t L^2|\Delta Z_{i+1}|^2 + \left|R^i_{z,3}\right|^2\right]} \\
& + \left|\frac{2\alpha-1}{\alpha}\right| L\Big(\sqrt{h_t\mathbb{E}^x_{t_i}\left[|\Delta Y_{i+1}|^2\right]} + \sqrt{h_t\mathbb{E}^x_{t_i}\left[|\Delta Z_{i+1}|^2\right]}\Big) + \mathbb{E}^x_{t_i}\left[|\Delta Z_{i+1}|\right] + \left|\frac{2}{h_t}R^i_z\right|. \quad (4.22)
\end{aligned}
$$

Squaring (4.22), we deduce

$$
\begin{aligned}
|\Delta Z_i|^2 \leq & \Bigg(\frac{20}{h_t} + \frac{10(1-\alpha)h_t L^2}{\alpha^2}\Big(4 + 4\alpha^2 h_t L^2 + 8\alpha h_t L + \frac{2}{\alpha h_t} + 2\alpha h_t L^2 + 4L\Big) \\
& + \frac{10(2\alpha-1)^2 h_t L^2}{\alpha^2}\Bigg)\mathbb{E}^x_{t_i}\left[|\Delta Y_{i+1}|^2\right]
\end{aligned}
$$

$$+\left(\frac{10(1-\alpha)h_tL^2}{\alpha^2}\left(4\alpha^2h_t^2L^2+2\alpha h_tL^2\right)+\frac{10(2\alpha-1)^2h_tL^2}{\alpha^2}+5\right)\mathbb{E}_{t_i}^x\left[|\Delta Z_{i+1}|^2\right]$$
$$+\frac{20(1-\alpha)h_tL^2}{\alpha^2}\left(\left|\widetilde{R}_{yf}^i\right|^2+\left|R_{z,3}^i\right|^2\right)+5\left|\frac{2}{h_t}R_z^i\right|^2. \tag{4.23}$$

Next, we deal with the term $|Y_{t_i}-Y_i^\pi|$. From (2.13) and (2.14), we obtain

$$\begin{aligned}|\Delta Y_i| &\le \left|\mathbb{E}_{t_i}^x[\Delta Y_{i+1}]\right|+\left|\frac{h_t}{2\alpha}\mathbb{E}_{t_i}^x\left[\widetilde{f}_{t_{i+1-\alpha}}-\widetilde{f}_{i+1-\alpha}^\pi\right]\right|+\left|\frac{(2\alpha-1)h_t}{2\alpha}\mathbb{E}_{t_i}^x\left[f_{t_{i+1}}-f_{i+1}^\pi\right]\right|+\left|R_y^i\right|\\
&\le \left|\mathbb{E}_{t_i}^x[\Delta Y_{i+1}]\right|+\frac{h_t}{2\alpha}L\mathbb{E}_{t_i}^x\left[\left|\widetilde{Y}_{t_{i+1-\alpha}}-Y_{i+1-\alpha}^\pi\right|+\left|\widetilde{Z}_{t_{i+1-\alpha}}-Z_{i+1-\alpha}^\pi\right|\right]\\
&\quad+\left|\frac{2\alpha-1}{2\alpha}\right|h_tL\mathbb{E}_{t_i}^x[|\Delta Y_{i+1}|+|\Delta Z_{i+1}|]+\left|R_y^i\right|.\end{aligned} \tag{4.24}$$

By the definition of $\widetilde{Y}_{t_{i+1-\alpha}}$ and (2.6), we get

$$\begin{aligned}&\left|\widetilde{Y}_{t_{i+1-\alpha}}-Y_{i+1-\alpha}^\pi\right|\\
&=\left|\mathbb{E}_{t_{i+1-\alpha}}^x[Y_{t_{i+1}}+\alpha h_tf_{t_{i+1}}]-\mathbb{E}_{t_{i+1-\alpha}}^x\left[Y_{i+1}^\pi+\alpha h_tf_{i+1}^\pi\right]\right|\\
&\le(1+\alpha h_tL)\mathbb{E}_{t_i}^x[|\Delta Y_{i+1}|]+\alpha h_tL\mathbb{E}_{t_i}^x[|\Delta Z_{i+1}|].\end{aligned} \tag{4.25}$$

By the definition of $\widetilde{Z}_{t_{i+1-\alpha}}$ and (2.12), we deduce

$$\begin{aligned}&\left|\widetilde{Z}_{t_{i+1-\alpha}}-Z_{i+1-\alpha}^\pi\right|\\
&\le\left|\mathbb{E}_{t_{i+1-\alpha}}^x\left[\frac{1}{\alpha h_t}\Delta Y_{i+1}\Delta W_{i+1-\alpha,i+1}^\top+L(|\Delta Y_{i+1}|+|\Delta Z_{i+1}|)\Delta W_{i+1-\alpha,i+1}^\top\right]\right|.\end{aligned} \tag{4.26}$$

Substituting (4.25) and (4.26) into (4.24), we obtain

$$\begin{aligned}|\Delta Y_i| &\le\left(1+\frac{h_tL}{2\alpha}(1+\alpha h_tL)+\left|\frac{2\alpha-1}{\alpha}\right|h_tL\right)\mathbb{E}_{t_i}^x[|\Delta Y_{i+1}|]\\
&\quad+\left(\frac{h_t^2L^2}{\alpha^2}+\left|\frac{2\alpha-1}{\alpha}\right|h_tL\right)\mathbb{E}_{t_i}^x[|\Delta Z_{i+1}|]+\left(\frac{L}{2\alpha^2}+\frac{h_tL^2}{2\alpha}\right)\mathbb{E}_{t_i}^x\left[\left|\Delta Y_{i+1}\Delta W_{i+1-\alpha,i+1}^\top\right|\right]\\
&\quad+\frac{h_tL^2}{2\alpha}\mathbb{E}_{t_i}^x\left[\left|\Delta Z_{i+1}\Delta W_{i+1-\alpha,i+1}^\top\right|\right]+\left|R_y^i\right|.\end{aligned} \tag{4.27}$$

Squaring (4.27), we deduce

$$\begin{aligned}|\Delta Y_i|^2 &\le 5\left(4+\frac{h_t^2L^2}{\alpha^2}+h_t^4L^4+\frac{4(2\alpha-1)^2}{\alpha^2}h_t^2L^2+\frac{h_tL^2}{2\alpha^3}+\frac{h_t^3L^4}{2\alpha}\right)\mathbb{E}_{t_i}^x\left[|\Delta Y_{i+1}|^2\right]\\
&\quad+5\left(\frac{h_t^4L^4}{2}+\frac{2(2\alpha-1)^2}{\alpha^2}h_t^2L^2+\frac{h_t^3L^4}{4\alpha}\right)\mathbb{E}_{t_i}^x\left[|\Delta Z_{i+1}|^2\right]+5\left|R_y^i\right|^2.\end{aligned} \tag{4.28}$$

Adding (4.23) to (4.28), we deduce

$$\begin{aligned}&|\Delta Y_i|^2+h_t|\Delta Z_i|^2\\
&\le 5\Bigg(8+\frac{4(1-\alpha)}{\alpha^3}L^2h_t+\frac{6(2\alpha-1)^2+1+8(1-\alpha)+8L(1-\alpha)}{\alpha^2}L^2h_t^2\\
&\quad+\frac{L+32(1-\alpha)+8L(1-\alpha)}{2\alpha}L^3h_t^3+\left(1+8(1-\alpha)\right)L^4h_t^4\Bigg)\mathbb{E}_{t_i}^x\left[|\Delta Y_{i+1}|^2\right]\end{aligned}$$

$$
\begin{aligned}
&+5\Big(h_t+\frac{4(2\alpha-1)^2}{\alpha^2}L^2h_t^2+\frac{16(1-\alpha)+1}{4\alpha}L^4h_t^3+\frac{16(1-\alpha)+1}{2}L^4h_t^4\Big)\mathbb{E}_{t_i}^x\left[|\Delta Z_{i+1}|^2\right]\\
&+5\big|R_y^i\big|^2+\frac{20(1-\alpha)h_t^2L^2}{\alpha^2}\Big(\big|\widetilde{R}_{yf}^i\big|^2+\big|R_{z,3}^i\big|^2\Big)+\frac{20}{h_t}\big|R_z^i\big|^2\\
=\;&40(1+C_1h_t)\mathbb{E}_{t_i}^x\left[|\Delta Y_{i+1}|^2\right]+C_2h_t\mathbb{E}_{t_i}^x\left[|\Delta Z_{i+1}|^2\right]\\
&+5\big|R_y^i\big|^2+\frac{20(1-\alpha)h_t^2L^2}{\alpha^2}\Big(\big|\widetilde{R}_{yf}^i\big|^2+\big|R_{z,3}^i\big|^2\Big)+\frac{20}{h_t}\big|R_z^i\big|^2,
\end{aligned}
\tag{4.29}
$$

where

$$
\begin{aligned}
C_1=\;&\frac{1}{8}\Big(\frac{4(1-\alpha)}{\alpha^3}L^2+\frac{6(2\alpha-1)^2+1+8(1-\alpha)+8L(1-\alpha)}{\alpha^2}L^2h_t\\
&+\frac{L+32(1-\alpha)+8L(1-\alpha)}{2\alpha}L^3{h_t}^2+\big(1+8(1-\alpha)\big)L^4{h_t}^3\Big),\\
C_2=\;&5\Big(1+\frac{4(2\alpha-1)^2}{\alpha^2}L^2h_t+\frac{16(1-\alpha)+1}{4\alpha}L^4{h_t}^2+\frac{16(1-\alpha)+1}{2}L^4{h_t}^3\Big).
\end{aligned}
$$

There exist two positive constants C_0 and C_3 such that

$$
\begin{aligned}
&C_0\left|\Delta Y_i\right|^2+h_t\left|\Delta Z_i\right|^2\\
\le\;&(1+C_3h_t)\Big(C_0\mathbb{E}_{t_i}^x\Big[\big|\Delta Y_{i+1}^\pi\big|^2\Big]+C_2h_t\mathbb{E}_{t_i}^x\Big[\big|\Delta Z_{i+1}^\pi\big|^2\Big]\Big)\\
&+C_3\big|R_y^i\big|^2+C_3h_t^2\Big(\big(\widetilde{R}_{yf}^i\big)^2+\big|R_{z,3}^i\big|^2\Big)+\frac{C_3}{h_t}\big|R_z^i\big|^2.
\end{aligned}
\tag{4.30}
$$

From (4.30) and the tower property of the conditional expectations, we deduce

$$
\begin{aligned}
&C_0\mathbb{E}\big[|\Delta Y_i|^2\big]+(1-C_2)h_t\sum_{\ell=i}^{N-1}(1+C_3h_t)^{\ell-i}\mathbb{E}\big[|\Delta Z_\ell|^2\big]\\
\le\;&(1+C_3h_t)^{N-i}\big(C_0\mathbb{E}\big[|\Delta Y_N|^2\big]+C_2h_t\mathbb{E}\big[|\Delta Z_N|^2\big]\big)\\
&+\sum_{\ell=i}^{N-1}(1+C_3h_t)^{\ell-i}\mathbb{E}\Big[C_3\big|R_y^\ell\big|^2+C_3h_t^2\Big(\big(\widetilde{R}_{yf}^\ell\big)^2+\big|R_{z,3}^\ell\big|^2\Big)+\frac{C_3}{h_t}\big|R_z^\ell\big|^2\Big].
\end{aligned}
\tag{4.31}
$$

Furthermore, we have

$$
\begin{aligned}
&\mathbb{E}\big[|\Delta Y_i|^2\big]+h_t\sum_{\ell=i}^{N-1}\mathbb{E}\big[|\Delta Z_\ell|^2\big]\\
\le\;&C\big(\mathbb{E}\big[|\Delta Y_N|^2\big]+h_t\mathbb{E}\big[|\Delta Z_N|^2\big]\big)\\
&+\sum_{\ell=i}^{N-1}C\mathbb{E}\Big[\big|R_y^\ell\big|^2+h_t^2\Big(\big(\widetilde{R}_{yf}^\ell\big)^2+\big|R_{z,3}^\ell\big|^2\Big)+\frac{1}{h_t}\big|R_z^\ell\big|^2\Big].
\end{aligned}
\tag{4.32}
$$

From Lemmas 4.1-4.3, we have

$$
\mathbb{E}\Big[\big|R_y^i\big|^2+h_t^2\Big(\big|\widetilde{R}_{yf}^i\big|^2+\big|R_{z,3}^i\big|^2\Big)+\frac{1}{h_t}\big|R_z^i\big|^2\Big]=\mathcal{O}\big(h_t^5\big).
\tag{4.33}
$$

From

$$
\max\big\{\mathbb{E}\big[\big|Y_{t_N}-Y_N^\pi\big|\big],\mathbb{E}\big[\big|Z_{t_N}-Z_N^\pi\big|\big]\big\}=\mathcal{O}\big(h_t^2\big)
$$

and (4.33), we have the following estimates:

$$\mathbb{E}\big[|\Delta Y_i|^2\big] + \sum_{\ell=i}^{N-1} \mathbb{E}\big[h_t|\Delta Z_\ell|^2\big] \le Ch_t^4.$$

The proof is complete. □

Remark 4.1. Through the proof provided in Theorem 4.1, we have successfully demonstrated that the proposed one step scheme achieves a second order convergence rate. This result is further confirmed by the numerical examples constructed in the following section.

Remark 4.2. This paper differs from [7,29] in that our one step scheme is fully explicit for both Y and Z, whereas the schemes in [7,29] are explicit for Y but implicit for Z. Moreover, when $\alpha = 1$, our method reduces to the Crank Nicolson method, as discussed in [7,29], demonstrating that the Crank Nicolson method is a special case within our framework.

5. Numerical Experiments

In this section, we illustrate the accuracy and effectiveness of the aforementioned scheme through numerical experiments. The time interval $[0, T]$ is uniformly divided into N parts, giving a time step size $h_t = T/N$, where T represents the terminal time. The time step sizes adopted in our experiments are $N = 1/2^m, m = 3, \dots, 7$. To compute the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ within the proposed scheme, we utilize the Gauss-Hermite quadrature method, together with spatial interpolation. By choosing a sufficiently large number of Gauss-Hermite quadrature points K, the local truncation error becomes insignificant; in the numerical examples provided, we set $K = 12$. We apply cubic spline interpolation for the spatial interpolation process.

In the tables below, the errors $\mathrm{err}_y := |Y_0 - Y_0^\pi|$ and $\mathrm{err}_z := |Z_0 - Z_0^\pi|$ denote the differences between the exact solution (Y_t, Z_t) of Eq. (1.1) at $t = 0$ and the corresponding numerical results (Y_0^π, Z_0^π) obtained from the scheme. Let CR denote the convergence rate with respect to the time step size h_t, which can be calculated using the least squares regression method. All the numerical tests are implemented in Matlab R2023b on a desktop computer with Intel Core i5-12600KF 10-Core Processor (4.9 GHz) and 32 GB DDR4 RAM (3600 MHz). The solution process is presented in the Algorithm 5.1.

Example 5.1. Consider the BSDE as below

$$\begin{cases} -dY_t = \big(-Y_t^3 + 2.5Y_t^2 - 1.5Y_t\big)dt - Z_t dW_t, & t \in [0, T], \\ Y_T = \dfrac{\exp(W_T + T)}{\exp(W_T + T) + 1}. \end{cases} \tag{5.1}$$

The solution to the BSDE (5.1) is

$$Y_t = \frac{\exp(W_t + t)}{\exp(W_t + t) + 1}, \quad Z_t = \frac{\exp(W_t + t)}{\big(\exp(W_t + t) + 1\big)^2}.$$

In this example, we set $T = 1$ and the exact solution of the Eq. (5.1) at $t = 0$ is $(Y_0, Z_0) = (0.5, 0.25)$. Table 5.1 presents the absolute errors, err_y and err_z, along with their convergence

Algorithm 5.1: The Algorithm for Solving FBSDEs Based on Scheme (2.16).

Input: $K, \alpha, N, Y_N^\pi, Z_N^\pi$.
for $i = N-1$ **to** 1 **do**
  Compute the interpolation points at time level t_{n+1}.
  for $k = 1$ **to** K **do**

$$Y_{i+1-\alpha}^\pi = \mathbb{E}_{t_{i+1-\alpha}}^x\left[Y_{i+1}^\pi + \alpha h_t f_{i+1}^\pi\right],$$

$$Z_{i+1-\alpha}^\pi = \mathbb{E}_{t_{i+1-\alpha}}^x\left[\frac{1}{\alpha h_t} Y_{i+1}^\pi \Delta W_{i+1-\alpha,i+1}^\top + f_{i+1}^\pi \Delta W_{i+1-\alpha,i+1}^\top\right].$$

  end
  for $k = 1$ **to** K **do**

$$Y_i^\pi = \mathbb{E}_{t_i}^x\left[Y_{i+1}^\pi + \frac{h_t}{2\alpha}\widetilde{f}_{i+1-\alpha}^\pi + h_t\left(1 - \frac{1}{2\alpha}\right) f_{i+1}^\pi\right],$$

$$Z_i^\pi = \mathbb{E}_{t_i}^x\left[\frac{2}{h_t} Y_{i+1}^\pi \Delta W_{i,i+1}^\top + \frac{1}{\alpha}\widetilde{f}_{i+1-\alpha}^\pi \Delta W_{i,i+1-\alpha}^\top + \frac{2\alpha-1}{\alpha} f_{i+1}^\pi \Delta W_{i,i+1}^\top - Z_{i+1}^\pi\right].$$

  end
end
Output: Y_0^π, Z_0^π.

Table 5.1: Error values and convergence rates for the scheme in Example 5.1.

N	$\alpha = 1/4$		$\alpha = 1/2$		$\alpha = 3/4$		$\alpha = 1$	
	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$
8	1.3590E-04	2.4418E-05	1.2017E-04	1.0366E-04	1.0258E-04	1.9060E-04	8.2793E-05	2.8535E-04
16	3.4884E-05	4.9078E-06	3.0986E-05	2.6713E-05	2.6797E-05	4.9519E-05	2.2350E-05	7.3383E-05
32	8.8581E-06	1.1203E-06	7.8802E-06	6.8085E-06	6.8659E-06	1.2627E-05	5.8187E-06	1.8580E-05
64	2.2179E-06	2.3749E-07	1.9729E-06	1.6882E-06	1.7234E-06	3.1556E-06	1.4697E-06	4.6399E-06
128	5.5778E-07	5.2154E-08	4.9651E-07	4.1841E-07	4.3466E-07	7.8672E-07	3.7230E-07	1.1572E-06
CR	1.9833	2.2111	1.9811	1.9889	1.9724	1.9813	1.9520	1.9875

Table 5.2: Run time(s) of the scheme for Example 5.1.

α \ N	8	16	32	64	128
1/4	4.5610	9.0057	18.0326	36.4851	74.9443
1/2	4.5909	9.0988	18.1107	36.1546	75.3838
3/4	4.5842	9.0513	18.6429	38.0595	74.9173
1	4.5079	8.9543	17.9163	36.0204	74.6518

rates, for various values of α using the described method. The run time of the scheme for Example 5.1 is provided in Table 5.2. Fig. 5.1 shows the log-log plots of err_y and err_z against time interval h_t.

Example 5.2. The FitzHugh-Nagumo (FHN) equation is widely used in biology and genetics, particularly in the mathematical modeling of electrophysiological systems and the mathematical

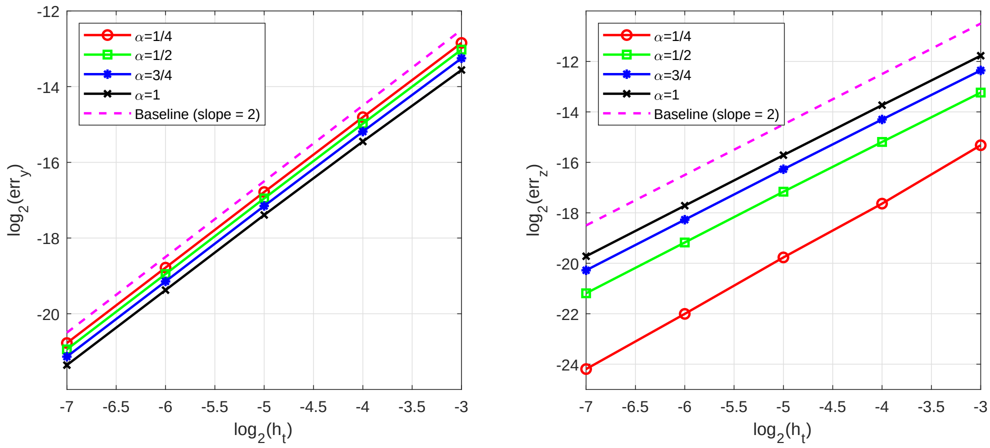

Fig. 5.1. The plot of err_y against time interval h_t in Example 5.1.

modeling of neuronal dynamics.The following is the partial differential equation (PDE) form of a simplified case of the FHN equation (taken from [19]):

$$\begin{cases} -\partial_t u - \dfrac{1}{2}\Delta u - (cu^3 + bu^2 - au) = 0, \\ u(T,x) = g(x). \end{cases} \tag{5.2}$$

Let $c = -1, b = 1 + a, a \in \mathbb{R}$, and $g(x) = (1 + e^x)^{-1}$, then

$$u(t,x) = \big(1 + \exp\{x - (0.5 - a)(T - t)\}\big)^{-1} \in C_b^\infty([0,T] \times \mathbb{R}) \tag{5.3}$$

is the solution of the Eq. (5.2). The corresponding FBSDEs is given by

$$\begin{cases} X_t = x_0 + \displaystyle\int_0^t dW_s, \\ Y_t = g(X_T) + \displaystyle\int_t^T \big[-Y_s^3 + (1+a)Y_s^2 - aY_s \big] ds - \int_t^T Z_s dW_s. \end{cases} \tag{5.4}$$

From Itô-Taylor formula, the analytical solutions of the BSDE (5.4) can be represented in the following form:

$$\begin{aligned} Y_t &= \frac{1}{1 + \exp\{X_t - (0.5 - a)(T - t)\}}, \\ Z_t &= -\frac{\exp\{X_t - (0.5 - a)(T - t)\}}{\big(1 + \exp\{X_t - (0.5 - a)(T - t)\}\big)^2}. \end{aligned}$$

Let $T = 1, a = -0.5$, and $x_0 = 1$, then the exact solution of the Eq. (5.4) at $t = 0$ is $(Y_0, Z_0) = (0.5, -0.25)$. Table 5.3 presents the absolute errors err_y and err_z obtained using the aforementioned method for the various values of α, accompanied by their respective convergence rates. The run time of the scheme for Example 5.2 is provided in Table 5.4. Fig. 5.2 shows the log-log plots of err_y and err_z against time interval h_t.

Let $T = 1, a = -1$, and $x_0 = 1.5$, then the exact solution of the Eq. (5.4) at $t = 0$ is $(Y_0, Z_0) = (0.5, 0.25)$. Table 5.5 presents the absolute errors err_y and err_z obtained using the aforementioned method for the various values of α, accompanied by their respective convergence rates. The run time of the scheme for Example 5.2 is provided in Table 5.6. Fig. 5.3 shows the log-log plots of err_y and err_z against time interval h_t.

From the errors, convergence rates and log-log plots listed in Tables 5.1-5.6 and Figs. 5.1-5.3, we conclude that

1. In the examples provided, the scheme demonstrates a convergence rate close to order two, as illustrated by Tables 5.1, 5.3 and 5.5. This finding aligns with the second order error estimate presented in Theorem 4.1.

2. As illustrated in Figs. 5.1-5.3, the slopes of the fitted curves closely align with the baseline slope of 2, regardless of the value of α. The specific outcomes for different α are influenced by the particular examples used.

3. The run time in Tables 5.2, 5.4 and 5.6 demonstrate that, regardless of the value of α, the scheme exhibits approximately the same efficiency. In the examples discussed, for a fixed N, the run time for the four different values of α show minimal variation.

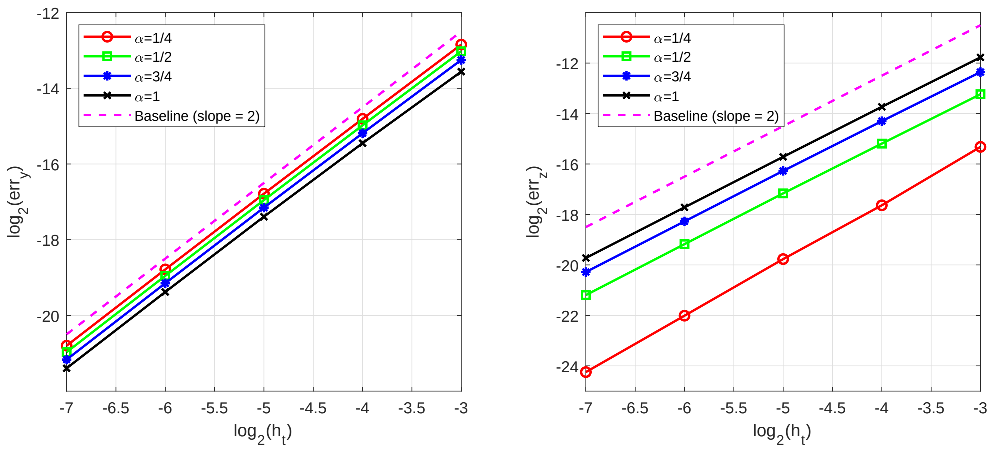

Fig. 5.2. Log-log plot of err_y against time interval h_t in Example 5.2 ($a = -0.5$, and $x_0 = 1$).

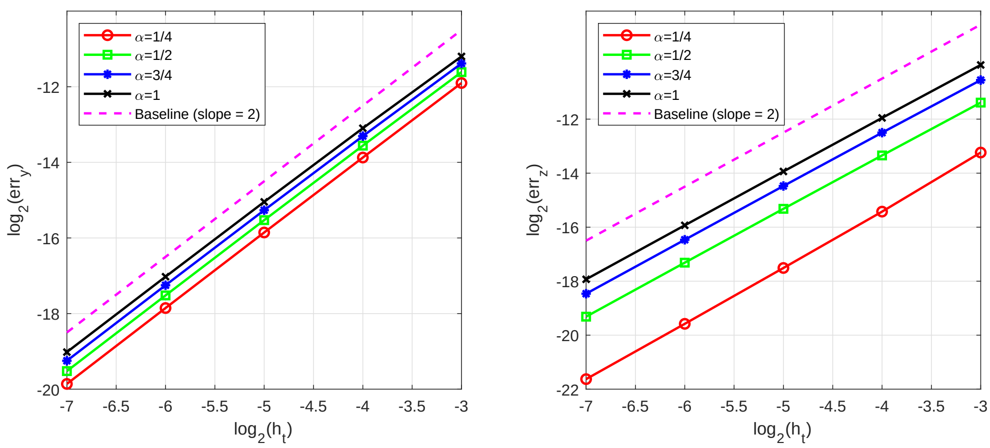

Fig. 5.3. Log-log plot of err_y against time interval h_t in Example 5.2 ($a = -1$, and $x_0 = 1.5$).

Table 5.3: Error values and convergence rates for the scheme in Example 5.2 ($a = -0.5$, and $x_0 = 1$).

N	$\alpha = 1/4$		$\alpha = 1/2$		$\alpha = 3/4$		$\alpha = 1$	
	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$
8	1.3590E-04	2.4418E-05	1.2017E-04	1.0366E-04	1.0258E-04	1.9060E-04	8.2793E-05	2.8535E-04
16	3.4883E-05	4.9075E-06	3.0985E-05	2.6713E-05	2.6795E-05	4.9519E-05	2.2349E-05	7.3383E-05
32	8.8556E-06	1.1198E-06	7.8777E-06	6.8080E-06	6.8634E-06	1.2626E-05	5.8162E-06	1.8579E-05
64	2.2129E-06	2.3651E-07	1.9679E-06	1.6873E-06	1.7184E-06	3.1546E-06	1.4648E-06	4.6389E-06
128	5.4779E-07	5.0245E-08	4.8655E-07	4.1652E-07	4.2471E-07	7.8485E-07	3.6235E-07	1.1553E-06
CR	1.9888	2.2225	1.9873	1.9903	1.9795	1.9820	1.9603	1.9880

Table 5.4: Run time(s) of the scheme for Example 5.2 ($a = -0.5$, and $x_0 = 1$).

α \ N	8	16	32	64	128
1/4	4.4760	8.9910	17.5931	35.7508	72.3901
1/2	4.4913	8.8813	17.8321	35.4145	72.4596
3/4	4.4974	8.9137	17.7552	35.3493	73.3658
1	4.5146	8.8281	17.9691	36.1171	71.7899

Table 5.5: Error values and convergence rates for the scheme in Example 5.2 ($a = -1$, and $x_0 = 1.5$).

N	$\alpha = 1/4$		$\alpha = 1/2$		$\alpha = 3/4$		$\alpha = 1$	
	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$
8	2.6130E-04	1.0356E-04	3.1917E-04	3.7300E-04	3.7406E-04	6.6506E-04	4.2542E-04	9.8084E-04
16	6.6635E-05	2.2732E-05	8.2900E-05	9.6144E-05	9.8711E-05	1.7250E-04	1.1412E-04	2.5192E-04
32	1.6869E-05	5.3519E-06	2.1155E-05	2.4429E-05	2.5385E-05	4.3883E-05	2.9564E-05	6.3724E-05
64	4.2222E-06	1.2762E-06	5.3214E-06	6.1340E-06	6.4137E-06	1.1040E-05	7.4996E-06	1.5993E-05
128	1.0505E-06	3.0869E-07	1.3289E-06	1.5341E-06	1.6063E-06	2.7656E-06	1.8830E-06	4.0030E-06
CR	1.9897	2.0935	1.9777	1.9822	1.9671	1.9785	1.9567	1.9851

Table 5.6: Run time(s) of the scheme for Example 5.2 ($a = -1$, and $x_0 = 1.5$).

α \ N	8	16	32	64	128
1/4	4.3599	8.4507	17.0215	34.1154	69.5319
1/2	4.2851	8.5262	16.9074	33.9132	69.1714
3/4	4.2347	8.3933	16.5688	33.4880	69.3673
1	4.2455	8.3685	16.9407	33.5637	68.5035

6. Conclusions

We have proposed a novel explicit one step scheme for solving decoupled forward backward stochastic differential equations. The stability analysis of the proposed numerical scheme has been conducted. We subsequently provided a rigorous proof that the proposed scheme achieves the second order convergence rates. Additionally, when $\alpha = 1$, our scheme degenerates into the Crank-Nicolson method as presented in [7, 29], making it a special case within our framework. Furthermore, this work provides a reference for how we can develop a new one step higher order scheme.

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