Journal of Computational Mathematics
Vol.xx, No.x, 2025, 1–19.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2505-m2024-0260

A NOVEL SCHEME FOR THE NONLINEAR SUBDIFFUSION EQUATION WITH A VARIABLE EXPONENT*

Chaobao Huang and Yujie Yu
School of Statistics and Mathematics, Shandong University of Finance and Economics, Jinan 250014, China
Emails: huangcb@sdufe.edu.cn, cinta_@163.com
Na An
School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China
Email: annabetter@sdnu.edu.cn
Hu Chen[1)]
School of Mathematical Sciences, Ocean University of China, Qingdao 266100, China
Email: chenhu@ouc.edu.cn

Abstract

In this work, a nonlinear subdiffusion initial-boundary value problem with a variable exponent is considered, whose solution behaves a weak singularity at initial time. By adding a corrected term to the nonuniform L1 scheme, a novel scheme is investigated to approximate the time-fractional Caputo derivative with a variable exponent. This scheme allows us to use a smaller grading parameter r to obtain a similar level of accuracy as that of the L1 method. Combining the proposed scheme with the finite element method in space and the Newton linearization for the nonlinear term, a fully discrete scheme is constructed. To obtain the unconditional optimal error estimate, the temporal-spatial splitting technique is adopted to derive the boundedness of the computed solution U_h^n in L^∞-norm. With the help of this bound and the discrete fractional Gronwall inequality, the optimal error analysis without certain temporal restrictions dependent on the spatial mesh size is derived. Furthermore, by using a simple postprocessing technique of the computed solution, the convergence order in the spatial direction is improved. Finally, numerical experiments are presented to verify the theoretical findings.

Mathematics subject classification: 65M60, 65M12, 35R11.
Key words: The nonlinear subdiffusion equation, Variable exponents, Weak singularity, The corrected L1 scheme, Graded meshes.

1. Introduction

The subdiffusion equation represents a class of mathematical models that describe transport processes occurring at a slower rate than classical diffusion. In contrast to normal diffusion, subdiffusion is characterized by a linear relationship between the mean squared displacement of particles and time (as described by Fick's second law and leading to a Gaussian probability distribution). This behavior is often observed in complex media with obstacles or traps that hinder the motion of particles, such as porous materials, biological tissues, and polymer networks. However, the subdiffusion equation with a variable exponent extends the standard

* Received November 23, 2024 / Revised version received April 19, 2025 / Accepted May 12, 2025 / Published online September 23, 2025 /
1) Corresponding author

subdiffusion equation by allowing the exponent of subdiffusion to vary in space or time. This generalization allows these equations to model many problems ranging from science and engineering fields, such as diffusion processes with changing diffusion modes [20], processing of geographical data [2], and signature verification [19].

This paper considers the numerical simulation for the following nonlinear subdiffusion initial-boundary value problem with a variable exponent:

$$D_t^{\alpha(t)}u(x,t) - \kappa^2\Delta u(x,t) = f\big(u(x,t)\big) + g(x,t), \quad (x,t) \in Q := \Omega \times (0,T], \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.1b}$$

$$u|_{\partial\Omega} = 0, \quad 0 < t \le T, \tag{1.1c}$$

where $0 < \alpha(t) \le \alpha^* < 1, \kappa$ is a given constant, $\Omega \subset \mathbb{R}^d$ $(d \in \{1,2,3\}), u_0 \in C(\bar{\Omega})$, $f \in C^2(\mathbb{R})$ and $g \in C(\bar{Q})$. Moreover, the variable exponent time-fractional operator $D_t^{\alpha(t)}$ given in (1.1a) is defined by

$$D_t^{\alpha(t)}v(t) := \frac{1}{\Gamma\big(1-\alpha(t)\big)}\int_0^t (t-s)^{-\alpha(t)}v'(s)\,ds, \quad v \in C^1[0,T]. \tag{1.2}$$

In recent years, the subdiffusion equation with weakly singular solutions has attracted a lot of attention. Many efficient numerical methods have been developed for this problem, such as the k-step BDF (BDF represents the backward difference formula) convolution quadrature [9], the nonuniform L1 method [11, 16], the nonuniform Alikhanov method [12, 25], the L1 method based on the change of variable [14], and the corrected L1 method [21]. However, numerical simulation for the subdiffusion equation with a variable exponent is relatively less. Kheirkhah *et al.* [10] developed a highly accurate numerical method to solve the time-fractional advection-reaction-subdiffusion equation with a variable exponent, which used a third-order weighted-shifted Grünwald scheme in the temporal direction and a fourth-order compact finite difference scheme in the spatial direction. Zheng and Wang [24] proposed a fully discrete finite element method to solve the variable exponent Caputo and Riemann-Liouville mobile-immobile subdiffusion equation with non-smooth solutions, and obtained the optimal convergent result on a uniform temporal partition. Quintana-Murillo and Yuste [15] proposed an adaptive finite difference scheme for subdiffusion equations with a variable exponent, where the Caputo derivative was approximated by the L1 scheme on nonhomogeneous time steps. Ye *et al.* [22] combined the quadratic spline collocation (QSC) method and the L1$^+$ formula to propose a QSC-L1$^+$ scheme for the variable exponent time fractional mobile/immobile diffusion. Tan [17] introduced a novel temporal second-order fully discrete approach of the finite element method (FEM) and its fast two-grid FEM on nonuniform meshes to solve nonlinear time-fractional variable exponent mobile/immobile equations with a solution exhibiting weak regularity, where the averaged L1 formula on graded meshes in the temporal domain was utilized. Zheng [23] developed a convolution method to study the well-posedness, regularity, an inverse problem, and numerical approximation for the subdiffusion equation with variable exponent.

Actually, the error analysis of the nonlinear subdiffusion equation (1.1) with a variable exponent is relatively less. Huang and Chen [5] presented the nonuniform L1 scheme to solve the linear subdiffusion equation with a variable exponent, and $\min\{2-\alpha^*, r\alpha^*\}$ order was achieved in the temporal direction. This convergent result implies that achieving global accuracy of order $2-\alpha^*$ for smaller α^* requires choosing larger r, which necessitates a very dense grid near the initial time. To overcome this difficulty, this paper constructs the corrected L1 scheme on nonuniform meshes to approximate the Caputo derivative with variable exponent, enabling sparser temporal discretization near the initial time. By combining the corrected L1 scheme for temporal discretization, the finite element method for spatial discretization, and the Newton linearization for the nonlinear term, we develop a novel numerical scheme for the problem (1.1). By means of the temporal-spatial splitting technique, optimal error estimates of the proposed scheme can be obtained without relying on temporal-spatial step size restrictions.

The main contribution of this paper is as follows:

- A novel scheme on graded meshes is constructed for discretizing the time-fractional Caputo derivative with a variable exponent, which allows the grid closer to the initial time to be sparser.
- With the help of the temporal-spatial splitting technique, the unconditional optimal convergent result of the proposed fully discrete scheme is given.
- By using a simple postprocessing technique of the computed solution, the convergence order in the spatial direction is improved.

The paper is organized as follows. In Section 2, a corrected L1 (CL1) scheme for the variable exponent Caputo derivative is investigated, and then a fully discrete CL1-FEM scheme is constructed. In Section 3, by using the temporal-spatial splitting technique, the optimal convergence analysis for the proposed scheme is developed. In Section 4, numerical experiments are presented to verify the theoretical analysis. In Section 5, a concluding remark is given.

Notation. The generic constant C is independent of the mesh, and it can take different values in different places. Let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the $L^2(\Omega)$ norm and its associated inner product, respectively. For any integer $q \geq 1$, $\|\cdot\|_q$ and $|\cdot|_q$ represent the norm and seminorm on the Sobolev space $H^q(\Omega)$.

2. The Construction of the Fully Discrete Finite Element Method

In this section, we will investigate the corrected L1 scheme for the variable exponent time-fractional derivative and then construct the fully discrete scheme for the initial boundary value problem (1.1).

Define temporal graded meshes by $t_n := T(n/N)^r$ for $n = 0, 1, \dots, N$, where N is a positive integer and $r \geq 1$ controls the mesh grading. Set $\tau_n = t_n - t_{n-1}$ for $n = 1, \dots, N$. It is obvious that

$$\tau_n = T\left(\frac{n}{N}\right)^r - T\left(\frac{n-1}{N}\right)^r \leq rTN^{-r}n^{r-1}, \quad n = 2, 3, \dots, N. \tag{2.1}$$

For $n = 1, 2, \dots, N$, the variable exponent Caputo derivative $D_t^{\alpha(t_n)}v(t_n)$ can be estimated by nonuniform L1 schemes

$$D_t^{\alpha(t_n)}v(t_n) \approx \delta_N^{\alpha(t_n)}v^n := \Theta_0^{(n)}v^n - \Theta_{n-1}^{(n)}v^0 + \sum_{j=1}^{n-1} v^{n-j}\big(\Theta_j^{(n)} - \Theta_{j-1}^{(n)}\big), \tag{2.2}$$

where

$$\Theta_{j-1}^{(n)} := \frac{1}{\Gamma\big(1-\alpha(t_n)\big)\tau_{n-j+1}} \int_{t_{n-j}}^{t_{n-j+1}} (t_n - s)^{-\alpha(t_n)}\, ds, \quad j = 1, \dots, n.$$

It is clear that we have $\Theta_j^{(n)} < \Theta_{j-1}^{(n)}$ for $1 \leq j \leq n-1$ from the mean value theorem.

Applying the notation of Eq. (2.2), we define the following complementary discrete kernels $\mathbb{P}_j^{(n)}$ by:

$$\mathbb{P}_0^{(n)} = \frac{1}{\Theta_0^{(n)}},$$

$$\mathbb{P}_j^{(n)} = \frac{1}{\Theta_0^{(n-j)}} \sum_{k=0}^{j-1} \big(\Theta_{j-k-1}^{(n-k)} - \Theta_{j-k}^{(n-k)}\big)\mathbb{P}_{n-k}^{(n)}, \quad n = 1, \ldots, N, \quad j = 1, \ldots, n-1.$$

Define $\lambda_N := 1/\ln N$. Imitating [4, Corollary 4.1], one has

$$\sum_{k=1}^{n} t_k^{-\alpha(t_k)} \mathbb{P}_{n-k}^{(n)} \leq \frac{e^r \max_{1\leq k\leq n} \Gamma\big(1+\lambda_N - \alpha(t_k)\big)}{\Gamma(1+\lambda_N)}, \tag{2.3}$$

$$\sum_{k=1}^{n} \mathbb{P}_{n-k}^{(n)} \leq \frac{e^r t_n^{\alpha^*} \max_{1\leq k\leq n} \Gamma\big(1+\lambda_N - \alpha(t_k)\big)}{\Gamma(1+\lambda_N)}. \tag{2.4}$$

From [6, Lemma 4.1], we have

$$\big(\delta_N^{\alpha(t_n)} v^n, v^n\big) \geq \frac{1}{2} \delta_N^{\alpha(t_n)} \|v^n\|^2, \quad v^n = v(\cdot, t^n) \in L^2(\Omega), \quad n = 1, 2, \ldots, N, \tag{2.5}$$

where the $L_2(\Omega)$ inner product is defined by $(v, w) := \int_\Omega vw\,dx$ for any $v, w \in L_2(\Omega)$.

Define

$$\Phi(n,\sigma) := \begin{cases} t_1^{-\alpha(t_1)} N^{-r\sigma}, & \text{if} \quad n = 1, \\ t_n^{-\alpha(t_n)} N^{-\min\{2-\alpha^*, r\sigma\}}, & \text{if} \quad 2 \leq n \leq N. \end{cases}$$

Form [5, Lemma 2.1], we obtain that the truncation error of the L1 scheme (2.2) satisfies

$$\big\|D_t^{\alpha(t_n)} v^n - \delta_N^{\alpha(t_n)} v^n\big\| \leq C\Phi(n,\sigma), \quad |v^{(l)}(t)| \leq Ct^{\sigma - l}, \quad l = 0, 1, 2. \tag{2.6}$$

Now we construct the corrected L1 scheme, which allows us to use a smaller r than the L1 scheme to achieve higher accuracy. The corrected L1 scheme for the Caputo derivative at $t = t_n$ is defined by

$$\mathbb{D}_N^{\alpha(t_n)} v^n = \delta_N^{\alpha(t_n)} v^n + W_n^\sigma (v^1 - v^0), \quad n \geq 1, \tag{2.7}$$

where the weight W_n^σ is chosen such that $\mathbb{D}_N^{\alpha(t_n)} v^n = D_t^{\alpha(t_n)} v(t_n)$ for $v = t^\sigma$. By directly calculating, we have

$$W_n^\sigma = t_1^{-\sigma} \left[\frac{\Gamma(1+\sigma)}{\Gamma\big(1+\sigma-\alpha(t_n)\big)} t_n^{\sigma - \alpha(t_n)} - \Theta_0^{(n)} t_n^\sigma - \sum_{k=1}^{n-1} t_{n-k}^\sigma \big(\Theta_k^{(n)} - \Theta_{k-1}^{(n)}\big) \right]. \tag{2.8}$$

In our later analysis, we should keep in mind that the solution of the problem (1.1) satisfies the following assumption.

Assumption 2.1. Let $0 < \sigma_1 < 1, \sigma_1 < \sigma_2$. Suppose that the solution u of the problem (1.1) can be decomposed into $u = \xi_u(x,t) + \eta_u(x,t)$, where $\xi_u(x,t)$ and $\eta_u(x,t)$ satisfy

$$\xi_u(x,t) = b_0(x) + b_1(x) t^{\sigma_1}, \quad \|b_k\|_2 \lesssim 1, \quad k = 0, 1,$$

and

$$\big\|\partial_t^l \eta_u\big\|_2 \leq Ct^{\sigma_2 - l}, \quad l = 0, 1, 2.$$

Furthermore, suppose that the solution u satisfies

$$\|u\|_2 + \big\|D_t^{\alpha(t)} u\big\|_2 \leq C_1^*.$$

Remark 2.1. The recent paper [23] established the regularity of the solution u to the problem (1.1) with $f=0$, demonstrating the derivative bound $\|\partial_t u\| \le Ct^{\alpha(0)-1}$. This analytical result confirms that solutions of the problem (1.1) with $f=0$ satisfy Assumption 2.1 with parameter choice $\sigma_1 = \alpha(0)$. Notably, the solution decomposition of Assumption 2.1 provides a general framework for solutions to problem (1.1). Under certain suitable assumption on g and u_0, Gracia *et al.* [3] verified that the solution of the problem (1.1) with $\alpha(t)=\alpha_0$ (the constant $\alpha_0 \in (0,1)$) satisfies the solution decomposition of Assumption 2.1.

Next we will state a priori bound of $W_n^{\sigma_1}$ and the truncation of the CL1 scheme in following lemma.

Lemma 2.1. *Suppose that the regularity of solution for the problem* (1.1) *satisfies the Assumption* 2.1, *then one has*

$$\left|W_n^{\sigma_1}\right| \le C_T t_1^{-\sigma_1}\Phi(n,\sigma_1), \qquad 1\le n\le N, \tag{2.9}$$

$$\left\|D_t^{\alpha(t_n)}u^n - \mathbb{D}_N^{\alpha(t_n)}u^n\right\|_2 \le C_T\Phi(n,\sigma_2) + C_T t_1^{\sigma_2-\sigma_1}\Phi(n,\sigma_1), \quad 1\le n\le N. \tag{2.10}$$

Proof. Applying the definition of W_n^{σ} given in (2.8) yields

$$\left|W_n^{\sigma_1}\right| = t_1^{-\sigma}\left|D_t^{\alpha(t_n)}t_n^{\sigma_1} - \delta_N^{\alpha(t_n)}(t^{\sigma_1})^n\right| \le Ct_1^{-\sigma_1}\Phi(n,\sigma_1),$$

where (2.6) is used. Hence, the proof of (2.9) is complete.

By using the definition of the corrected L1 scheme (2.7), we have

$$\begin{aligned}
&\left\|D_t^{\alpha(t_n)}u^n - \mathbb{D}_N^{\alpha(t_n)}u^n\right\|_2\\
&= \left\|D_t^{\alpha(t_n)}\xi_u^n - \mathbb{D}_N^{\alpha(t_n)}\xi_u^n + D_t^{\alpha(t_n)}\eta_u^n - \mathbb{D}_N^{\alpha(t_n)}\eta_u^n\right\|_2\\
&= \left\|D_t^{\alpha(t_n)}\eta_u^n - \delta_N^{\alpha(t_n)}\eta_u^n - W_n^{\sigma_1}\left(\eta_u^1-\eta_u^0\right)\right\|_2\\
&\le C\Phi(n,\sigma_2) + Ct_1^{\sigma_2-\sigma_1}\Phi(n,\sigma_1),
\end{aligned}$$

where (2.6) and (2.9) are used. Finally, we finish the proof. □

At each time level $t=t_n$, the nonlinear term $f(u)$ can be approximated by the following Newton linearization:

$$f(u^n) \approx f(u^{n-1}) + f'(u^{n-1})(u^n-u^{n-1}). \tag{2.11}$$

Define the truncation error of this linearization by

$$\mathcal{R}^n := f(u^n) - f(u^{n-1}) - f'(u^{n-1})(u^n-u^{n-1}).$$

Next we will state the boundness of $\mathcal{R}^n$ in following lemma.

Lemma 2.2. *The truncation error of the Newton linearization satisfies*

$$\|\nabla\mathcal{R}^n\| \le \begin{cases} C_T N^{-2r\sigma_1}, & \text{if} \quad n=1,\\ C_T N^{-\min\{2,2r\sigma_1\}}, & \text{if} \quad 2\le n\le N. \end{cases} \tag{2.12}$$

Proof. Applying the Taylor expansion, one has

$$\mathcal{R}^n = \frac{1}{2}(u^n-u^{n-1})^2\int_0^1 f''\big(u^{n-1}+s(u^n-u^{n-1})\big)\,ds.$$

Hence,

$$\begin{aligned}\|\nabla\mathcal{R}^n\| &\le \left\|(u^n-u^{n-1})\nabla(u^n-u^{n-1})\int_0^1 f''\big(u^{n-1}+s(u^n-u^{n-1})\big)\,ds\right\|\\ &\quad+\left\|\frac{1}{2}(u^n-u^{n-1})^2\int_0^1 \nabla f''\big(u^{n-1}+s(u^n-u^{n-1})\big)\,ds\right\|\\ &\le C\|u^n-u^{n-1}\|_{L^4}\|\nabla(u^n-u^{n-1})\|_{L^4}+C\|u^n-u^{n-1}\|_{L^\infty}^2\\ &\le C\left\|\int_{t_{n-1}}^{t_n} u_t\,dt\right\|_1\left\|\int_{t_{n-1}}^{t_n} u_t\,dt\right\|_2+C\left\|\int_{t_{n-1}}^{t_n} u_t\,dt\right\|_2^2\\ &\le C\int_{t_{n-1}}^{t_n}\|u_t\|_1\,dt\int_{t_{n-1}}^{t_n}\|u_t\|_2\,dt+C\left(\int_{t_{n-1}}^{t_n}\|u_t\|_2\,dt\right)^2,\end{aligned}$$

where Minkowski's inequality and $f\in C^2(\mathbb{R})$ are used. By using Assumption 2.1, we arrive at

$$\|\nabla\mathcal{R}^1\|\le 2C\left(\int_{t_0}^{t_1} t^{\sigma_1-1}\ dt\right)^2\le 2Ct_1^{2\sigma_1}\le C_TN^{-2r\sigma_1}.$$

For $2\le n\le N$, one has

$$\begin{aligned}\|\nabla\mathcal{R}^n\| &\le 2C\left(\int_{t_{n-1}}^{t_n} t^{\sigma_1-1}dt\right)^2\le 2C\big(\tau_n t_{n-1}^{\sigma_1-1}\big)^2\\ &\le 2Cr^2T^{2\sigma_1}\big[N^{-r(\sigma_1-1)}(n-1)^{r(\sigma_1-1)}N^{-r}n^{r-1}\big]^2\\ &= C_T\big[N^{-r\sigma_1}n^{r\sigma_1-1}\big]^2,\end{aligned}$$

where (2.1) is used. Hence,

$$\|\nabla\mathcal{R}^n\|\le\begin{cases}CN^{-2}, & \text{if}\quad r\sigma_1\ge 1,\quad n=2,3,\dots,N,\\ CN^{-2r\sigma_1}, & \text{if}\quad r\sigma_1<1,\quad n=2,3,\dots,N.\end{cases}$$

Combining this bound with the case $n=1$, we finish the proof. □

Now applying the nonuniform CL1 scheme (2.7) to discretise the variable exponent Caputo derivative of (1.1), and then using Newton linearization (2.11) to handle the nonlinear term, we have

$$\mathbb{D}_N^{\alpha(t_n)}U^n-\kappa^2\Delta U^n=f(U^{n-1})+f'(U^{n-1})(U^n-U^{n-1})+g^n,\quad n=1,\dots,N,\tag{2.13}$$

where we set $U^0=u_0$ and $g^n(\cdot):=g(\cdot,t_n)$.

Fix $k\ge 0$. Let $V_h\subset H_0^1(\Omega)$ be the finite element space of polynomials in d variables that have degree at most k in each variable, with a mesh size h.

Next we will present the following two operators. Define the L^2 projection $P_h: L^2(\Omega)\to V_h$ and the Ritz projector $R_h: H_0^1(\Omega)\to V_h$ by $(P_hw,v_h)=(w,\nabla v_h)$ and $(\nabla R_hw,\nabla v_h)=(\nabla w,\nabla v_h)$ for all $v_h\in V_h$, respectively. From [18, Lemma 1.1], we have

$$\|w-R_hw\|+h\|w-R_hw\|_1\le C_\Omega h^2|w|_2,\quad\forall\, w\in H_0^1(\Omega)\cap H^2(\Omega).\tag{2.14}$$

Discretising (2.13) in space using the finite element method, we obtain a fully discrete scheme (which we call CL1-FEM): Find $U_h^n\in V_h$ such that

$$\mathbb{D}_N^{\alpha(t_n)}U_h^n-\kappa^2\Delta_hU_h^n=P_h\big[f\big(U_h^{n-1}\big)+f'\big(U_h^{n-1}\big)\big(U_h^n-U_h^{n-1}\big)+g^n\big],\quad n=1,\dots,N\tag{2.15}$$

with $U_h^0=R_hu_0$, where Δ_h is the discrete Laplacian operator defined by $(\Delta_hv,w)=-(\nabla v,\nabla w)$ for any $v,w\in V_h$.

3. Unconditional Error Analysis of the CL1-FEM

In this section, we will investigate a bound for the numerical solution in L^∞-norm by using the temporal-spatial splitting technique. With the help of this bound, the unconditional optimal convergent result in $L^\infty(L^2)$ is obtained.

Define $e^n := u^n - U^n$ and $\varphi^n := \mathbb{D}_N^{\alpha(t_n)} u^n - D_t^{\alpha(t_n)} u^n$. From (1.1) and (2.13), we derive the following time discrete system:

$$\mathbb{D}_N^{\alpha(t_n)} e^n - \kappa^2 \Delta e^n = \varphi^n + \mathcal{R}^n + Q^n e^{n-1} + f'(U^{n-1}) e^n, \quad n = 1, \ldots, N \tag{3.1}$$

with $e^0 = 0$ and $e^n|_{\partial\Omega} = 0$, where Q^n is defined by

$$\begin{aligned} Q^n = & \int_0^1 f'\big((1-s)u^{n-1} + sU^{n-1}\big)\, ds + (u^n - u^{n-1}) \\ & \times \int_0^1 f''\big((1-s)u^{n-1} + sU^{n-1}\big)\, ds - f'(U^{n-1}). \end{aligned}$$

Now we state a fractional Gronwall inequality for the L1 scheme in the following lemma.

Lemma 3.1. *Let the λ_s be nonnegative constants with $0 < \sum_{s=0}^n \gamma_s \le \Lambda$ for $n \ge 1$, where Λ is some positive constant independent of n. Suppose that the nonnegative sequences $\{\mu_n, \nu_n : n \ge 1\}$ are bounded and the grid function $\{ w^n \,|\, n \ge 0\}$ satisfies*

$$\delta_N^{\alpha(t_n)} (w^n)^2 \le \sum_{s=0}^n \gamma_{n-s} (w^s)^2 + \mu^n w^n + (\nu^n)^2, \quad n \ge 1. \tag{3.2}$$

If the number of the nonuniform grid satisfies

$$N \ge \frac{1}{Tr} \max_{1 \le j \le N} \big[2\pi_A \Gamma\big(2 - \alpha(t_j)\big)\Lambda\big]^{-\frac{1}{\alpha(t_j)}},$$

then

$$w^n \le 2 \max_{1 \le j \le n} E_{\alpha(t_j)}\Big(2\pi_A \Lambda t_j^{\alpha(t_j)}\Big) \left[w^0 + \max_{1 \le k \le n} \sum_{j=1}^k \mathbb{P}_{k-j}^{(k)} (\mu^j + \nu^j) + \max_{1 \le j \le n} \{\nu^j\} \right] \tag{3.3}$$

for $n = 1, \ldots, N$, where π_A is a constant.

Proof. Define

$$\begin{aligned} & F_n := 2 \max_{1 \le j \le n} E_{\alpha(t_j)} \left(2\pi_A \Lambda t_j^{\alpha(t_j)}\right), \\ & G_n := w_0 + \max_{1 \le k \le n} \sum_{j=1}^k \mathbb{P}_{k-j}^{(k)} (\mu^j + \nu^j) + \max_{1 \le j \le n} \{\nu^j\}. \end{aligned}$$

Hence, the inequality (3.3) is equivalent to $w^n \le F_n G_n$.

Now we replace the index n by j in (3.2), multiplying it by $\mathbb{P}_{n-j}^{(n)}$ and sum over j to obtain

$$\sum_{j=1}^n P_{n-j}^{(n)} \delta_N^{\alpha(t_j)} (w^j)^2 \le \sum_{j=1}^n \mathbb{P}_{n-j}^{(n)} \left(\sum_{s=1}^j \gamma_{j-s} (w^s)^2 \right) + \sum_{j=1}^n \mathbb{P}_{n-j}^{(n)} \big(\mu^j w^j + (\nu^j)^2\big). \tag{3.4}$$

Imitating [12, Eq. (3.4)], one has

$$\sum_{j=1}^{n} P_{n-j}^{(n)} \delta_N^{\alpha(t_j)} (w^j)^2 = (w^n)^2 - (w^0)^2.$$

Inserting the above result into (3.4) yields

$$(w^n)^2 \le (w^0)^2 + \sum_{j=1}^{n} \mathbb{P}_{n-j}^{(n)} \left(\sum_{s=1}^{j} \gamma_{j-s} (v^{s,\theta_1})^2 \right) + \sum_{j=1}^{n} \mathbb{P}_{n-j}^{(n)} \big(\mu^j w^j + (\nu^j)^2 \big). \tag{3.5}$$

Imitating the induction of [8, Lemma 4.1], we can use the mathematical induction to prove $w^n \le F^n G^n$ by virtue of (3.5). □

Next we will state two useful properties in following two lemmas.

Lemma 3.2. *Suppose that $F(v(x),x) \in C^2(\mathbb{R}\times\Omega)$ and the function $v(x)$ satisfying $\|v\|_2 \le C_F$. Then there exists a positive constant C_f such that*

$$\big\|F\big(v(x),x\big)w\big\| \le C_f \|w\|, \quad \big\|\Delta F\big(v(x),x\big)w\big\| \le C_f \|\Delta w\|, \quad w \in H_0^1(\Omega)\cap H^2(\Omega).$$

Proof. Imitating the proof of [13, Lemma 4.1]. □

Define

$$\begin{aligned}
C_1 &:= 2C_T\left(\left[2+\frac{2C_T}{\Gamma(1+\sigma_1)}\right](1+T^{\sigma_2-\sigma_1}+T^{\alpha^*})\right) \\
&\quad \times \max_{1\le j\le N} E_{\alpha(t_j)}\left(8\pi_A C_f t_j^{\alpha(t_j)}\right) \frac{e^r \max_{1\le j\le N}\Gamma\big(1+\lambda_N-\alpha(t_j)\big)}{\Gamma(1+\lambda_N)}, \\
C_2^* &:= C_1 \frac{\Gamma(\sigma_1)}{\max_{0\le t\le T}\Gamma\big(1+\sigma_1-\alpha(t)\big)} + 2C_T T^{-\alpha_*} + \frac{2T^{-\alpha_*}C_1^*}{\max_{0\le t\le T}\Gamma\big(2-\alpha(t)\big)}, \\
N_1^* &:= \max\left\{ \frac{1}{Tr} \max_{1\le j\le N} \big[8\pi_A C_f \Gamma\big(2-\alpha(t_j)\big)\big]^{-\frac{1}{\alpha(t_j)}}, \right. \\
&\qquad \left. \max_{t\in[0,T]} \left(\frac{2C_f}{\Gamma(1+\sigma_1)}\right)^{\frac{1}{r\alpha(t)}} T^{\frac{1}{r}}, (C_\Omega C_1)^{\frac{1}{\min\{2-\alpha^*, r\sigma_2, 2r\sigma_1\}}} \right\}.
\end{aligned}$$

Next we will present the boundedness of U^n in the following lemma.

Lemma 3.3. *The time discrete system (2.13) has a unique solution U^n. If $N \ge N_1^*$ and $1 \le r \le \min\{2-\alpha^*, \sigma_2, 2\sigma_1\}/\alpha^*$, one has*

$$\|U^n\|_2 \le 1 + C_1^*, \quad 1 \le n \le N. \tag{3.6}$$

Furthermore, if $1 \le r \le \min\{2-\alpha^*, \sigma_2, 2\sigma_1\}/\alpha^*$, one has

$$\big\|\mathbb{D}_N^{\alpha(t_n)} U^n\big\|_2 \le C_2^*, \quad 1 \le n \le N. \tag{3.7}$$

Proof. Since the problem (2.13) is a linear elliptic problem, the existence and uniqueness of the solution U_h^n can be easily obtained. Multiplying (3.1) by $\Delta^2 e^n$, and integrating over Ω, one has

$$\left(\mathbb{D}_N^{\alpha(t_n)} \Delta e^n, \Delta e^n\right) + \kappa^2 \|\nabla\Delta e^n\|^2 = \big(\Delta\big(\varphi^n + \mathcal{R}^n + Q^n e^{n-1} + f'(U^{n-1})e^n\big), \Delta e^n\big).$$

Applying a Cauchy-Schwarz inequality, one has

$$\Big(\mathbb{D}_N^{\alpha(t_n)}\Delta e^n, \Delta e^n\Big) \le \big(\|\Delta\varphi^n\| + \|\Delta\mathcal{R}^n\| + \|\Delta(Q^n e^{n-1})\| + \big\|\Delta\big(f'(U^{n-1})e^n\big)\big\|\big)\|\Delta e^n\|. \quad (3.8)$$

Next we will prove (3.6) by using the mathematical induction. Firstly, we verify (3.6) holds for $n = 1$. By choosing $n = 1$ in (3.8), then applying (2.8) and $e^0 = 0$, one has

$$\frac{\Gamma(1+\sigma_1)}{\Gamma\big(1+\sigma_1-\alpha(t_1)\big)} t_1^{-\alpha(t_1)}\|\Delta e^1\| \le \|\Delta\varphi^1\| + \|\Delta\mathcal{R}^1\| + C_f\|\Delta e^1\|, \quad (3.9)$$

where we combine Lemma 3.2 with $U^0 = u_0$ and Assumption 2.1. By using (2.9), (2.10), and (2.12), (3.9) is reduced to

$$\begin{aligned}
\|\Delta e^1\| &\le \frac{2\Gamma\big(1+\sigma_1-\alpha(t_1)\big)}{\Gamma(1+\sigma_1)} t_1^{\alpha(t_1)}\big(C_T\Phi(1,\sigma_2) + C_T t_1^{\sigma_2-\sigma_1}\Phi(1,\sigma_1) + C_T N^{-2r\sigma_1}\big)\\
&\le \frac{2C_T\Gamma\big(1+\sigma_1-\alpha(t_1)\big)}{\Gamma(1+\sigma_1)} t_1^{\alpha(t_1)}\big(t_1^{-\alpha(t_1)}N^{-r\sigma_2} + t_1^{\sigma_2-\sigma_1}t_1^{-\alpha(t_1)}N^{-r\sigma_1} + N^{-2r\sigma_1}\big)\\
&\le \frac{2C_T(1+T^{\sigma_2-\sigma_1}+T^{\alpha^*})}{\Gamma(1+\sigma_1)} N^{-\min\{r\sigma_2,2r\sigma_1\}}\\
&\le C_1 N^{-\min\{r\sigma_2,2r\sigma_1\}},
\end{aligned}$$

where we used

$$N \ge \max_{t\in[0,T]}\left(\frac{2C_f}{\Gamma(1+\sigma_1)}\right)^{\frac{1}{r\alpha(t)}} T^{\frac{1}{r}}.$$

Furthermore, we have

$$\begin{aligned}
\|U^1\|_2 &\le \|e^1\|_2 + \|u^1\|_2 \le C_\Omega\|\Delta e^1\| + \|u^1\|_2\\
&\le C_\Omega C_1 N^{-\min\{r\sigma_2,2r\sigma_1\}} + \|u^1\|_2 \le 1 + C_1^*,
\end{aligned} \quad (3.10)$$

where we use $\|v\|_2 \le C_\Omega\|\Delta v\|$ and $N \ge (C_\Omega C_1)^{1/\min\{2-\alpha^*,r\sigma_2,2r\sigma_1\}}$. The above result displays that (3.6) holds for $n = 1$.

Fix $m \in \{2,3,\ldots,N\}$, assume that (3.6) is also valid for $1 \le n \le m-1$. Next we will verify (3.6) holds for $n = m$. By using (2.5), a Young's inequality, (2.9), (2.10) and (2.12), one has

$$\begin{aligned}
&\delta_N^{\alpha(t_m)}\|\Delta e^m\|^2\\
&\le 2\big(\|\Delta\varphi^m\| + \|\Delta\mathcal{R}^m\| + W_m^{\sigma_1}\|\Delta e^1\|\big)\|e^m\| + C_f\|\Delta e^{m-1}\|^2 + 3C_f\|\Delta e^m\|^2\\
&\le 2C_T\big(\Phi(m,\sigma_2) + t_1^{\sigma_2-\sigma_1}\Phi(m,\sigma_1) + N^{-\min\{2,2r\sigma_1\}} + t_1^{-\sigma_1}\Phi(m,\sigma_1)\|\Delta e^1\|\big)\|e^m\|\\
&\qquad + C_f\|\Delta e^{m-1}\|^2 + 3C_f\|\Delta e^m\|^2\\
&\le 2C_T\Bigg(t_m^{-\alpha(t_m)}N^{-\min\{2-\alpha^*,r\sigma_2\}} + t_1^{\sigma_2-\sigma_1}t_m^{-\alpha(t_m)}N^{-\min\{2-\alpha^*,r\sigma_1\}} + N^{-\min\{2,2r\sigma_1\}}\\
&\qquad\quad + \frac{2C_T(1+T^{\sigma_2-\sigma_1}+T^{\alpha^*})}{\Gamma(1+\sigma_1)} t_1^{-\sigma_1}t_m^{-\alpha(t_m)}N^{-\min\{2-\alpha^*,r\sigma_1\}}N^{-\min\{2r\sigma_1,r\sigma_2\}}\Bigg)\|e^m\|\\
&\qquad + C_f\|\Delta e^{m-1}\|^2 + 3C_f\|\Delta e^m\|^2\\
&\le 2C_T\Bigg(\left[2 + \frac{2C_T}{\Gamma(1+\sigma_1)}\right](1+T^{\sigma_2-\sigma_1}+T^{\alpha^*})t_m^{-\alpha(t_m)}N^{-\min\{2-\alpha^*,2r\sigma_1,r\sigma_2\}}\Bigg)\|e^m\|\\
&\qquad + C_f\|\Delta e^{m-1}\|^2 + 3C_f\|\Delta e^m\|^2,
\end{aligned} \quad (3.11)$$

where Lemma 3.2 combining with Assumption 2.1 and (3.6) is used.

Now we use Lemma 3.1 to the above inequality, obtaining

$$\begin{aligned}\|\Delta e^m\| &\le 2C_T \max_{1\le j\le m} E_\alpha(t_j)\Big(8\pi_A C_f t_j^{\alpha(t_j)}\Big)\\ &\quad\times \max_{1\le k\le m}\sum_{j=1}^{k}\mathbb{P}_{k-j}^{(k)}\left(\left[2+\frac{2C_T}{\Gamma(1+\sigma_1)}\right](1+T^{\sigma_2-\sigma_1}+T^{\alpha^*})t_j^{-\alpha(t_j)}\right)N^{-\min\{2-\alpha^*,2r\sigma_1,r\sigma_2\}}\\ &\le 2C_T\left(\left[2+\frac{2C_T}{\Gamma(1+\sigma_1)}\right](1+T^{\sigma_2-\sigma_1}+T^{\alpha^*})\right)\\ &\quad\times \max_{1\le j\le m} E_\alpha(t_j)\Big(8\pi_A C_f t_j^{\alpha(t_j)}\Big)\frac{e^r\max_{1\le j\le m}\Gamma\big(1+\lambda_N-\alpha(t_j)\big)}{\Gamma(1+\lambda_N)}N^{-\min\{2-\alpha^*,2r\sigma_1,r\sigma_2\}}\\ &\le C_1 N^{-\min\{2-\alpha^*,2r\sigma_1,r\sigma_2\}}, \end{aligned} \tag{3.12}$$

where we invoke (2.3) and (2.4). Furthermore, combining (3.12) with the regularity $\|u(\cdot,t)\|_2 \le C_1^*$ and $\|v\|_2 \le C_\Omega\|\Delta v\|$ given in [1, Section 5.5.1], we have

$$\|U^m\|_2 \le \|e^m\|_2 + \|u^m\|_2 \le C_\Omega C_1 N^{-\min\{2-\alpha^*,2r\sigma_1,r\sigma_2\}} + \|u^m\|_2 \le 1 + C_1^*, \tag{3.13}$$

where we use $N \ge (C_\Omega C_1)^{1/\min\{2-\alpha^*,2r\sigma_1,r\sigma_2\}}$. Therefore, (3.6) is also valid for $n = m$. By the principle of induction, we conclude that (3.6) is proved.

Now, we approximate (3.7). By using the definition of $\mathbb{D}_N^{\alpha(t_n)}e^n$ and (3.6), we arrive at

$$\begin{aligned}\big\|\mathbb{D}_N^{\alpha(t_n)}e^n\big\|_2 &\le \Theta_0^{(n)}\|e^n\|_2 + \sum_{j=1}^{n-1}\|e^{n-j}\|_2\big(\Theta_j^{(n)}-\Theta_{j-1}^{(n)}\big)\\ &\le 2\Theta_0^{(n)}C_1^* N^{-\min\{2-\alpha^*,r\sigma_2,2r\sigma_1\}}\\ &\le \frac{2}{\Gamma\big(2-\alpha(t_n)\big)\tau_n^{\alpha(t_n)}}C_1^* N^{-\min\{2-\alpha^*,r\sigma_2,2r\sigma_1\}}\\ &\le \frac{2T^{-\alpha^*}C_1^*}{\Gamma\big(2-\alpha(t_n)\big)}, \end{aligned} \tag{3.14}$$

where we use $\tau_n^{\alpha(t_n)} \ge \tau_1^{\alpha^*}$ and $1 \le r \le \min\{2-\alpha^*,\sigma_2,2\sigma_1\}/\alpha^*$. Hence, applying (2.9) and (3.14) yields

$$\begin{aligned}\big\|\mathbb{D}_N^{\alpha(t_n)}U^n\big\|_2 &\le \big\|D_t^{\alpha(t_n)}u(\cdot,t_n)\big\|_2 + \big\|\mathbb{D}_N^{\alpha(t_n)}u^n - D_t^{\alpha(t_n)}u(\cdot,t_n)\big\|_2 + \big\|\mathbb{D}_N^{\alpha(t_n)}e^n\big\|_2\\ &\le \frac{C_1}{\Gamma\big(1-\alpha(t_n)\big)}\int_0^{t_n}(t_n-s)^{-\alpha(t_n)}s^{\sigma_1-1}\,ds + C_T\Phi(n,\sigma_2)\\ &\quad + C_T t_1^{\sigma_2-\sigma_1}\Phi(n,\sigma_1) + \frac{2T^{-\alpha^*}C_1^*}{\Gamma(2-\alpha(t_n)}\\ &\le \frac{C_1\Gamma(\sigma_1)}{\max_{0\le t\le T}\Gamma\big(1+\sigma_1-\alpha(t)\big)} + 2C_T T^{-\alpha^*} + \frac{2T^{-\alpha^*}C_1^*}{\max_{0\le t\le T}\Gamma\big(2-\alpha(t)\big)} = C_2^*, \end{aligned}$$

where $1 \le r \le \min\{2-\alpha^*,\sigma_2,2\sigma_1\}/\alpha^*$ and Assumption 2.1 are used. Finally, we complete the proof. □

Now we split

$$U^n - U_h^n = \big(R_hU^n - U_h^n\big) + \big(U^n - R_hU^n\big) := \vartheta^n + \mu^n.$$

From (2.13) and (2.15), we get the following system:

$$\mathbb{D}_N^{\alpha(t_n)}\vartheta^n - \kappa^2\Delta_h\vartheta^n = P_h\Big(\mathbb{D}_N^{\alpha(t_n)}\mu^n + \Pi^n(\vartheta^{n-1}+\mu^{n-1}) + f'(U_h^{n-1})(\vartheta^n+\mu^n)\Big) \tag{3.15}$$

for $n = 1,\dots,N$, where

$$\Pi^n = \int_0^1 f'\big((1-s)U^{n-1} + sU_h^{n-1}\big)\,ds + (U^n - U^{n-1}) \times \int_0^1 f''\big((1-s)U^{n-1} + sU_h^{n-1}\big)\,ds - f'\big(U_h^{n-1}\big).$$

Define

$$C_3^* := 2C_\Omega\big(C_2^* + 2C_f(1+C_1^*)\big)\max_{1\le j\le N} E_{\alpha(t_j)}\Big(2\pi_A(1+C_f)^2 t_j^{\alpha(t_j)}\Big) \times \frac{e^r \max_{1\le j\le N}\Gamma\big(1+\lambda_N-\alpha(t_j)\big)}{\Gamma(1+\lambda_N)}\left(\frac{2C_T^2T^{\alpha^*}}{\Gamma(1+\sigma_1)} + T^{\alpha^*}\right),$$

$$N_2^* := \max\left\{N_1^*, \frac{1}{Tr}\max_{1\le j\le N}\left[2\pi_A\Gamma\big(2-\alpha(t_j)\big)(1+C_f)^2\right]^{-\frac{1}{\alpha(t_j)}}\right\},$$

$$h_1^* := \min\left\{(C_\Omega)^{-\frac{4}{7-2d}}, \left[\frac{2T^{\alpha^*}\big(C_2^*+2C_f(1+C_1^*)\big)C_\Omega}{\Gamma(1+\sigma_1)}\right]^{-\frac14}, (C_3^*)^{-\frac14}\right\},$$

$$K_1 := 1 + C_\Omega(1+C_1^*).$$

Next we will state a bound for the computed solution U_h^n in following lemma.

Lemma 3.4. *Assume $N \ge N_2^*, h \le h_1^*$, and $1 \le r \le (2-\alpha^*)/\sigma_1$. For $1 \le n \le N$, one has*

$$\big\|R_hU^n - U_h^n\big\| \le h^{\frac74}, \tag{3.16}$$

$$\big\|U_h^n\big\|_{L^\infty} \le K_1. \tag{3.17}$$

Proof. Multiplying (3.15) by ϑ^n and integrating over Ω, we have

$$\Big(\mathbb{D}_N^{\alpha(t_n)}\vartheta^n, \vartheta^n\Big) + \kappa^2\|\nabla\vartheta^n\|^2 = \Big(P_h\Big(\mathbb{D}_N^{\alpha(t_n)}\mu^n + \Pi^n(\vartheta^{n-1}+\mu^{n-1}) + f'\big(U_h^{n-1}\big)(\vartheta^n+\mu^n)\Big), \vartheta^n\Big).$$

Applying a Cauchy-Schwarz inequality, one has

$$\Big(\mathbb{D}_N^{\alpha(t_n)}\vartheta^n, \vartheta^n\Big) \le \Big(\big\|\mathbb{D}_N^{\alpha(t_n)}\mu^n\big\| + \|\Pi^n\vartheta^{n-1}\| + \|\Pi^n\mu^{n-1}\| + \big\|f'\big(U_h^{n-1}\big)\vartheta^n\big\| + \big\|f'\big(U_h^{n-1}\big)\mu^n\big\|\Big)\|\vartheta^n\|. \tag{3.18}$$

Next we will use the mathematical induction to prove (3.16). By choosing $n = 1$ in (3.18), then applying (2.8) and a Young's inequality, we arrive at

$$\frac{\Gamma(1+\sigma_1)}{\Gamma\big(1+\sigma_1-\alpha(t_1)\big)} t_1^{-\alpha(t_1)}\|\vartheta^1\| \le \big\|\mathbb{D}_N^{\alpha(t_n)}\mu^1\big\| + C_f\|\mu^0\| + C_f\big(\|\vartheta^1\| + \|\mu^1\|\big), \tag{3.19}$$

where Lemma 3.2 and $U_h^0 = R_hu_0$ are used. By using $N \ge N_2^*$, (2.14), (3.6), and (3.7), (3.19) is reduced to

$$\|\vartheta^1\| \le \frac{2\Gamma\big(1+\sigma_1-\alpha(t_1)\big)}{\Gamma(1+\sigma_1)} t_1^{\alpha(t_1)}\Big(C_\Omega h^2\big\|\mathbb{D}_N^{\alpha(t_n)}U^1\big\|_2 + C_fC_\Omega h^2\|U^0\|_2 + C_fC_\Omega h^2\|U^1\|_2\Big) \le \frac{2T^{\alpha^*}}{\Gamma(1+\sigma_1)}\big(C_2^* + 2C_f(1+C_1^*)\big)C_\Omega h^2 \le h^{\frac74},$$

where

$$h \le h_1 \le \left[\frac{2T^{\alpha^*}\big(C_2^* + 2C_f(1 + C_1^*)\big)C_\Omega}{\Gamma(1+\sigma_1)}\right]^{-\frac{1}{4}}$$

is used. This result implies (3.16) holds for $n = 1$.

Fix $m \in \{2, 3, \ldots, N\}$, we assume that (3.16) is valid for $n = 1, 2, \ldots, m-1$. Applying this assumption and (3.6), we have

$$\begin{aligned}\big\|U_h^n\big\|_{L^\infty} &\le \big\|R_hU^n - U_h^n\big\|_{L^\infty} + \big\|R_hU^n\big\|_{L^\infty} \\ &\le C_\Omega h^{-\frac{d}{2}}\big\|R_hU^n - U_h^n\big\| + C_\Omega\|U^n\|_2 \\ &\le C_\Omega h^{-\frac{d}{2}}h^{\frac{7}{4}} + C_\Omega(1 + C_1^*) \le K_1, \quad 0 \le n \le m-1,\end{aligned} \tag{3.20}$$

where $h \le h_1^* \le (C_\Omega)^{-4/(7-2d)}$ is used.

Next we will prove (3.16) holds for $n = m$. Now applying (2.5), a Young's inequality, (2.14), and $1 \le r \le (2-\alpha^*)/\sigma_1$, we arrive at

$$\begin{aligned}&\delta_N^{\alpha(t_m)}\|\vartheta^m\|^2 \\ &\le 2\big\|\mathbb{D}_N^{\alpha(t_m)}\mu^m\big\|\,\|\vartheta^m\| + \|\Pi^m\vartheta^{m-1}\|^2 + \|\vartheta^m\|^2 + 2\|\Pi^m\mu^{m-1}\|\,\|\vartheta^m\| \\ &\quad + 2\big\|f'\big(U_h^{m-1}\big)\mu^m\big\|\,\|\vartheta^m\| + 2\big\|f'\big(U_h^{m-1}\big)(\vartheta^m)^2\big\| + \big|W_m^{\sigma_1}\big|\,\|\vartheta^1\|\,\|\vartheta^m\| \\ &\le 2C_\Omega h^2\Big(\big\|\mathbb{D}_N^{\alpha(t_m)}U^m\big\|_2 + C_f\|U^{m-1}\| + C_f\|U^m\|\Big)\|\vartheta^m\| + (1+2C_f)\|\vartheta^m\|^2 \\ &\quad + C_f^2\|\vartheta^{m-1}\|^2 + t_1^{-\sigma_1}t_m^{-\alpha(t_m)}N^{-\min\{2-\alpha^*, r\sigma_1\}}\frac{2C_T^2T^{\alpha^*}}{\Gamma(1+\sigma_1)}\big(C_2^* + 2C_f(1+C_1^*)\big)C_\Omega h^2\|\vartheta^m\| \\ &\le 2C_\Omega h^2\big(C_2^* + 2C_f(1+C_1^*)\big)\left(1 + t_m^{-\alpha(t_m)}\frac{C_T^2T^{\alpha^*}}{\Gamma(1+\sigma_1)}\right)\|\vartheta^m\| \\ &\quad + C_f^2\|\vartheta^{m-1}\|^2 + (1+2C_f)\|\vartheta^m\|^2,\end{aligned}$$

where we combine Lemma 3.2 with Assumption 2.1 and (3.20).

Now applying Lemma 3.1 to the above inequality yields

$$\begin{aligned}\|\vartheta^m\| &\le 2C_\Omega h^2\big(C_2^* + 2C_f(1+C_1^*)\big)\max_{1\le j\le m}E_{\alpha(t_j)}\Big(2\pi_A(1+C_f)^2t_j^{\alpha(t_j)}\Big) \\ &\quad \times \max_{1\le k\le m}\sum_{j=1}^{k}\mathbb{P}_{k-j}^{(k)}\left(1 + t_j^{-\alpha(t_j)}\frac{2C_T^2T^{\alpha^*}}{\Gamma(1+\sigma_1)}\right) \\ &\le 2C_\Omega h^2\big(C_2^* + 2C_f(1+C_1^*)\big)\max_{1\le j\le m}E_{\alpha(t_j)}\Big(2\pi_A(1+C_f)^2t_j^{\alpha(t_j)}\Big) \\ &\quad \times \frac{e^r\max_{1\le j\le m}\Gamma\big(1+\lambda_N-\alpha(t_j)\big)}{\Gamma(1+\lambda_N)}\left(\frac{2C_T^2T^{\alpha^*}}{\Gamma(1+\sigma_1)} + t_n^{\alpha^*}\right) \\ &= C_3^*h^2 \le h^{\frac{7}{4}},\end{aligned}$$

where we use $\|\vartheta_u^0\| = \|R_hu^0 - U_h^0\| = 0$, (2.3), (2.4), and $h \le h_1^* \le (C_3^*)^{-1/4}$.

Furthermore,

$$\big\|U_h^m\big\|_{L^\infty} \le \|\vartheta^m\|_{L^\infty} + \big\|R_hU^m\big\|_{L^\infty} \le C_\Omega h^{-\frac{d}{2}}h^{\frac{7}{4}} + C_\Omega(1+C_1^*) \le K_1,$$

where $h \le h_1^* \le (C_\Omega)^{-4/(7-2d)}$ is used. Hence, (3.16) and (3.17) hold for $n = m$. By the principle of induction, we finish the proof. □

Now we split

$$u^n - U_h^n = \big(R_h u^n - U_h^n\big) + \big(u^n - R_h u^n\big) := \zeta^n + \rho^n.$$

Imitating [7, (5.5)], we obtain the following error equation:

$$\begin{aligned}
&\mathbb{D}_N^{\alpha(t_n)}\zeta^n - \kappa^2\Delta_h\zeta^n \\
&= P_h\left(D_t^{\alpha(t_n)}\rho^n + \mathcal{R}^n - R_h\varphi^n + \Psi^n(\zeta^{n-1}+\rho^{n-1}) + f'\big(U_h^{n-1}\big)(\zeta^n+\rho^n)\right),
\end{aligned} \tag{3.21}$$

where Ψ^n is defined by

$$\begin{aligned}
\Psi^n = &\int_0^1 f'\big((1-s)u^{n-1} + sU_h^{n-1}\big)\,ds + (u^n - u^{n-1}) \\
&\times \int_0^1 f''\big((1-s)u^{n-1} + sU_h^{n-1}\big)\,ds - f'\big(U_h^{n-1}\big).
\end{aligned}$$

Define

$$\begin{aligned}
C_4^* &:= \frac{(\sqrt{2}+1)KC_T}{\Gamma(1+\sigma_1)}(1 + T^{\sigma_2-\sigma_1} + T^{\alpha^*}),\\
C_5^* &:= \sqrt{\frac{5T^{\alpha^*}C_\Omega^2\big(1+2C_f^2\big)}{2\Gamma(1+\sigma_1)\kappa^2}},\\
C_6^* &:= \max_{1\le j\le N} E_{\alpha(t_j)}\left(\frac{5\pi_A}{\kappa^2}C_\Omega^2C_f^2t_j^{\alpha(t_j)}\right)\frac{C_4^*(1+T^{\alpha^*})e^r\max_{1\le j\le N}\Gamma\big(1+\lambda_N-\alpha(t_j)\big)}{\Gamma(1+\lambda_N)},\\
C_7^* &:= \max_{1\le j\le N} E_{\alpha(t_j)}\left(\frac{5\pi_A}{\kappa^2}C_\Omega^2C_f^2t_j^{\alpha(t_j)}\right)\left[\frac{1}{\sqrt{T^{\alpha^*}}} + \frac{T^{\alpha^*}e^r\max_{1\le j\le N}\Gamma\big(1+\lambda_N-\alpha(t_j)\big)}{\Gamma(1+\lambda_N)}\right.\\
&\qquad\left.\times\left(1+\frac{1}{\sqrt{T^{\alpha^*}}}\right)\right]C_5^*,\\
N_3^* &:= \max\left\{N_2^*, \frac{\max_{1\le j\le N}\big[10C_\Omega^2C_f^2\pi_A\Gamma\big(2-\alpha(t_j)\big)\big]^{-\frac{1}{\alpha(t_j)}}}{\kappa^2Tr}\right\}.
\end{aligned}$$

Now we present the optimal convergence analysis in $L^\infty(L^2)$ for the CL1-FEM (2.15).

Theorem 3.1. *Assume $N \ge N_3^*$ and $h \le h_1^*$. Let u^n and U_h^n be the solutions of* (2.13) *and* (2.15)*, respectively. Then for $n = 1, 2, \ldots, N$, we have*

$$\begin{aligned}
&\big\|u^n - U_h^n\big\| + \big\|\nabla R_h u^n - \nabla U_h^n\big\| \\
&\le (1 + C_\Omega)\big[(C_4^* + C_6^*)N^{-\min\{2-\alpha^*, 2r\sigma_1, r\sigma_2\}} + (C_5^* + C_7^* + 1)h^2\big].
\end{aligned} \tag{3.22}$$

Proof. Now multiplying (3.21) by $-\Delta_h\zeta^n$ and integrating it over Ω, one has

$$\begin{aligned}
&\left(\mathbb{D}_N^{\alpha(t_n)}\nabla\zeta^n, \nabla\zeta^n\right) + \kappa^2\big\|\Delta_h\zeta^n\big\|^2 \\
&= -\left(D_t^{\alpha(t_n)}\rho^n + \Psi^n(\rho^{n-1}+\zeta^{n-1}) + f'\big(U_h^{n-1}\big)(\rho^n+\zeta^n), \Delta_h\zeta^n\right) \\
&\quad + \big(\nabla P_h\big(\mathcal{R}^n - R_h\varphi^n\big), \nabla\zeta^n\big),
\end{aligned}$$

where the definition of the operators Δ_h and P_h are used. Applying a Cauchy-Schwarz inequality, a Young's inequality, $\|\nabla P_h w\| \le K\|\nabla w\|$, and $\|\nabla R_h w\| \le \|\nabla w\|$ yields

$$
\begin{aligned}
&\Big(\mathbb{D}_N^{\alpha(t_n)}\nabla\zeta^n,\nabla\zeta^n\Big)\\
&\le \frac{5}{4\kappa^2}\Big(\big\|D_t^{\alpha(t_n)}\rho^n\big\|^2+\|\Psi^n\rho^{n-1}\|^2+\|\Psi^n\zeta^{n-1}\|^2+\big\|f'\big(U_h^{n-1}\big)\rho^n\big\|^2+\big\|f'\big(U_h^{n-1}\big)\zeta^n\big\|^2\Big)\\
&\quad+\big(\big\|\nabla P_h\mathcal{R}^n\big\|+\big\|\nabla P_hR_h\varphi^n\big\|\big)\|\nabla\zeta^n\|\\
&\le \frac{5}{4\kappa^2}\Big(\big\|D_t^{\alpha(t_n)}\rho^n\big\|^2+C_f^2\|\rho^{n-1}\|^2+C_f^2\|\zeta^{n-1}\|^2+C_f^2\|\rho^n\|^2+C_f^2\|\zeta^n\|^2\Big)\\
&\quad+K(\|\nabla\mathcal{R}^n\|+\|\nabla\varphi^n\|)\|\nabla\zeta^n\|, \qquad (3.23)
\end{aligned}
$$

where we combine Lemma 3.2 with Assumption 2.1 and (3.17).

Now we verify the convergent result (3.22) holds for $n=1$. Applying the definition of the corrected L1 scheme (2.7), a Poincare inequality, and (2.14), one has

$$
\begin{aligned}
&\frac{\Gamma(1+\sigma_1)}{\Gamma\big(1+\sigma_1-\alpha(t_1)\big)}t_1^{-\alpha(t_1)}\|\nabla\zeta^1\|^2\\
&\le \frac{5C_\Omega^2\big(1+2C_f^2\big)}{4\kappa^2}h^4+\frac{5C_\Omega^2C_f^2}{4\kappa^2}\|\nabla\zeta^1\|^2\\
&\quad+KC_T\big(\Phi(1,\sigma_2)+t_1^{\sigma_2-\sigma_1}\Phi(1,\sigma_1)+N^{-2r\sigma_1}\big)\|\nabla\zeta^1\|\\
&\le \frac{5C_\Omega^2\big(1+2C_f^2\big)}{4\kappa^2}h^4+\frac{5C_\Omega^2C_f^2}{4\kappa^2}\|\nabla\zeta^1\|^2\\
&\quad+KC_T\big[t_1^{-\alpha(t_1)}(1+T^{\sigma_2-\sigma_1})N^{-r\sigma_2}+N^{-2r\sigma_1}\big]\|\nabla\zeta^1\|,
\end{aligned}
$$

where (2.9), (2.10), and (2.12) are used. Furthermore, applying

$$
N\ge \max_{t\in[0,T]}\left(\frac{2C_\Omega^2C_f^2}{\Gamma(1+\sigma_1)}\right)^{\frac{1}{r\alpha(t)}}T^{\frac{1}{r}},
$$

we arrive at

$$
\begin{aligned}
&\left(\|\nabla\zeta^1\|-\frac{KC_T}{\Gamma(1+\sigma_1)}\big[(1+T^{\sigma_2-\sigma_1})N^{-r\sigma_2}+T^{\alpha^*}N^{-2r\sigma_1}\big]\right)^2\\
&\le \frac{5T^{\alpha^*}C_\Omega^2\big(1+2C_f^2\big)}{2\Gamma(1+\sigma_1)\kappa^2}h^4+\frac{2K^2C_T^2}{\big(\Gamma(1+\sigma_1)\big)^2}\big[(1+T^{\sigma_2-\sigma_1})N^{-r\sigma_2}+T^{\alpha^*}N^{-2r\sigma_1}\big]^2,
\end{aligned}
$$

which implies

$$
\|\nabla\zeta^1\|\le C_4^*N^{-\min\{2r\sigma_1,r\sigma_2\}}+C_5^*h^2. \qquad (3.24)
$$

For $2\le n\le N$, applying (2.5), a Poincare inequality, and (2.14), (3.23) is reduced to

$$
\begin{aligned}
\frac{1}{2}\delta_N^{\alpha(t_n)}\|\nabla\zeta^n\|^2&\le\frac{5}{4\kappa^2}\Big(\big\|D_t^{\alpha(t_n)}\rho^n\big\|^2+C_f^2\big(\|\rho^{n-1}\|^2+\|\zeta^{n-1}\|^2\big)+C_f^2\big(\|\rho^n\|^2+\|\zeta^n\|^2\big)\Big)\\
&\quad+\big(\big\|\nabla P_h\big(R_h\varphi^n\big)\big\|+\big\|\nabla P_h\mathcal{R}^n\big\|\big)\|\nabla\zeta^n\|+\big|W_n^{\sigma_1}\big|\,\|\nabla\zeta^1\|\,\|\nabla\zeta^n\|\\
&\le\frac{5}{4\kappa^2}C_\Omega^2\big(1+C_f^2\big)h^4+\big(K(\|\nabla\varphi^n\|+\|\nabla\mathcal{R}^n\|)+\big|W_n^{\sigma_1}\big|\,\|\nabla\zeta^1\|\big)\|\nabla\zeta^n\|\\
&\quad+\frac{5}{4\kappa^2}C_\Omega^2C_f^2\big(\|\nabla\zeta^n\|^2+\|\nabla\zeta^{n-1}\|^2\big).
\end{aligned}
$$

Now using Lemma 3.1, the above inequality gives

$$
\|\nabla\zeta^n\|\le\max_{1\le j\le n}E_{\alpha(t_j)}\left(\frac{5\pi_A}{\kappa^2}C_\Omega^2C_f^2t_j^{\alpha(t_j)}\right)
$$

$$
\begin{aligned}
&\times \Bigg(\|\nabla\zeta^0\| + \max_{1\le k\le n} \sum_{j=1}^{k} \mathbb{P}_{k-j}^{(k)} \Big[K(\|\nabla\varphi^j\| + \|\nabla\mathcal{R}^j\|) + \big|W_j^{\sigma_1}\big| \, \|\nabla\zeta^1\| \\
&\qquad\qquad + \frac{C_5^*}{\sqrt{T^{\alpha^*}}} h^2 \Big] + \frac{C_5^*}{\sqrt{T^{\alpha^*}}} h^2 \Bigg) \\
&\le \max_{1\le j\le n} E_{\alpha(t_j)} \left(\frac{5\pi_A}{\kappa^2} C_\Omega^2 C_f^2 t_j^{\alpha(t_j)} \right) \\
&\times \Bigg(\max_{1\le k\le n} \sum_{j=1}^{k} \mathbb{P}_{k-j}^{(k)} \Big[K C_T \big(\Phi(j,\sigma_2) + t_1^{\sigma_2-\sigma_1} \Phi(j,\sigma_1) + N^{-\min\{2,2r\sigma_1\}} \big) \\
&\qquad + t_1^{-\sigma_1} \Phi(j,\sigma_1) \big(C_4^* N^{-\min\{2r\sigma_1, r\sigma_2\}} + C_5^* h^2 \big) \\
&\qquad + \frac{C_5^*}{\sqrt{T^{\alpha^*}}} h^2 \Big] + \frac{C_5^*}{\sqrt{T^{\alpha^*}}} h^2 \Bigg) \\
&\le \max_{1\le j\le n} E_{\alpha(t_j)} \left(\frac{5\pi_A}{\kappa^2} C_\Omega^2 C_f^2 t_j^{\alpha(t_j)} \right) \\
&\times \Bigg(\max_{1\le k\le n} \sum_{j=1}^{k} \mathbb{P}_{k-j}^{(k)} \Big[C_4^* t_j^{-\alpha(t_j)} N^{-\min\{2-\alpha^*, 2r\sigma_1, r\sigma_2\}} \\
&\qquad + \big(C_4^* N^{-\min\{2r\sigma_1, r\sigma_2\}} + C_5^* h^2 \big) + \frac{C_5^*}{\sqrt{T^{\alpha^*}}} h^2 \Big] + \frac{C_5^*}{\sqrt{T^{\alpha^*}}} h^2 \Bigg) \\
&\le \max_{1\le j\le n} E_{\alpha(t_j)} \left(\frac{5\pi_A}{\kappa^2} C_\Omega^2 C_f^2 t_j^{\alpha(t_j)} \right) \\
&\times \Bigg(\frac{C_4^* \big(1 + t_n^{\alpha^*}\big) e^r \max_{1\le j\le n} \Gamma\big(1 + \lambda_N - \alpha(t_j)\big)}{\Gamma(1+\lambda_N)} N^{-\min\{2-\alpha^*, 2r\sigma_1, r\sigma_2\}} \\
&\qquad + \left[\frac{1}{\sqrt{T^{\alpha^*}}} + \frac{t_n^{\alpha^*} e^r \max_{1\le j\le n} \Gamma\big(1+\lambda_N - \alpha(t_j)\big)}{\Gamma(1+\lambda_N)} \left(1 + \frac{1}{\sqrt{T^{\alpha^*}}} \right) \right] C_5^* h^2 \Bigg) \\
&\le C_6^* N^{-\min\{2-\alpha^*, 2r\sigma_1, r\sigma_2\}} + C_7^* h^2, \qquad (3.25)
\end{aligned}
$$

where (2.3), (2.4), (2.9), (2.10), (2.12), and $\|\nabla\zeta^0\| = \|\nabla(R_h u^0 - U_h^0)\| = 0$ are used.

Furthermore, applying (2.14), (3.24), (3.25), and the Poincaré inequality, one has

$$
\begin{aligned}
\big\|u^n - U_h^n\big\| &\le \big\|u^n - R_h u^n\big\| + \big\|R_h u^n - U_h^n\big\| \le C_\Omega h^2 + C_\Omega \big\|\nabla R_h u^n - \nabla U_h^n\big\| \\
&\le C_\Omega \big[(C_4^* + C_6^*) N^{-\min\{2-\alpha^*, 2r\sigma_1, r\sigma_2\}} + (C_5^* + C_7^* + 1) h^2 \big].
\end{aligned}
$$

Hence, the bound (3.22) follows. □

Now the convergence accuracy of (3.22) in the spatial direction can be improved on certain rectangular domains.

Corollary 3.1. *By organizing mesh elements into macroelements, a postprocessing of the numerical solution U_h^n on these meshes yields a piecewise quadratic solution $\pi_{2h} U_h^n$. If the bilinear element is used in space, then we have*

$$
\big\|\nabla u^n - \nabla \pi_{2h} U_h^n\big\| \le C\big(N^{-\min\{2-\alpha^*, 2r\sigma_1, r\sigma_2\}} + h^2 \big).
$$

Proof. Imitate the analysis of [7, Section 6]. □

4. Numerical Experiments

In this section, we will present some numerical examples to verify the convergent results given in Theorem 3.1 for the CL1-FEM (2.15).

In our computation, we use the bilinear element on a uniform rectangular partition of Ω with $M+1$ nodes in each spatial direction. By choosing $\sigma_1 = \alpha^*$, $\sigma_2 = 2\alpha^*$, and

$$r = \max\left\{\frac{2-\alpha^*}{2\alpha^*}, 1\right\},$$

Theorem 3.1 and Corollary 3.1 achieve the optimal rates of convergence viz., $\mathcal{O}(h^2 + N^{-(2-\alpha^*)})$ for $\|u^n - U_h^n\|$ and $\|\nabla u^n - \nabla \pi_{2,h} U_h^n\|$.

Example 4.1. We consider the problem (1.1) with

$$\alpha(t) = \alpha^* + (0.1 - \alpha^*)\left(1 - t - \frac{\sin\big(2\pi(1-t)\big)}{2\pi}\right),$$

$\kappa^2 = 0.1, T = 1, \Omega = (0, 0.5) \times (0, 0.5)$, and $f(u) = 0.1(u - u^3)$. The function $g(x, y, t)$ is chosen such that the exact solution of this problem is

$$u(x, y, t) = (t^{\alpha^*} + t^{2\alpha^*} + t^3)\sin(0.5x - x^2)\sin(0.5y - y^2).$$

Tables 4.1 and 4.2 present the errors and their associated orders of convergence for the $\max_{0\le n\le N}\|u^n - U_h^n\|$ and $\max_{0\le n\le N}\|\nabla u^n - \nabla\pi_{2,h}U_h^n\|$ with different α^*, where $M = 3N$ is taken to eliminate the effect from the spatial error. The results of these two tables indicate that the convergence order in the temporal direction is $2 - \alpha^*$, as predicted by Theorem 3.1 and Corollary 3.1.

Table 4.1: $\max_{0\le n\le N}\|u^n - U_h^n\|$ errors and rates of convergence in time for Example 4.1.

	$N=10$	$N=20$	$N=40$	$N=80$
$\alpha^*=0.4$	9.5883E-6	3.1919E-6	1.0578E-6	3.4967E-7
		1.5879	1.5924	1.5956
$\alpha^*=0.6$	1.1806E-5	4.1920E-6	1.5157E-6	5.5555E-7
		1.4937	1.4669	1.4509
$\alpha^*=0.8$	2.1178E-5	8.7369E-6	3.6744E-6	1.5637E-6
		1.2783	1.2518	1.2342

Table 4.2: $\max_{0\le n\le N}\|\nabla u^n - \nabla\pi_{2,h}U_h^n\|$ errors and rates of convergence in time for Example 4.1.

	$N=10$	$N=20$	$N=40$	$N=80$
$\alpha^*=0.4$	8.5521E-5	2.8434E-5	9.4160E-6	3.1110E-6
		1.5900	1.5920	1.5988
$\alpha^*=0.6$	1.0521E-4	3.7317E-5	1.3484E-5	4.9407E-6
		1.4931	1.4662	1.4503
$\alpha^*=0.8$	1.8845E-4	7.7703E-5	3.2670E-5	1.3902E-5
		1.2747	1.2486	1.2342

Table 4.3: $\alpha^* = 0.4$; $\max_{0\le n\le N}\|u^n - U_h^n\|$ and $\max_{0\le n\le N}\|\nabla u^n - \nabla\pi_{2,h}U_h^n\|$ errors and convergence rates in space for Example 4.1.

Polynomial	M	$\max_{0\le n\le N}\|u^n - U_h^n\|$	Order	$\max_{0\le n\le N}\|\nabla u^n - \nabla\pi_{2,h}U_h^n\|$	Order
Q_1	10	2.9126E-5	-	2.6489E-4	-
	20	7.2413E-6	2.0079	6.5815E-5	2.0098
	40	1.8119E-6	2.0002	1.6465E-5	1.9956
	80	4.5732E-7	1.9857	4.1539E-6	1.9912

Table 4.3 shows the $\max_{0\le n\le N}\|u^n - U_h^n\|$ and $\max_{0\le n\le N}\|\nabla u^n - \nabla\pi_{2,h}U_h^n\|$ errors and their associated orders of convergence in space for $\alpha^* = 0.4$, where $N = 1000$ is taken. The obtained results display that the convergence order in the spatial direction attains 2 orders, as consistent with Theorem 3.1 and Corollary 3.1.

Example 4.2. Considering the initial-boundary value problem (1.1) with $\kappa^2 = 0.1, T = 1$, $\Omega = (0,1)\times(0,1)$, and $f(u) = 0.1(u - u^2)$. We consider different choices of $\alpha(t)$ which are presented in Table 4.4. The function $g(x,y,t)$ satisfies that the exact solution of this problem is

$$u(x,y,t) = (1 + t^{\alpha^*} + t^{2\alpha^*})\big(\cos(x - x^2) - 1\big)\big(\cos(y - y^2) - 1\big),$$

where $\alpha^* = \max_{0\le t\le T}\alpha(t)$.

In this example, we only pay attention to the convergent result in time. Tables 4.4 and 4.5 display the error $\max_{1\le n\le N}\|u^n - U^n\|$ and $\max_{1\le n\le N}\|\nabla u^n - \nabla\pi_{2,h}U^n\|$ for different $\alpha(t)$, where $M = 3N$ is chosen to eliminate the effect from the spatial error. The displayed results of these two tables imply that the rates of convergence in time are $\mathcal{O}(N^{-(2-\alpha^*)})$, as predicted by Theorem 3.1 and Corollary 3.1.

Table 4.4: $\max_{0\le n\le N}\|u^n - U_h^n\|$ errors and rates of convergence in time for Example 4.2.

	$N = 16$	$N = 32$	$N = 64$	$N = 128$
$\alpha(t) = 0.4e^{t^3-t}$	7.3994E-7	2.5451E-7	8.6902E-8	2.9389E-8
		1.5396	1.5502	1.5641
$\alpha(t) = 0.6\cos(\pi(t-t^2))$	2.3750E-6	9.8090E-7	3.9440E-7	1.5562E-7
		1.2757	1.3144	1.3416
$\alpha(t) = 0.8 - 0.2\sin(\pi t/2)$	2.9205E-6	1.2412E-6	5.2512E-7	2.2193E-7
		1.2344	1.2410	1.2425

Table 4.5: $\max_{0\le n\le N}\|\nabla u^n - \nabla\pi_{2,h}U_h^n\|$ errors and rates of convergence in time for Example 4.2.

	$N = 16$	$N = 32$	$N = 64$	$N = 128$
$\alpha(t) = 0.4e^{t^3-t}$	3.5382E-6	1.2060E-6	4.1258E-7	1.4036E-7
		1.5527	1.5475	1.5554
$\alpha(t) = 0.6\cos(\pi(t-t^2))$	1.0921E-5	4.5277E-6	1.8357E-6	7.3156E-7
		1.2703	1.3024	1.3272
$\alpha(t) = 0.8 - 0.2\sin(\pi t/2)$	1.3257E-5	5.6110E-6	2.3696E-6	1.0005E-6
		1.2404	1.2436	1.2438

5. Conclusions

By adding a corrected term to the L1 scheme, a corrected L1 scheme is developed to approximate the Caputo derivative with a variable exponent. Combining the proposed scheme with the finite element method in space and the Newton linearization for the nonlinear term, a fully discrete CL1-FEM is constructed. The boundeness of the computed solution is proved by adopting the temporal-spatial splitting technique. With the help of this bound, the unconditional optimal convergent result for the proposed scheme is obtained. Finally, the theoretical findings are confirmed by numerical examples. In the future, we will pay attention to the fast scheme and the pointwise-in-time error analysis of the proposed scheme for the problem (1.1).

Acknowledgments. The research of C.B. Huang is supported in part by the National Natural Science Foundation of China (Grant Nos. 12101360, 12171278), by the Support Plan for Outstanding Youth Innovation Team in Shandong Higher Education Institutions (Grant No. 2022KJ184), and by the Natural Science Foundation of Shandong Province (Grant No. ZR2022MA068). The research of H. Chen is supported in part by the National Natural Science Foundation of China (Grant No. 11801026), by the Natural Science Foundation of Shandong Province (Grant No. ZR2023MA077), and by the Fundamental Research Funds for the Central Universities (No. 202264006).

References

[1] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, in: *Texts in Applied Mathematics*, Vol. 15, Springer, 2008.

[2] G.R.J. Cooper and D.R. Cowan, Filtering using variable order vertical derivatives, *Comput. Geosci.*, **30**:5 (2004), 455–459.

[3] J.L. Gracia, E. O'Riordan, and M. Stynes, A fitted scheme for a Caputo initial-boundary value problem, *J. Sci. Comput.*, **76**:1 (2018), 583–609.

[4] C.B. Huang, N. An, H. Chen, and X.J. Yu, α-robust error analysis of two nonuniform schemes for subdiffusion equations with variable-order derivatives, *J. Sci. Comput.*, **97**:2 (2023), 43.

[5] C.B. Huang and H. Chen, Superconvergence analysis of finite element methods for the variable-order subdiffusion equation with weakly singular solutions, *Appl. Math. Lett.*, **139** (2023), 108559.

[6] C.B. Huang and M. Stynes, Optimal spatial H^1-norm analysis of a finite element method for a time-fractional diffusion equation, *J. Comput. Appl. Math.*, **367** (2020), 112435.

[7] C.B. Huang and M. Stynes, Superconvergence of a finite element method for the multi-term time-fractional diffusion problem, *J. Sci. Comput.*, **82**:1 (2020), 10.

[8] C.B. Huang and M. Stynes, A sharp α-robust $L^\infty(H^1)$ error bound for a time-fractional Allen-Cahn problem discretised by the Alikhanov $L2-1_\sigma$ scheme and a standard FEM, *J. Sci. Comput.*, **91**:2 (2022), 43.

[9] B.T. Jin, B.Y. Li, and Z, Zhou, Correction of high-order BDF convolution quadrature for fractional evolution equations, *SIAM J. Sci. Comput.*, **39**:6 (2017), A3129–A3152.

[10] F. Kheirkhah, M. Hajipour, and D. Baleanu, The performance of a numerical scheme on the variable-order time-fractional advection-reaction-subdiffusion equations, *Appl. Numer. Math.*, **178** (2022), 25–40.

[11] N. Kopteva, Error analysis of the L1 method on graded and uniform meshes for a fractional-derivative problem in two and three dimensions, *Math. Comp.*, **88**:319 (2019), 2135–2155.

[12] H.L. Liao, W. McLean, and J.W. Zhang, A discrete Grönwall inequality with applications to numerical schemes for subdiffusion problems, *SIAM J. Numer. Anal.*, **57**:1 (2019), 218–237.

[13] H.L. Liao, Y.G. Yan, and J.W. Zhang, Unconditional convergence of a fast two-level linearized algorithm for semilinear subdiffusion equations, *J. Sci. Comput.*, **80**:1 (2019), 1–25.

[14] H.Y. Qin, D.F. Li, and Z.M. Zhang, A novel scheme to capture the initial dramatic evolutions of nonlinear subdiffusion equations, *J. Sci. Comput.*, **89**:3 (2021), 65.

[15] J. Quintana-Murillo and S.B. Yuste, An adaptive difference method for variable-order diffusion equations, *Mediterr. J. Math.*, **21**:5 (2024), 145.

[16] M. Stynes, E. O'Riordan, and J.L. Gracia, Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation, *SIAM J. Numer. Anal.*, **55**:2 (2017), 1057–1079.

[17] Z.J. Tan, Second-order non-uniform and fast two-grid finite element methods for non-linear time-fractional mobile/immobile equations with weak regularity, *Appl. Math. Comput.*, **486** (2025), 129043.

[18] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, in: *Springer Series in Computational Mathematics*, Vol. 25, Springer, 2006.

[19] C.C. Tseng, Design of variable and adaptive fractional order fir differentiators, *Signal Process.*, **86**:10 (2006), 2554–2566.

[20] S. Umarov and S. Steinberg, Variable order differential equations with piecewise constant order-function and diffusion with changing modes, *Z. Anal. Anwend.*, **28**:4 (2009), 431–450.

[21] Z. Yang and F.H. Zeng, A corrected L1 method for a time-fractional subdiffusion equation, *J. Sci. Comput.*, **95**:3 (2023), 85.

[22] X. Ye, J. Liu, B.Y. Zhang, H.F. Fu, and Y. Liu, High order numerical methods based on quadratic spline collocation method and averaged L1 scheme for the variable-order time fractional mobile/immobile diffusion equation, *Comput. Math. Appl.*, **171** (2024), 82–99.

[23] X.C. Zheng, Two methods addressing variable-exponent fractional initial and boundary value problems and Abel integral equation, *arXiv:2404.09421*, 2024.

[24] X.C. Zheng and H. Wang, Optimal-order error estimates of finite element approximations to variable-order time-fractional diffusion equations without regularity assumptions of the true solutions, *IMA J. Numer. Anal.*, **41**:2 (2021), 1522–1545.

[25] B.Y. Zhou, X.L. Chen, and D.F. Li, Nonuniform Alikhanov linearized Galerkin finite element methods for nonlinear time-fractional parabolic equations, *J. Sci. Comput.*, **85**:2 (2020), 39.