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SUPERCONVERGENCE OF DIFFERENTIAL STRUCTURE FOR FINITE ELEMENT METHODS ON PERTURBED SURFACE MESHES*

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Abstract

Superconvergence of differential structure on discretized surfaces is studied in this paper. The newly introduced geometric supercloseness provides us with a fundamental tool to prove the superconvergence of gradient recovery on deviated surfaces. An algorithmic framework for gradient recovery without exact geometric information is introduced. Several numerical examples are documented to validate the theoretical results.

Mathematics subject classification: 41A25, 65N15, 65N30.
Key words: Superconvergence, Differential structure, Discretized surfaces with deviation, Geometric supercloseness, Gradient recovery.

1. Introduction

The numerical solution of partial differential equations on manifolds or more general surfaces has been the subject of much systematic investigation. Many efficient numerical methods have been developed since the pioneering work of Dziuk [14]. However, their numerical analysis including the a priori error analysis of surface finite element methods [4,15] and the a posteriori error analysis [11,25] usually requires exact information of the surfaces. For instance, many methods ask that the vertices of discrete surfaces are located on the underlying surfaces and the exact unit normal vectors at the given vertices are known. This is neither theoretically complete nor practically available since the exact geometric information is often blind to users in reality. Therefore, it is of interest and also practically meaningful to investigate the problems where the exact geometric information is not given. We particularly pay attention to the cases when solutions contain differential structures of the surfaces. First-order differential structures involve tangential spaces and normal spaces of the surfaces, while the former is our focus in this paper. Typical examples are tangential vector fields on surfaces and gradients

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of scalar functions on surfaces. In such situations, it is desired to know the conditions for geometric discretization to guarantee optimal convergence rates either in the a priori or the a posteriori error analysis. Fundamental questions here are like (i) to what extent that the errors of geometric approximations will affect the total errors of the numerical methods, and (ii) what is the hypothesis on the geometric discretization in order to have optimal convergence of numerical solutions or superconvergence of the differential structure of surfaces.

This paper aims to provide some insight into these questions. Such problems have been open in the community for a while. For instance in [25], gradient recovery schemes on general surfaces have been systematically investigated, and superconvergence rates of several recovery schemes were proven provided that the exact geometry is given. The supercloseness of the numerical data has played a crucial role in establishing the theoretical results in [25]. Moreover, the following two interesting questions arose in [25]:

(i) How to design gradient recovery algorithms given no exact information of the surfaces, i.e. no exact normal vectors and no exact vertices?

(ii) Is it possible to preserve the superconvergence rates of gradient recovery schemes using triangulated meshes whose vertices are not located on the exact surfaces but in a $\mathcal{O}(h^2)$ neighborhoods of the underlying surfaces? Here h is the scale of the mesh size.

These questions partially motivate the research here. In particular, superconvergence of gradient recovery on surfaces is connected to the concept of geometric supercloseness which we propose in this paper. Gradient recovery techniques for data defined in Euclidean domain have been intensively investigated [1, 3, 16, 17, 19, 26–29], and also find many interesting applications, e.g. [6, 18, 22, 23]. The methods for data on discretized surfaces have been studied, e.g. in [13, 25]. Using the idea of tangential projection, many of the recovery algorithms in the setting of the Euclidean domain have been generalized to the setting of surfaces. However, there are certain restrictions in the existing approaches as many of them require the exact geometry (exact vertices, exact normal vectors) either for designing algorithms or for proving superconvergent rates. In [11] a novel gradient recovery scheme for data defined on discretized surfaces was proposed, which is called the parametric polynomial preserving recovery (PPPR) method. PPPR does not rely on the exact geometry-prior, and it was proven to be able to achieve superconvergence under mildly structured meshes, including high curvature cases. That can be thought of partially answered the first open question in [25]. However, the theoretical proof for the superconvergence result in [11] still requires that the vertices are located on the underlying exact surfaces, though numerically the superconvergence has been observed when this condition is violated.

In this paper, we first construct some examples to show that there exist cases where the superconvergence of gradient recovery on surfaces is not guaranteed given barely the $\mathcal{O}(h^2)$ vertex condition. In particular, the examples show that data supercloseness does not guarantee superconvergence of the recovered gradient, in contrast to the exact nodal points case. We introduce a new concept called geometric supercloseness, which gives the property of superconvergence of differential structure on deviated discretizations of surfaces. Especially, we provide conditions of the discretized meshes under which the geometric supercloseness property can be proven. With the tool of superconvergence of differential structure, we provide complete answers to the two open questions in [25]. To do this, we generalize the idea from [11]. That is the idea of local parametric polynomials can be further developed to cover other methods, e.g. superconvergence