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A NEW PROJECTION-BASED STABILIZED VIRTUAL ELEMENT APPROXIMATION FOR THREE-FIELD POROELASTICITY*

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Abstract

In this paper, we develop a fully discrete virtual element scheme based on the local pressure projection stabilization for a three-field poroelasticity problem with a storage coefficient $c_0 \geq 0$. We not only provide the well-posedness of the proposed scheme by proving a weaker form of the discrete inf-sup condition, but also show optimal error estimates for all unknowns, whose generic constants are independent of the Lamé coefficient λ. Moreover, our proposed scheme avoids pressure oscillation and applies to general polygonal elements, including hanging-node elements. Finally, we numerically validate the good performance of our virtual element scheme.

Mathematics subject classification: 65N12, 65N15, 65N30, 74F10, 76S05.
Key words: Stabilized virtual element method, Three-field poroelasticity problem, Well-posedness, Optimal error estimates, General polygonal meshes.

1. Introduction

The poroelasticity theory, which describes an interaction between the deformation of an elastic porous medium and its internal fluid flow, was originally proposed by Terzaghi in his early one-dimensional work [69]. Later, Biot [24–26] developed a three-dimensional mathematical model in which the solid-to-fluid coupling was also considered (not just the fluid-to-solid coupling), so that one can successfully deal with both slightly compressible and incompressible fluids. A further development of Biot's model can be found in [74–76]. Since then, Biot's poroelastic theory has been used in a great variety of science and engineering fields, ranging from geomechanics, environment and reservoir engineering to, more recently, biomechanical engineering and food processing, such as soil consolidation, geological carbon sequestration, petroleum production, perfusion of bones and soft living tissues [34, 40, 52, 55, 65, 67, 70, 71]. At the same time, many different numerical methods for solving Biot's model have also been developed, with the finite difference [36, 39, 58] and finite element methods being the common

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choices. The earlier works about the finite element methods for Biot's model were introduced as follows: Murad [56, 57] proposed and analyzed the discrete schemes for two-field model using Stokes finite elements for the displacement and pressure; Phillips, Yi, Hu, Rivière and others combined the mixed finite element methods for the pressure-velocity pair and a continuous [59, 60]/nonconforming [41, 72]/discontinuous [61, 63]/weak Galerkin [66] method for the displacement, different versions of these methods can also be found in [22, 37, 43, 44].

Because of existence of complex geometrical features in practical poroelastic problems (particularly in faulted and heterogeneous media) and consideration of interfaces when Biot's model is coupled with other models, one must develop numerical methods that allow for use of general meshes (including meshes with hanging nodes). The virtual element method (VEM) introduced in [8] is a new method for the discretization of partial differential equations and can handle these features. In general, the VEM avoids explicit expressions of basis functions and implements their discrete forms with the corresponding degrees of freedom and projection operators. More specific implementations, including how to calculate the H^1- and L^2-projections and discrete bilinear forms solely by using degrees of freedom, can be found in [10]. The basic VEM, combined with other numerical methods, has been successively applied to second-order elliptic equations to form the mixed VEM [28], nonconforming VEM [5], discontinuous VEM [30], $H(\text{div})/H(\text{curl})$-VEM [11] and serendipity VEM [12,15]. In addition, thanks to its firm theoretical foundation and concise program implementation, the VEM has also widely been applied to solving two- and three-dimensional elasticity [4,9,38], incompressible fluids [1,18,46,49,50], plate bending [3, 29, 73], discrete fracture networks [20, 21, 23], and Cahn-Hillard problems [2, 47, 48].

In this paper, we develop a fully discrete virtual element scheme with arbitrary approximation accuracy for solving three-field Biot's model, which is different from the scheme proposed in [35] for the two-field model. More specifically, we use appropriate mixed and conforming virtual element approximations for the pressure, fluid velocity and displacement, instead of the finite volume and virtual element approximations for the pressure and displacement in [35]. The introduction of a new variable - the fluid velocity avoids the postprocessing of this variable and the material stress, and allows an application of physically meaningful boundary conditions at interfaces when considering the coupling between a fluid and a poroelastic structure [6]. Meanwhile, it is easier to maintain the mass conservation for a fluid phase by using continuous elements for the fluid velocity variable. Moreover, different from the low-order case in [68], our discrete scheme is designed based on the local pressure projection stabilization method to overcome the pressure oscillation, which is parameter-free and does not require the calculation of higher-order derivatives or edge-based data structures in contrast to the stabilization procedure in [22]. And we can prove the existence and uniqueness of solutions to our semi and fully discrete schemes by combing the weaker form of the discrete inf-sup condition with the theory of differential algebraic equations. Furthermore, we show the optimal convergence analysis independent of the Lamé coefficient λ under different norms. Finally, in the spirit of the VEM, this new scheme allows the use of general polygonal meshes, including the meshes with hanging nodes.

The rest of the paper is organized as follows. In Section 2, we describe fundamental settings for Biot's consolidation model. In Section 3, we introduce some discrete settings and formulate the proposed virtual element scheme. In Section 4, we show the well-posedness of the semi-discrete problem and derive the optimal error estimates for the virtual element approximations. In Section 5, we extend these results to the corresponding fully discrete scheme. Finally, numerical results are presented to validate the robustness of the proposed scheme.