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ON POLYNOMIAL PMLs FOR HELMHOLTZ SCATTERING PROBLEMS WITH HIGH WAVE-NUMBERS*

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Abstract

This paper presents a simple proof for the stability of circular perfectly matched layer (PML) methods for solving acoustic scattering problems in two and three dimensions. The medium function of PML allows arbitrary-order polynomials, and can be extended to general nondecreasing functions with a slight modification of the proof. In the regime of high wavenumbers, the inf-sup constant for the PML truncated problem is shown to be $\mathcal{O}(k^{-1})$. Moreover, the PML solution converges to the exact solution exponentially, with a wavenumber-explicit rate, as either the thickness or medium property of PML increases. Numerical experiments are presented to verify the theories and performances of PML for variant polynomial degrees.

Mathematics subject classification: 65N12, 65N15, 78A40.
Key words: Helmholtz equation, Perfectly matched layer, High wavenumber, Arbitrary polynomial degrees, Scattering problem.

1. Introduction

In this paper, we study the perfectly matched layer method for the Helmholtz scattering problem in $\mathbb{R}^d$ $(d = 2, 3)$,

$$\Delta u + k^2 u = 0 \qquad \text{in } D_e := \mathbb{R}^d \backslash \overline{D}, \tag{1.1a}$$

$$u = g \qquad \text{on } \Gamma, \tag{1.1b}$$

$$|\partial_r u - \mathbf{i} k u| = o\big(r^{\frac{1-d}{2}}\big), \quad r = |\boldsymbol{x}| \ \to \ \infty. \tag{1.1c}$$

Here $D \subset \mathbb{R}^d$ denotes the scatterer and is a bounded and star-shaped domain with a Lipschitz-continuous boundary $\Gamma := \partial D$. The Dirichlet boundary condition $g \in H^{-1/2}(\Gamma)$ is usually set by incident waves. Moreover, $\partial_r u$ represents the partial derivative in the radial direction. The wavenumber k is a positive constant. Since we are interested in the high wavenumber regime, $k \geq 1$ is assumed throughout this paper.

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The idea of PML is to enclose the domain where the solution is concerned by a wave-absorbing layer. This specially designed layer is capable of exponentially attenuating all waves propagating outward from the computational domain [1, 2]. One can truncate the unbounded domain into a bounded one and impose homogeneous boundary condition on the truncation boundary. Two important questions arise from the truncation:

1) Does the truncated problem have a unique solution?

2) Does the approximate solution converge to the exact solution as the thickness of PML increases?

Clearly the second question depends crucially on the answer to the first question, namely, the stability of the PML method.

Two kinds of PMLs are widely studied in the literature, that is,

- the circular PML which is defined with a complex coordinate transformation along the radial direction,
- the uniaxial PML or Cartesian PML which is defined with d complex transformations along d Cartesian coordinate directions, respectively.

Bramble and Pasciak [3, 4] utilized circular PML to truncate the unbounded domain and proved the well-posedness of the PML problem, providing that the medium function is constant outside of a large ball. Chen and Zheng [11] extended the results to electromagnetic scattering problems in a two-layer medium and proved the well-posedness of the PML problem. Li and Wu [15] extended the results to high-frequency acoustic scattering problems, and presented a k-explicit theory (k denotes the wavenumber) for the stability and convergence of circular PMLs. They assumed that the medium function is constant in the PML. Chen and Zheng [10] studied the uniaxial PML method for the Helmholtz scattering problem in a two-layer medium. They proved the well-posedness of the PML truncated problem, providing that the medium functions used for defining complex coordinate transformations are constant along their respective coordinate directions. Later on, Bramble and Pasciak [5, 6] established the well-posedness of PML truncated problems for both Helmholtz equations and Maxwell's equations, based on the assumption that the medium functions, outside of a large box, are constant along their respective coordinate directions. To enhance the flexibility of PMLs and reduce the reflection of outgoing waves by the interface between the PML region and the non-dissipative region, it is desirable to establish the stability theory for polynomial medium functions of arbitrary degrees.

The second issue is concerning high-frequency scattering problems. In the case of large wavenumbers, k-explicit theories for PML methods are much more attractive than k-implicit ones, since practitioners can determine the thickness of PML accurately to avoid redundant computational quantities. Chen and Xiang [9] proposed a source-transfer-domain-decomposition method for solving source scattering problems. They proved that, for uniaxial PML method, the lower bound of the inf-sup constant is $\mathcal{O}(k^{-3/2})$. Later, Li and Wu [15] improved this result for circular PML and constant medium function. The lower bound of the inf-sup constant is improved to $\mathcal{O}(k^{-1})$. In 2022, Chaumont-Frelet *et al.* [8] studied circular PML methods for two-dimensional (2D) acoustic scattering problems by bounded obstacles. The PML medium can be defined by general nondecreasing functions. They proved the k-explicit stability with lower bound of the inf-sup constant being $\mathcal{O}(k^{-1})$. The obstacle is assumed to be star-shaped and has a smooth boundary so that the H^2-regularity holds for the solutions of elliptic equations.