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ON AN INCREMENTAL VERSION OF THE CHEBYSHEV METHOD FOR THE MATRIX P-TH ROOT*

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Abstract

The aim of this paper is to present an improvement of the incremental Newton method proposed by Iannazzo [SIAM J. Matrix Anal. Appl., 28:2 (2006), 503–523] for approximating the principal p-th root of a matrix. We construct and analyze an incremental Chebyshev method with better numerical behavior. We present a convergence and numerical analysis of the method, where we compare it with the corresponding incremental Newton method. The new method has order of convergence three and is stable and more efficient than the incremental Newton method.

Mathematics subject classification: 65F45.
Key words: Matrix p-th root, Incremental Newton method, Chebyshev's method, Order of convergence, Convergence, Efficiency.

1. Introduction

The Newton and the Halley methods are iterative techniques widely used to find approximations of zeros of functions in numerical analysis. These methods can be adapted to approximate the p-th root of a matrix. The p-th root of a matrix is fundamental in applications related to the solution of linear systems and the calculation of matrix exponentials, as well as used in spectral analysis. Approximating this root accurately extends to approximating other matrix functions, enabling efficient solutions to a wide range of problems in engineering, physics, and data analysis.

While the Newton and the Halley methods are certainly popular choices due to their simplicity and effectiveness, other iterative methods and specialized algorithms tailored for matrix root approximation may also be employed, especially in cases where specific matrix structures or computational constraints dictate alternative approaches. However, a good alternative of both methods for this problem is given by Chebyshev's method. In our experience, this methods is more efficient from a computational point of view. We present a particular example related to

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the discritization of partial differential equations (see [1] for more comparisons): We consider the matrix

$$A = \begin{pmatrix} 1-2\lambda & \lambda & 0 & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & 0 & \cdots & 0 \\ 0 & \lambda & 1-2\lambda & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda & 1-2\lambda & \lambda \\ 0 & \cdots & 0 & 0 & \lambda & 1-2\lambda \end{pmatrix},$$

and the equation $X^p - A = 0$. We consider $\dim(A) = 100$, $p = 2, 4, 6, 8, 10$ and $\lambda = 2 \cdot 10^{-3}$. In Figs. 1.1 and 1.2, we observe a better numerical behavior of the Chebyshev method.

On the other hand, there are several versions of this type of methods that are stable for the approximation of the p-th root of a matrix. But, we remember that the classical versions of the methods are not stable. One version of the Newton method that has attracted attention during years is the incremental version of this method. Motivated by the previous examples, in this work, we are interested in constructing and analyzing an incremental version of the Chebyshev method and see that this new version is both stable and more efficient than the Newton incremental version.

For solving the matrix equation $X^p - A = 0$, where $A \in \mathbb{C}^{n \times n}$, it is easy to consider first the matrix iteration

$$X_{k+1} = \frac{1}{p}\big((p-1)X_k + AX_k^{1-p}\big), \quad k \geq 0 \tag{1.1}$$

with an initial value X_0 satisfying $AX_0 = X_0A$, that is obtained from the Newton method by applying this method to the equation $X^p - A = 0$. Method (1.1) is usually referred as the simplified Newton iteration. One of the most interesting solutions of $X^p - A = 0$ is the principal p-th root $A^{1/p}$ whose eigenvalues lie in the set $\{z \in \mathbb{C}\backslash\{0\} : -\pi/p < \arg z < \pi/p\}$.

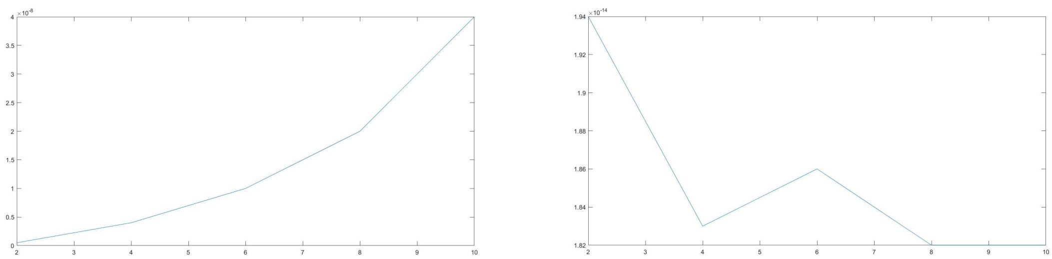

Fig. 1.1. The errors of the Newton method (on the left) and the Chebyshev method (on the right) fixed the CPU time.

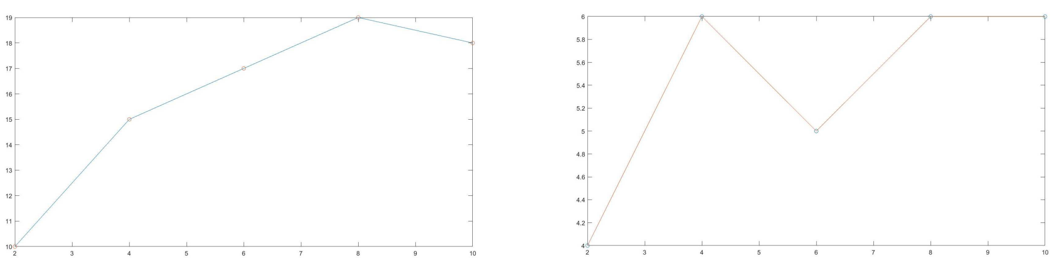

Fig. 1.2. The CPU times of the Halley method (on the left) and the Chebyshev method (on the right) using the stopping criterium $\|X_{k+1} - X_k\|_1 \leq 10^{-13}$.