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SUPERCONVERGENCE ERROR ESTIMATES OF THE LOWEST-ORDER RAVIART-THOMAS GALERKIN MIXED FINITE ELEMENT METHOD FOR NONLINEAR THERMISTOR EQUATIONS*

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Abstract

This paper is concerned with the superconvergence error estimates of a classical mixed finite element method for a nonlinear parabolic/elliptic coupled thermistor equations. The method is based on a popular combination of the lowest-order rectangular Raviart-Thomas mixed approximation for the electric potential/field $(\phi, \boldsymbol{\theta})$ and the bilinear Lagrange approximation for temperature u. In terms of the special properties of these elements above, the superclose error estimates with order $\mathcal{O}(h^2)$ are obtained firstly for all three components in such a strongly coupled system. Subsequently, the global superconvergence error estimates with order $\mathcal{O}(h^2)$ are derived through a simple and effective interpolation post-processing technique. As by a product, optimal error estimates are acquired for potential/field and temperature in the order of $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$, respectively. Finally, some numerical results are provided to confirm the theoretical analysis.

Mathematics subject classification: 65M15, 65M30, 65N15, 65N30.
Key words: Nonlinear thermistor equations, Galerkin mixed finite element method, Interpolation post-processing technique, Superclose and superconvergence error estimates.

1. Introduction

In this paper, we focus on superconvergence error analysis of the lowest-order Raviart-Thomas mixed finite element method for nonlinear and coupled thermistor equations, which are modeled as a coupled system of nonlinear partial differential equations with a quadratic growth on the gradient of one of the unknowns, defined by

$$u_t - \Delta u = \sigma(u)|\nabla\phi|^2, \qquad (\boldsymbol{x},t) \in \Omega \times J, \tag{1.1}$$

$$-\nabla\cdot(\sigma(u)\nabla\phi) = 0, \qquad (\boldsymbol{x},t) \in \Omega \times J, \tag{1.2}$$

$$u(\boldsymbol{x},t) = 0, \quad \phi(\boldsymbol{x},t) = g(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \partial\Omega \times J, \tag{1.3}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^2$ is a rectangular domain with boundary $\partial\Omega, \boldsymbol{x} = (x,y), J = (0,T]$. The system (1.1)-(1.4) models the electric heating of a conducting body, which plays an important role in many micro-electromechanical systems. The unknowns $\phi = \phi(\boldsymbol{x},t)$ and $u = u(\boldsymbol{x},t)$ are the

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distributions of the electrical potential and the temperature in Ω, respectively. $\sigma(u)$ is the temperature dependent electrical conductivity, $\sigma(u)|\nabla\phi|^2$ is the Joule heating. Moreover, u_0 and g are given smooth functions.

A lot of theoretical and numerical analysis have been devoted to system (1.1)-(1.4) by many authors due to its wide applications (see [2–5, 12, 16, 19–22, 32, 49–52, 56, 59–62, 64] and the references therein). More precisely, for theoretical analysis, the existence of time-dependent thermistor equations was shown by means of supersolutions and subsolutions, the maximum principle and fixed point argument in [4]. The existence of weak solutions was studied with Faedo-Galerkin method for an arbitrarily large interval of time in [12]. The existence and uniqueness of C^α solution for thermistor problem with mixed boundary conditions was established in [60]. For numerical analysis, a linearized Euler Galerkin scheme with linear finite element approximation applied in spatial direction was presented and analyzed in [61]. Due to some pollution arising from the approximation used the nonlinear term $\sigma(u)|\nabla\phi|^2$, only a sub-optimal error estimate was obtained. In [16], optimal error estimate was established based on the duality argument under the time-step condition $\tau = \mathcal{O}(h^{d/6})$, where d is the dimension, for completely discrete scheme with minimal regularity assumptions. Based on a standard finite element method used in spatial direction and the combinations of rational implicit and explicit multistep schemes used in temporal direction, some higher-order linearly implicit finite element schemes were developed in [2] and optimal error estimates were proved under the time-step condition $\tau = \mathcal{O}(h^{3/(2p)})$ and $r \geq 2$, where p is the order of discretization in time and r is the degree of piecewise polynomial approximations used in space, respectively. In terms of an error splitting technique proposed in [33, 34], the unconditionally optimal error estimates for Lagrange finite element methods with different time approximation schemes were established in [2, 19, 20, 32]. Subsequently, the superconvergence error estimates were derived in [49–52] with the help of the high precision integral identity technique under the appropriate restriction between time step size and space step size.

Since mixed finite element methods allow simultaneous computation of the original variable and its gradient, both of them being equally accurate [18], and these methods have been applied to many problems [6, 7, 10, 13, 23, 25–27] and the references therein. As pointed out in [22], $\boldsymbol{\theta} = \sigma(u)\nabla\phi$ denotes the electric field, which is more important in physics, it is natural to solve the system with a mixed finite element method to approximate potential/field and temperature $(\phi, \boldsymbol{\theta}, u)$. Mixed finite element methods may produce a better approximation to the electric field $\boldsymbol{\theta}$ and the nonlinear source term $\sigma(u)|\nabla\phi|^2$. In [62], based on the Raviart-Thomas mixed finite element approximation used for the electric potential/field $(\phi, \boldsymbol{\theta})$ and classical Lagrange finite element approximation applied for the temperature u, a mixed finite element scheme was proposed and investigated. It should be pointed out that a higher-order mixed finite element space was required in [62] to obtain optimal error estimate for temperature, in which the lowest-order Raviart-Thomas mixed finite element space is excluded. As we known, the lowest-order Raviart-Thomas mixed finite element space is the most popular and widely used in practical applications [8, 43, 53] due to the ease of implementation and less computational costs. In terms of an H^{-1}-norm estimate of a classical mixed finite element method, which the lowest Raviart-Thomas mixed used to approximate the electric potential/field $(\phi, \boldsymbol{\theta})$ and the linear Lagrange element used to approximate the temperature u, and a nonclassical elliptic map, optimal error estimates were derived in [22]. Meanwhile, a simple one-step recovery technique with one-order Raviart-Thomas mixed finite element space was developed to obtain a new numerical electric potential/field of second-order accuracy.