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# DECENTRALIZED DOUGLAS-RACHFORD SPLITTING METHODS FOR SMOOTH OPTIMIZATION OVER COMPACT SUBMANIFOLDS*

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**Abstract**

We study decentralized smooth optimization problems over compact submanifolds. Recasting it as a composite optimization problem, we propose a decentralized Douglas-Rachford splitting algorithm (DDRS). When the proximal operator of the local loss function does not have a closed-form solution, an inexact version of DDRS (iDDRS), is also presented. Both algorithms rely on careful integration of the nonconvex Douglas-Rachford splitting algorithm with gradient tracking and manifold optimization. We show that our DDRS and iDDRS achieve the convergence rate of $\mathcal{O}(1/k)$. The main challenge in the proof is how to handle the nonconvexity of the manifold constraint. To address this issue, we utilize the concept of proximal smoothness for compact submanifolds. This ensures that the projection onto the submanifold exhibits convexity-like properties, which allows us to control the consensus error across agents. Numerical experiments on the principal component analysis are conducted to demonstrate the effectiveness of our decentralized DRS compared with the state-of-the-art ones.

*Mathematics subject classification:* 65N06, 65B99.
*Key words:* Decentralized optimization, Compact submanifold, Douglas-Rachford splitting, Proximal smoothness, Convergence rate.

## 1. Introduction

Owing to concerns about privacy and robustness, decentralized optimization over manifolds has garnered significant attention in machine learning, optimization control, and signal processing. Examples include principal component analysis [8,36,49], low-rank matrix completion [5,19,29], and low-dimension subspace learning [19,29]. The problem can be mathematically

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formulated as follows:

$$\begin{aligned} \min_{x_1,\cdots,x_n} \quad & \sum_{i=1}^{n} f_i(x_i) \\ \text{s.t.} \quad & x_1 = \cdots = x_n, \quad x_i \in \mathcal{M}, \quad \forall\, i = 1, \dots, n, \end{aligned} \tag{1.1}$$

where $f_i : \mathbb{R}^{d\times r} \to \mathbb{R}$ is a continuously differentiable function held privately by the $i$-th agent, and $\mathcal{M}$ is a compact submanifold of $\mathbb{R}^{d\times r}$, e.g. Stiefel manifold, Oblique manifold [1,4,21].

While numerous algorithms [3,18,26,37,41,42,45,51] have been explored for decentralized optimization with nonconvex objective functions, there are only a few papers dealing with the nonconvex constraint. This is an important issue because there is a frequent interest in optimizing nonconvex functions over nonconvex sets, especially compact submanifolds, see, e.g. [8,19,29,36,49]. Such nonconvex constraint introduces additional challenges in the implementation and analysis of decentralized optimization algorithms. These scenarios often require global solutions for a series of nonconvex constrained optimization problems (e.g. projections to the nonconvex manifold), potentially obstructing the use of conventional tools (e.g. the one-Lipschitz continuity of the projection mapping to the closed convex set) for algorithmic complexity analysis.

The Douglas-Rachford splitting (DRS) is recognized as a famous and efficient splitting algorithm in solving convex and nonconvex optimization problems. The DRS is closely related to the more popular alternating direction method of multipliers (ADMM). The authors in [14] first showed that ADMM is an application of the Douglas-Rachford splitting method (DRSM) to the dual problem when the primal problem is convex. A recent study in [43] shows the primal equivalence between DRS and ADMM in the nonconvex case and demonstrates the convergence of both methods using the Douglas-Rachford envelope. This leads to the following question: Can we design provably convergent decentralized DRS methods for solving (1.1)?

### 1.1. Our contributions

In this paper, we leverage a novel fusion of gradient tracking and DRS, presenting two decentralized DRS algorithms to solve the decentralized manifold optimization problem (1.1).

- **An easy-to-implement paradigm of decentralized DRS.** By utilizing the decentralized communication graph to construct an inexact projection to the consensus set, we develop a decentralized DRS method (DDRS). To mitigate potential consensus distortions caused by the nonconvexity of the manifold constraints, the communication graph needs to be well-connected (which corresponds to a large enough number of communication rounds, i.e. $t$, in Algorithm 3.1). Moreover, for cases where the proximal operator of the loss function lacks a closed-form solution, we present an inexact decentralized DRS method (iDDRS), where the inexactness of evaluating the proximal operator associated with the loss function gradually decreases. Numerical results on eigenvalue problems demonstrate the superior efficacy of our algorithm compared with state-of-the-art methods. DDRS and iDDRS are the first splitting algorithms for solving decentralized manifold optimization problems.

- **Harnessing convex-like properties for best-known convergence complexity.** Compared to algorithms for convex constraints, the main challenge in the convergence analysis of our algorithms arises from the nonconvexity of manifold constraints. To address