Journal of Computational Mathematics
Vol.43, No.6, 2025, 1604–1620.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2407-m2023-0131

SIMPLIFIED EXPLICIT EXPONENTIAL RUNGE-KUTTA METHODS WITHOUT ORDER REDUCTION*

Begoña Cano[1)]
Departmento de Matemática Aplicada, IMUVA, Universidad de Valladolid, Spain
Email: bcano@uva.es
María Jesús Moreta
Departmento de Análisis Económico y Economía Cuantitativa, IMUVA,
Universidad Complutense de Madrid, Spain
Email: mjesusmoreta@ccee.ucm.es

Abstract

In a previous paper, a technique was suggested to avoid order reduction with any explicit exponential Runge-Kutta method when integrating initial boundary value nonlinear problems with time-dependent boundary conditions. In this paper, we significantly simplify the full discretization formulas to be applied under conditions which are nearly always satisfied in practice. Not only a simpler linear combination of φ_j-functions is given for both the stages and the solution, but also the information required on the boundary is so much simplified that, in order to get local order three, it is no longer necessary to resort to numerical differentiation in space. In many cases, even to get local order 4. The technique is then shown to be computationally competitive against other widely used methods with high enough stiff order through the standard method of lines.

Mathematics subject classification: 65M12, 65M20.
Key words: Exponential Runge-Kutta methods, Avoiding order reduction in time, Efficiency.

1. Introduction

Exponential methods have become valuable in the last two decades due to the recent development of Krylov methods to calculate exponential functions of matrices applied over vectors in reasonable computational time [3–5, 7, 12, 13]. Integrating in time reaction-diffusion problems with standard Runge-Kutta (RK) methods is well-known to require an implicit integration so that stability is observed without imposing impractical Courant-Friedrichs-Lewy (CFL) conditions. However, exponential Runge-Kutta methods manage to integrate those problems in an "explicit" way, just at the cost of calculating iteratively exponential functions of matrices over vectors. It is justified in the literature [16] that there are situations where it is cheaper to calculate the latter than solving the linear systems in the stages of implicit standard RK methods. This corresponds to the case where a good preconditioner is not available to solve efficiently those linear systems.

On the other hand, when using the standard method of lines to integrate initial boundary value problems, order reduction in time turns up with both standard and exponential RK methods. For the latter, stiff order conditions are given in [6, 10] which allow to construct

* Received June 20, 2023 / Revised version received February 26, 2024 / Accepted July 8, 2024 / Published online September 24, 2024 /
1) Corresponding author

methods which avoid that order reduction. However, those restrictions may imply that the computational cost is bigger to get a desired accuracy, even that the required number of stages to get a given stiff order of accuracy increases with respect to that necessary for the classical one. In contrast, the technique which is suggested in [1] allows to avoid order reduction with any given explicit exponential RK method (EERK) just by adding in the formulas some information of the problem at the boundary. Since the number of nodes at the boundary is negligible compared to the total number of nodes, the computational cost of adding those terms is also negligible. Furthermore, as it will be seen in the numerical experiments, adding those terms may imply that Krylov subroutines converge more quickly (see [2] for an explanation for that).

However, there are still a couple of items which can be improved in the technique which is suggested in [1].

On the one hand, although analytic expressions are given for the data of the problem (i.e. the boundary conditions and the source term and their derivatives), numerical differentiation in space may be necessary to approximate the required information on the boundary to get local order 3. With Robin/Neumann boundary conditions, even numerical differentiation in time may be required. Furthermore, to get local order 4, numerical differentiation in time and space may be necessary with both Dirichlet and Robin/Neumann boundary conditions. Although using numerical differentiation at the nodes on the boundary is not expensive, it would be better not to use it so as to avoid possible instabilities when the space grid or the time stepsize are very small.

On the other hand, using space numerical differentiation to calculate those boundary values implies that a CFL condition has been necessary in [1] to prove the required order of convergence. Fortunately, this CFL condition is much weaker than that required to prove convergence with an explicit RK method, but it would also be better not having to impose it.

In this paper we will justify that, under some conditions on the coefficients of the method, which most EERK methods satisfy, some simplifications on the required boundary values to avoid order reduction can be performed. Thanks to them, no numerical differentiation in space will be required to achieve local orders 3 and 4 with both Dirichlet and Robin/Neumann boundary conditions, so that the weak CFL condition is no longer required. Although the conditions are a little more restrictive to get local order 4 than to get local order 3, all methods we have seen in the literature satisfy them.

The paper is then structured as follows. The first section gives some preliminaries. The second one justifies the simplification of the full discretization formulas under certain assumptions. The third one proves that some of the simplifying assumptions are almost always satisfied. Finally, some numerical experiments are given with both Dirichlet and Robin/Neumann boundary conditions, which confirm that using the suggested technique is cheaper than using the standard method of lines with other EERK methods which have been especially constructed to get the desired stiff order.

2. Preliminaries

In this paper, we are interested in integrating an initial boundary value problem of this type

$$\begin{aligned} &u'(t) = Au(t) + f\big(t, u(t)\big), \quad 0 \le t \le T, \\ &u(0) = u_0 \in X, \\ &\partial u(t) = g(t) \in Y, \qquad\qquad 0 \le t \le T, \end{aligned} \tag{2.1}$$