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INVERSE NODAL PROBLEM FOR THE STURM-LIOUVILLE OPERATOR WITH THREE DISCONTINUITIES*

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Abstract

In this work, the inverse nodal problem for the Sturm-Liouville operator with three discontinuities is studied. It is proved that the dense nodes of the eigenfunctions can uniquely determine the potential on the whole interval and some parameters, and a reconstruction algorithm for the solution is presented. Finally, numerical examples were provided, and the effectiveness of the algorithm was verified through numerical calculations.

Mathematics subject classification: 34A55, 34B24, 47E05.
Key words: Inverse nodal problems, Discontinuities, Uniqueness theorem, Reconstruction algorithm.

1. Introduction

Consider the following Sturm-Liouville problem:

$$ly := -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0, \pi) \tag{1.1}$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \tag{1.2}$$
$$V(y) := y'(\pi) + Hy(\pi) = 0, \tag{1.3}$$

and the jump conditions ($i = 1, 2, 3$)

$$\begin{cases} y(d_i + 0) = a_i y(d_i - 0), \\ y'(d_i + 0) = a_i^{-1} y'(d_i - 0) + b_i y(d_i - 0). \end{cases} \tag{1.4}$$

Here $\lambda = \rho^2$ is the spectral parameter, $q(x)$ is a real-valued function in $L^2(0, \pi), h, H, a_i, b_i, d_i$ ($i = 1, 2, 3$) are real, and $d_1 = \pi/4, d_2 = \pi/2, d_3 = 3\pi/4$. The problem (1.1)-(1.4), denoted by

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$L = L(q, h, H, a_i, b_i, d_i)$, is called a boundary value problem for the Sturm-Liouville operator with three discontinuities.

McLaughlin [13] was the first scholar who proved the uniqueness for the Sturm-Liouville operator with the Dirichlet boundary condition by using the node points as spectral data. The problems solved in this way are called inverse node problems (see, e.g. [4, 10, 15–17, 21, 22]). That is, the uniqueness of operators can be proved by a dense set of the node points of eigenfunctions. In 1989, Hald and McLaughlin [8] extended the results of McLaughlin to the case in which the boundary conditions are general separated boundary conditions and the conclusion stilled stands. In 1998, Law and Yang [11] solved the inverse nodal problem of reconstructing the potential and the boundary condition using the nodal points together with the integral average of the potential. Guo and Wei [6] considered the Sturm-Liouville problems with general separated boundary conditions and showed that a twin dense subset of the nodal set in interior subinterval uniquely determined the potential and the boundary conditions through two different case. Zhang *et al.* [23] studied the inverse nodal problem with a weight and the jump condition at the middle point. It is shown that the dense nodes of the eigenfunctions can uniquely determine the potential on the whole interval and some parameters.

This results about the inverse nodal problems are generalized to the boundary value problem with discontinuous conditions which is related to discontinuous material characters (see [1–3, 5, 7, 9, 20]). In 2008, Shieh and Yurko [18] discussed the discontinuous Sturm-Liouville operator with the Robin boundary condition at $\pi/2$, proved the uniqueness theorem and provided a constructive procedure.

In this paper, we present the inverse nodal problem for the Sturm-Liouville operator $L = L(q, h, H, a_i, b_i, d_i)$ with three discontinuities and prove that the potential and some parameters can be uniquely determined by a dense nodes of the eigenfunctions (see Fig. 1.1).

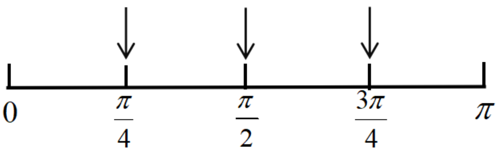

Fig. 1.1. The three discontinuities.

2. Preliminaries

In this section, we mainly present the spectral property of $L = L(q, h, H, a_i, b_i, d_i)$, especially the asymptotic estimation of eigenvalues. Let $\varphi(x, \lambda)$ be the solutions of Eq. (1.1), satisfying the initial conditions $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = h$ and the jump condition (1.4). We can obtain the following lemma.

Lemma 2.1. *The following asymptotic relations hold as* $|\rho| \to \infty$*:*

$$\varphi(x,\lambda) = \cos\rho x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin\rho x}{\rho} + o\left(\frac{e^{|\tau|x}}{\rho}\right), \quad 0 < x < \frac{\pi}{4}, \tag{2.1}$$

$$\begin{aligned}\varphi(x,\lambda) = {} & C_1^+ \cos\rho x + C_1^- \cos\rho\left(\frac{\pi}{2} - x\right) + \xi_1(x)\frac{\sin\rho x}{\rho} \\ & + \xi_2(x)\frac{\sin\rho(\pi/2 - x)}{\rho} + o\left(\frac{e^{|\tau|x}}{\rho}\right), \qquad \frac{\pi}{4} < x < \frac{\pi}{2},\end{aligned} \tag{2.2}$$

$$\begin{aligned}\varphi(x,\lambda) &= C_1^+C_2^+\cos\rho x + C_1^+C_2^-\cos\rho(\pi-x) + a_2C_1^-\cos\rho\left(\frac{\pi}{2}-x\right)\\ &\quad + A_1(x)\frac{\sin\rho x}{\rho} + A_2(x)\frac{\sin\rho(\pi-x)}{\rho}\\ &\quad + A_3(x)\frac{\sin\rho(\pi/2-x)}{\rho} + o\left(\frac{e^{|\tau|x}}{\rho}\right), \quad \frac{\pi}{2}<x<\frac{3\pi}{4}, \end{aligned} \tag{2.3}$$

$$\begin{aligned}\varphi(x,\lambda) &= C_1^+C_2^+C_3^+\cos\rho x + \left(C_1^-a_2C_3^+ + C_1^+C_2^-C_3^-\right)\cos\rho\left(\frac{\pi}{2}-x\right)\\ &\quad + \left(C_1^-a_2C_3^- + C_1^+C_2^-C_3^+\right)\cos\rho(\pi-x) + C_1^+C_2^+C_3^-\cos\rho\left(\frac{3\pi}{2}-x\right)\\ &\quad + B_1(x)\frac{\sin\rho x}{\rho} + B_2(x)\frac{\sin\rho(3\pi/2-x)}{\rho} + B_3(x)\frac{\sin\rho(\pi-x)}{\rho}\\ &\quad + B_4(x)\frac{\sin\rho(\pi/2-x)}{\rho} + o\left(\frac{e^{|\tau|x}}{\rho}\right), \quad \frac{3\pi}{4}<x<\pi, \end{aligned} \tag{2.4}$$

where $C_i^{\pm} = (a_i \pm a_i^{-1})/2, \tau = \mathrm{Im}\rho$ *and*

$$\begin{aligned}
\xi_1(x) &= C_1^+\left(h + \frac{1}{2}\int_0^x q(t)dt\right) + \frac{b_1}{2},\\
\xi_2(x) &= C_1^-\left(h - \frac{1}{2}\int_{\frac{\pi}{4}}^x q(t)dt + \int_0^{\frac{\pi}{4}} q(t)dt\right) - \frac{b_1}{2},\\
A_1(x) &= C_1^+C_2^+h + \frac{b_1C_2^+}{2} + \frac{b_2C_1^+}{2} + \frac{C_1^+C_2^+}{2}\int_0^x q(t)dt,\\
A_2(x) &= -C_1^+C_2^-h - \frac{b_1C_2^-}{2} + \frac{b_2C_1^+}{2} + \frac{C_1^+C_2^-}{2}\int_0^x q(t)dt,\\
A_3(x) &= C_1^-a_2^{-1}h - \frac{a_2^{-1}b_1}{2} - b_2C_1^- - \frac{C_1^-C_2^-}{2}\int_0^{\frac{\pi}{4}} q(t)dt - \frac{C_1^-C_2^+}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} q(t)dt - \frac{a_2C_1^-}{2}\int_{\frac{\pi}{2}}^x q(t)dt,\\
B_1(x) &= C_1^+C_2^+C_3^+h + \frac{b_1C_2^+C_3^+}{2} + \frac{b_2C_1^+C_3^+}{2} + \frac{b_3C_1^+C_2^+}{2} + \frac{C_1^+C_2^+C_3^+}{2}\int_0^x q(t)dt,\\
B_2(x) &= C_1^+C_2^+C_3^-h + \frac{b_1C_2^+C_3^-}{2} + \frac{b_2C_2^+C_3^-}{2} - \frac{b_3C_1^+C_2^+}{2} + \frac{C_1^+C_2^+C_3^-}{2}\int_0^x q(t)dt,\\
B_3(x) &= -C_1^+C_2^-C_3^+h - \frac{b_1C_2^-C_3^+}{2} + \frac{b_2C_1^+C_3^+}{2} - \frac{b_3C_1^+C_2^-}{2} - \frac{b_3a_2C_1^-}{2}\\
&\quad + \frac{C_1^+C_2^-C_3^+}{2}\int_0^x q(t)dt + \frac{C_1^-a_2C_3^-}{2}\int_{\frac{\pi}{2}}^x q(t)dt + \frac{C_1^-a_2^{-1}C_3^-}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} q(t)dt,\\
B_4(x) &= C_1^+C_2^-C_3^-h + \frac{b_1C_2^-C_3^-}{2} - \frac{b_2C_1^+C_3^-}{2} + C_1^-a_2^{-1}C_3^+h - \frac{a_2^{-1}b_1C_3^+}{2}\\
&\quad - b_2C_1^-C_3^+ - \frac{b_3C_1^+C_2^-}{2} - \frac{b_3a_2C_1^-}{2} + \frac{C_1^-C_2^-C_3^-}{2}\int_0^x q(t)dt\\
&\quad + \frac{C_1^-a_2C_3^+}{2}\int_{\frac{\pi}{2}}^x q(t)dt - \frac{C_1^-a_2^{-1}C_3^+}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} q(t)dt - \frac{C_1^+C_2^-C_3^+}{2}\int_0^{\frac{\pi}{4}} q(t)dt.
\end{aligned}$$

Proof. The proofs of (2.1) and (2.2) are similar to [24, Lemma 2.1], so we only prove (2.3). Considering $\varphi(x,\lambda)$ has the following representation on $(\pi/2, 3\pi/4)$:

$$\varphi(x,\lambda) = D_1(\lambda)\cos\rho x + D_2(\lambda)\sin\rho x + \int_0^x \frac{\sin\rho(x-t)}{\rho}q(t)\varphi(t,\lambda)dt. \tag{2.5}$$

Clearly, $\varphi(x,\lambda)$ satisfies the conditions (1.4) at $\pi/2$. Together with the expression (2.2) on $(\pi/4,\pi/2)$, we have

$$\begin{aligned}
D_1(\lambda) &= C_1^+C_2^+ + C_1^+C_2^-\cos\rho\pi + a_2C_1^-\cos\frac{\pi}{2}\rho \\
&+\left(\frac{C_1^+C_2^-h}{\rho}+\frac{b_1C_2^-}{2\rho}-\frac{b_2C_1^+}{2\rho}\right)\sin\rho\pi \\
&+\left(\frac{C_1^-a_2^{-1}h}{\rho}-\frac{a_2^{-1}b_1}{2\rho}\frac{b_2C_1^-}{\rho}\right)\sin\frac{\pi}{2}\rho+O\left(\frac{e^{\frac{\pi}{2}|\tau|}}{\rho}\right), \qquad (2.6)\\
D_2(\lambda) &= C_1^+C_2^-\sin\rho\pi + a_2C_1^-\sin\frac{\pi}{2}\rho+\frac{C_1^+C_2^+h}{\rho}+\frac{b_1C_2^+}{2\rho}+\frac{b_2C_1^+}{2\rho} \\
&+\left(-\frac{C_1^+C_2^-h}{\rho}-\frac{b_1C_2^-h}{2\rho}+\frac{b_2C_1^+h}{2\rho}\right)\cos\rho \\
&-\left(\frac{C_1^-a_2^{-1}h}{\rho}-\frac{a_2^{-1}b_1}{2\rho}-\frac{b_2C_1^-}{\rho}\right)\cos\frac{\pi}{2}\rho+O\left(\frac{e^{\frac{\pi}{2}|\tau|}}{\rho}\right) \qquad (2.7)
\end{aligned}$$

as $|\rho|\to\infty$. Substituting asymptotics (2.6) and (2.7) into (2.5), we arrive at (2.3). $\square$

Denote

$$\Delta(\lambda)=V(\varphi). \qquad (2.8)$$

The function $\Delta(\lambda)$ is called the characteristic function of L, which is entire in λ, and it has the zeros $\{\lambda_n\}_{n\geq 0}$. From (1.3) and (2.4), we get that as $|\rho|\to\infty$,

$$\begin{aligned}
\Delta(\lambda)&=\rho\left[C_1^+C_2^+C_3^+\sin\rho\pi+\left(C_1^-a_2C_3^+-C_1^+a_2^{-1}C_3^-\right)\sin\frac{\pi}{2}\rho\right]\\
&+\omega_1-\omega_2\cos\rho\pi+\omega_3\cos\frac{\pi}{2}\rho+o\big(\exp(\pi|\tau|)\big), \qquad (2.9)
\end{aligned}$$

where

$$\begin{aligned}
\omega_1 &= -B_3(\pi)+H\left(a_2C_1^-C_3^-+C_1^+C_2^-C_3^+\right), \qquad (2.10)\\
\omega_2 &= B_1(\pi)+HC_1^+C_2^+C_3^+, \qquad (2.11)\\
\omega_3 &= -B_2(\pi)-B_4(\pi)+Ha_2\left(C_1^+C_3^-+C_1^-C_3^+\right). \qquad (2.12)
\end{aligned}$$

Lemma 2.2. *The boundary value problem (BVP) L has real eigenvalues*

$$\{\lambda_n\}_{n\geq0}:=\{\lambda_{2n}\}_{n\geq0}\cup\{\lambda_{2n+1}\}_{n\geq0}.$$

For $n\to+\infty$, there holds

$$\rho_{2n}=\rho_{2n}^0+\frac{\theta_{2n}}{\rho_{2n}^0}+\frac{\kappa_{2n}}{\rho_{2n}^0}, \qquad (2.13)$$

$$\rho_{2n+1}=\rho_{2n+1}^0+\frac{\theta_{2n+1}}{\rho_{2n+1}^0}+\frac{\kappa_{2n+1}}{\rho_{2n+1}^0}, \qquad (2.14)$$

where $\lambda_{2n}=\rho_{2n}^2,\lambda_{2n+1}=\rho_{2n+1}^2$, and $\{\kappa_{2n}\},\{\kappa_{2n+1}\}\in l^2$. And $\lambda_{2n}^0=(\rho_{2n}^0)^2$, $\lambda_{2n+1}^0=(\rho_{2n+1}^0)^2$ are zeros of the function

$$\Delta_0(\lambda):=\rho\Big[A\sin(\rho\pi)+B\sin\left(\frac{\pi}{2}\rho\right)\Big]. \qquad (2.15)$$

Here,

$$
\begin{aligned}
&A := C_1^+ C_2^+ C_3^+, \quad B := C_1^- a_2 C_3^+ - C_1^+ a_2^{-1} C_3^-, \\
&\theta_{2n} = \frac{-\omega_1 + \omega_2 \cos\left(\rho_{2n}^0 \pi\right) - \omega_3 \cos\left(\pi \rho_{2n}^0/2\right)}{2\dot{\Delta}_0\left(\lambda_{2n}^0\right)} = \frac{-\omega_1 + \omega_2 - (-1)^n \omega_3}{A\pi + (-1)^n B\pi/2}, \\
&\theta_{2n+1} = \frac{-\omega_1 + \omega_2 \cos\left(\rho_{2n+1}^0 \pi\right) - \omega_3 \cos\left(\pi \rho_{2n+1}^0/2\right)}{2\dot{\Delta}_0\left(\lambda_{2n+1}^0\right)},
\end{aligned} \tag{2.16}
$$

and

$$
\dot{\Delta}_0(\lambda_n^0) = \left(\frac{d}{d\lambda}\Delta_0(\lambda)\right)\Bigg|_{\lambda=\lambda_n^0} = \frac{1}{2}\left[A\pi\cos\left(\rho_n^0\pi\right) + \frac{B\pi}{2}\cos\left(\frac{\pi}{2}\rho_n^0\right)\right].
$$

Proof. Take the proof of (2.13) for example. The expression of ρ_{2n+1} could be obtained by following the similar procedure, here the proof is list as follows.

From (2.15), we can calculate that the zeros are

$$
\rho_{2n}^0 = 2n, \quad \rho_{2n+1}^0 = \frac{2}{\pi}\arccos\left(-\frac{B}{2A}\right) + 4n.
$$

According to the Rouché's theorem, it follows from (2.10) and (2.15) that

$$
\rho_{2n} = \rho_{2n}^0 + \delta_n = 2n + \delta_n, \quad n \ \to \ +\infty, \tag{2.17}
$$

where $\delta_n = o(1)$. Combing $\Delta(\lambda_{2n}) = 0$ and (2.17), it yields

$$
A\sin(\rho_{2n}\pi) + B\sin\left(\rho_{2n}\frac{\pi}{2}\right) = \mathcal{O}\left(\frac{1}{n}\right). \tag{2.18}
$$

Considering

$$
\Delta_0\left(\lambda_{2n}^0\right) = \rho[A\sin(2n\pi) + B\sin(n\pi)] = 0, \tag{2.19}
$$

and substituting ρ_{2n} into (2.18), we obtain

$$
\delta_n\left(A\pi\cos(2n\pi) + \frac{B\pi}{2}\cos(n\pi)\right) = \delta_n\left(A\pi + (-1)^n\frac{B\pi}{2}\right) = \mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}(\delta_n^2).
$$

Together (2.18) and (2.19), we get

$$
\delta_n\left(A\pi + (-1)^n\frac{B\pi}{2}\right) = \delta_n\dot{\Delta}_0\left(\lambda_{2n}^0\right) = \mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}(\delta_n^2). \tag{2.20}
$$

Then

$$
\delta_n = \mathcal{O}\left(\frac{1}{n}\right). \tag{2.21}
$$

Taking $\Delta(\lambda_n) = 0$ into account and using (2.21), we have

$$
\delta_n = \frac{\theta_{2n}}{\rho_{2n}^0} + \frac{\kappa_{2n}}{2n} = \frac{-\omega_1 + \omega_2 - (-1)^n\omega_3}{2n(A\pi + (-1)^n B\pi/2)} + \frac{\kappa_{2n}}{2n}.
$$

This completes the proof. □

3. Main Results

Together with the problem $L = L(q, h, H, a_i, b_i, d_i)$, we consider a boundary value problem $\tilde{L} = L(\tilde{q}, \tilde{h}, \tilde{H}, \tilde{a}_i, \tilde{b}_i, \tilde{d}_i)$ of the same form but with the different coefficients $\tilde{q}(x), \tilde{h}, \tilde{H}, \tilde{a}_i, \tilde{b}_i, \tilde{d}_i$. We agree that if a certain symbol υ denotes an object related to L, then $\tilde{\upsilon}$ denotes the analogous object related to $\tilde{L}$. Denote $\varphi(x, \lambda_{2n}) = \varphi_{2n}(x)$ and $\varphi(x, \lambda_{2n+1}) = \varphi_{2n+1}(x)$ are real-valued functions. For brevity, we only consider the eigenvalue $\{\lambda_{2n}\}_{n\geq 0}$ and present the following corresponding lemmas about $\varphi_{2n}(x)$. Assume that

$$\begin{aligned}
&\Gamma_{1n} := C_1^+ + (-1)^n C_1^- \neq 0,\\
&\Gamma_{2n} := C_1^+ C_2^+ + C_1^+ C_2^- + (-1)^n a_2 C_1^- \neq 0,\\
&\Gamma_{3n} := C_1^+ a_2 C_3^+ + C_1^- a_2 C_3^- + (-1)^n \big(C_1^- a_2 C_3^- + C_1^+ C_2^- C_3^+ + C_1^+ C_2^+ C_3^-\big) \neq 0.
\end{aligned}$$

Define

$$\begin{aligned}
&\Gamma_{10} := \Gamma_{1n} = C_1^+ + C_1^- = a_1,\\
&\Gamma_{20} := \Gamma_{2n} = C_1^+ C_2^+ + C_1^+ C_2^- + a_2 C_1^- = a_1 a_2,\\
&\Gamma_{30} := \Gamma_{3n} = C_1^+ a_2 C_3^+ + 2C_1^- a_2 C_3^- + C_1^+ C_2^- C_3^+ + C_1^+ C_2^+ C_3^-,
\end{aligned}$$

when n is even.

Lemma 3.1. *The following asymptotic formulaes hold as $n \to +\infty$:*

$$\varphi_{2n}(x) = \cos(2nx) + \frac{1}{4n}\left(2h - 2\theta_{2n}x + \int_0^x q(t)dt\right)\sin(2nx) + o\left(\frac{1}{n}\right), \quad 0 < x < \frac{\pi}{4}, \tag{3.1}$$

$$\begin{aligned}
\varphi_{2n}(x) = {}& \Gamma_{1n}\cos(2nx) + \frac{1}{2n}\left[\xi_1(x) - (-1)^n\xi_2(x) - \theta_{2n}\left(C_1^+ x - (-1)^n C_1^-\left(\frac{\pi}{2} - x\right)\right)\right]\\
&\times \sin(2nx) + o\left(\frac{1}{n}\right), \quad \frac{\pi}{4} < x < \frac{\pi}{2},
\end{aligned}\tag{3.2}$$

$$\begin{aligned}
\varphi_{2n}(x) = {}& \Gamma_{2n}\cos(2nx) + \frac{1}{2n}\Bigg[A_1(x) - A_2(x) - (-1)^n A_3(x)\\
&\quad - \theta_{2n}\left(C_1^+ C_2^+ x - C_1^+ C_2^-(\pi - x) - (-1)^n a_2 C_1^-\left(\frac{\pi}{2} - x\right)\right)\Bigg]\\
&\times \sin(2nx) + o\left(\frac{1}{n}\right), \quad \frac{\pi}{2} < x < \frac{3\pi}{4},
\end{aligned}\tag{3.3}$$

$$\begin{aligned}
\varphi_{2n}(x) = {}& \Gamma_{3n}\cos(2nx) + \frac{1}{2n}\Bigg[B_1(x) - (-1)^n B_2(x) - B_3(x) - (-1)^n B_4(x)\\
&\quad - \theta_{2n}\Bigg(C_1^+ C_2^+ C_3^+ x - (-1)^n\big(C_1^- a_2 C_3^- + C_1^+ C_2^- C_3^+\big)\left(\frac{\pi}{2} - x\right)\\
&\quad - \big(C_1^- a_2 C_3^- + C_1^+ C_2^- C_3^+\big)(\pi - x)\\
&\quad - (-1)^n C_1^+ C_2^+ C_3^-\left(\frac{3\pi}{2} - x\right)\Bigg)\Bigg]\\
&\times \sin(2nx) + o\left(\frac{1}{n}\right), \quad \frac{3\pi}{4} < x < \pi.
\end{aligned}\tag{3.4}$$

Here $\theta_{2n}, \xi_1(x), \xi_2(x), A_1(x), A_2(x), A_3(x), B_1(x), B_2(x), B_3(x), B_4(x)$ are given in Lemma 2.1.

Proof. Take the proof of the representation (3.3) for $\varphi_{2n}(x)$ on $(\pi/2, 3\pi/4)$ for example. Putting the expression of ρ_{2n} into (2.3), it yields $n \to +\infty$,

$$\begin{aligned}
\varphi_{2n}(x) &= C_1^+C_2^+\cos[(2n+\delta_n)x] + C_1^+C_2^-\cos[(2n+\delta_n)(\pi-x)] \\
&\quad + a_2C_1^-\cos\left[(2n+\delta_n)\left(\frac{\pi}{2}-x\right)\right] + A_1(x)\frac{\sin[(2n+\delta_n)x]}{2n} \\
&\quad + A_2(x)\frac{\sin[(2n+\delta_n)(\pi-x)]}{2n} + A_3(x)\frac{\sin[(2n+\delta_n)(\pi/2-x)]}{2n} + o\left(\frac{1}{n}\right) \\
&= C_1^+C_2^+\cos(2nx) - C_1^+C_2^+\frac{\theta_{2n}x}{2n}\sin(2nx) + C_1^+C_2^-\cos(2nx) \\
&\quad + C_1^+C_2^-\frac{\theta_{2n}}{2n}(\pi-x)\sin(2nx) + (-1)^n a_2C_1^-\cos(2nx) \\
&\quad + (-1)^n a_2C_1^-\frac{\theta_{2n}}{2n}\left(\frac{\pi}{2}-x\right)\sin(2nx) + \frac{A}{2n}\sin(2nx) - \frac{B}{2n}\sin(2nx) \\
&\quad - (-1)^n\frac{C}{2n})\sin(2nx) + o\left(\frac{1}{n}\right) \\
&= \Gamma_{2n}\cos(2nx) + \left[-C_1^+C_2^+\frac{\theta_{2n}}{2n}x + C_1^+C_2^-\frac{\theta_{2n}}{2n}(\pi-x) + (-1)^n a_2C_1^-\frac{\theta_{2n}}{2n}\left(\frac{\pi}{2}-x\right)\right. \\
&\qquad\qquad \left. + \frac{A}{2n} - \frac{B}{2n} - (-1)^n\frac{C}{2n}\right]\sin(2nx) + o\left(\frac{1}{n}\right).
\end{aligned}$$

Simplifying the above formula, we get the expression of $\varphi_{2n}(x)$ on $(\pi/2, 3\pi/4)$. The same method can be used to obtain representations of $\varphi_{2n}(x)$ on other intervals. $\square$

Lemma 3.2. *For sufficiently large n, the eigenfunction $\varphi_{2n}(x)$ exactly $2n$ nodes*

$$\{x_{2n}^s : s = \overline{1, 2n}\}$$

in the interval $(0, \pi)$, which are simple.

Proof. First, consider the function

$$F(x) := \Gamma_{2n}\cos(2nx).$$

By calculating, we can get that the zeros of $F(x)$ are

$$\tau_{2n}^s := \frac{(s-1/2)\pi}{2n}.$$

It follows from the representation of $\varphi_{2n}(x)$ on $(0, \pi)$ that the zeros of $\varphi_{2n}(x)$ are

$$x_{2n}^s = \tau_{2n}^s + \epsilon_n = \frac{(s-1/2)\pi}{2n} + \epsilon_n, \quad 0 < x_{2n}^s < \pi, \tag{3.5}$$

where $\epsilon_n = o(1)$. Since $0 < x_{2n}^s < \pi/4$, together with (3.5), we can get $s = \overline{1, [n/2]}$; similarly, when $\pi/4 < x_{n1}^s < \pi/2$, it yields $s = \overline{[n/2]+1, n}$; when $\pi/2 < x_{n1}^s < 3\pi/4$, it yields

$s = \overline{n+1, [3n/2]}$; when $3\pi/4 < x_{n1}^s < \pi$, it yields $s = \overline{[3n/2]+1, 2n}$. That is,

$$s = \begin{cases} \overline{1, \left[\frac{n}{2}\right]}, & 0 < x_{2n}^s < \frac{\pi}{4}, \\ \overline{\left[\frac{n}{2}\right]+1, n}, & \frac{\pi}{4} < x_{2n}^s < \frac{\pi}{2}, \\ \overline{n+1, \left[\frac{3n}{2}\right]}, & \frac{\pi}{2} < x_{2n}^s < \frac{3\pi}{4}, \\ \overline{\left[\frac{3n}{2}\right]+1, 2n}, & \frac{3\pi}{4} < x_{2n}^s < \pi. \end{cases}$$

The proof is complete. □

Next, denote by $X_L := \{x_{2n}^s\}_{n\geq 1, s=\overline{1,2n}}$ the set of the nodal points of the problem L and

$$\tau_{2n}^s := \frac{(s-1/2)\pi}{2n},$$

we can obtain the following estimations about nodes.

Lemma 3.3. *For sufficiently large n, the following asymptotic formulas for nodes hold:*

$$\begin{aligned} x_{2n}^s = \tau_{2n}^s + \frac{1}{8n^2}\left(2h - 2\theta_{2n}\tau_{2n}^s + \int_0^{\tau_{2n}^s} q(t)dt\right) \\ + o\left(\frac{1}{n^2}\right), \quad 0 < x < \frac{\pi}{4}, \quad s = \overline{1, \left[\frac{n}{2}\right]}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} x_{2n}^s = \tau_{2n}^s + \frac{1}{4n^2}\frac{1}{\Gamma_{1n}}\left[\xi_1(\tau_{2n}^s) - (-1)^n\xi_2(\tau_{2n}^s) - \theta_{2n}\left(\Gamma_{1n}\tau_{2n}^s - (-1)^n C_1^- \frac{\pi}{2}\right)\right] \\ + o\left(\frac{1}{n^2}\right), \quad \frac{\pi}{4} < x < \frac{\pi}{2}, \quad s = \overline{\left[\frac{n}{2}\right]+1, n}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} x_{2n}^s = \tau_{2n}^s + \frac{1}{4n^2}\frac{1}{\Gamma_{2n}}\Big[A_1(\tau_{2n}^s) - A_2(\tau_{2n}^s) - (-1)^n A_3(\tau_{2n}^s) \\ - \theta_{2n}\left(\Gamma_{2n}\tau_{2n}^s - C_1^+C_2^-\pi - (-1)^n a_2 C_1^- \frac{\pi}{2}\right)\Big] \\ + o\left(\frac{1}{n^2}\right), \quad \frac{\pi}{2} < x < \frac{3\pi}{4}, \quad s = \overline{n+1, \left[\frac{3n}{2}\right]}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} x_{2n}^s = \tau_{2n}^s + \frac{1}{4n^2}\frac{1}{\Gamma_{3n}}\Big[B_1(\tau_{2n}^s) - (-1)^n B_2(\tau_{2n}^s) - B_3(\tau_{2n}^s) - (-1)^n B_4(\tau_{2n}^s) \\ - \theta_{2n}\Big(\Gamma_{3n}\tau_{2n}^s - (-1)^n\left(C_1^- a_2 C_3^- + C_1^+ C_2^- C_3^+\right)\frac{\pi}{2} \\ - \left(C_1^- a_2 C_3^- + C_1^+ C_2^- C_3^+\right)\pi - (-1)^n C_1^+ C_2^+ C_3^- \frac{3\pi}{2}\Big)\Big] \\ + o\left(\frac{1}{n^2}\right), \quad \frac{3\pi}{4} < x < \pi, \quad s = \overline{\left[\frac{3n}{2}\right]+1, 2n}. \end{aligned} \tag{3.9}$$

Proof. Here we only prove (3.8), since the relations (3.6)-(3.9) can be calculated by the similar methods.

Substituting (3.5) into (3.3), we have $n\to+\infty$

$$
\begin{aligned}
0 &= \Gamma_{1n}\cos\left[2n\left(\tau^s_{2n}+\epsilon_n\right)\right]\\
&\quad+\frac{1}{2n}\left[\xi_1\left(\tau^s_{2n}\right)-(-1)^n\xi_2\left(\tau^s_{2n}\right)-\theta_{2n}\left(C_1^+\tau^s_{2n}-(-1)^nC_1^-\left(\frac{\pi}{2}-\tau^s_{2n}\right)\right)\right]\\
&\quad\times\sin\left[2n\left(\tau^s_{2n}+\epsilon_n\right)\right]+o\left(\frac{1}{n}\right)\\
&= -2n\epsilon_n\Gamma_{1n}\sin\left(2n\tau^s_{2n}\right)\\
&\quad+\frac{1}{2n}\left[\xi_1\left(\tau^s_{2n}\right)-(-1)^n\xi_2\left(\tau^s_{2n}\right)-\theta_{2n}\left(C_1^+\tau^s_{2n}-(-1)^nC_1^-\left(\frac{\pi}{2}-\tau^s_{2n}\right)\right)\right]\\
&\quad\times\sin\left(2n\tau^s_{2n}\right)+o\left(\frac{1}{n^2}\right).
\end{aligned}
$$

That is, when $n\to+\infty$,

$$
\begin{aligned}
\epsilon_n\Gamma_{1n} = \frac{1}{4n^2}\Bigg[&\xi_1\left(\tau^s_{2n}\right)-(-1)^n\xi_2\left(\tau^s_{2n}\right)\\
&-\theta_{2n}\left(C_1^+\tau^s_{2n}-(-1)^nC_1^-\left(\frac{\pi}{2}-\tau^s_{2n}\right)\right)\Bigg]+o\left(\frac{1}{n^2}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\epsilon_n &= \frac{1}{\Gamma_{1n}}\frac{1}{4n^2}\Bigg[\xi_1\left(\tau^s_{2n}\right)-(-1)^n\xi_2\left(\tau^s_{2n}\right)\\
&\qquad-\theta_{2n}\left(C_1^+\tau^s_{2n}-(-1)^nC_1^-\left(\frac{\pi}{2}-\tau^s_{2n}\right)\right)\Bigg]+o\left(\frac{1}{n^2}\right)\\
&= \frac{1}{\Gamma_{1n}}\frac{1}{4n^2}\Bigg[\xi_1\left(\tau^s_{2n}\right)-(-1)^n\xi_2\left(\tau^s_{2n}\right)\\
&\qquad-\theta_{2n}\left(\Gamma_{1n}\tau^s_{2n}-(-1)^nC_1^-\frac{\pi}{2}\right)\Bigg]+o\left(\frac{1}{n^2}\right),\quad n\to+\infty.
\end{aligned}
$$

We can get (3.6). The proof is complete. □

Note that X_L is dense on $(0,\pi)$. Give the following auxiliary theorem.

Theorem 3.1. *Fix $x\in(0,\pi)$. There exists a subsequences $\{x^{s_n}_{2n}\}\in X_L$ such that $\lim_{j\to\infty}x^{s_n}_{2n}=x$. Then the following equations exist:*

$$
\begin{aligned}
f_1(x) &:= \lim_{n\to\infty}\frac{2}{\Gamma_{10}}\Bigg\{C_1^+h+b_1-C_1^-h-C_1^-\int_0^{\frac{\pi}{4}}q(t)dt+\frac{C_1^+}{2}\int_0^{\frac{\pi}{4}}q(t)dt\\
&\qquad-\theta_0\left[C_1^+\tau^{s_n}_{2n}-C_1^-\left(\frac{\pi}{2}-\tau^{s_n}_{2n}\right)\right]\Bigg\}\\
&= \frac{2}{\Gamma_{10}}\left\{a_1^{-1}h+b_1-C_1^-\int_0^{\frac{\pi}{4}}q(t)dt+\frac{C_1^+}{2}\int_0^{\frac{\pi}{4}}q(t)dt-\theta_0\left[\Gamma_{10}x-C_1^-\frac{\pi}{2}\right]\right\},
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
f_2(x) &:= \lim_{n\to\infty}\frac{2}{\Gamma_{20}}\Bigg\{C_1^+C_2^+h+\frac{b_1C_2^+}{2}+\frac{b_2C_1^+}{2}+\frac{C_1^+C_2^+}{2}\int_0^{\frac{\pi}{2}}q(t)dt\\
&\qquad+C_1^+C_2^-h+\frac{b_1C_2^-}{2}-\frac{b_2C_1^+}{2}-\frac{C_1^+C_2^-}{2}\int_0^{\frac{\pi}{2}}q(t)dt-C_1^-a_2^{-1}h
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{a_2^{-1}b_1}{2}+b_2C_1^-+\frac{C_1^-C_2^-}{2}\int_0^{\frac{\pi}{4}}q(t)dt+\frac{C_1^-C_2^+}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}q(t)dt\\
&-\theta_0\left[C_1^+C_2^+\tau_{2n}^{s_n}-C_1^+C_2^-\left(\pi-\tau_{2n}^{s_n}\right)-a_2C_1^-\left(\frac{\pi}{2}-\tau_{2n}^{s_n}\right)\right]\Bigg\}\\
=&\frac{2}{\Gamma_{20}}\Bigg\{C_1^+a_2h+\frac{b_1a_2}{2}+\frac{C_1^+a_2^{-1}}{2}\int_0^{\frac{\pi}{2}}q(t)dt-C_1^-a_2^{-1}h\\
&+b_2C_1^-+\frac{a_2^{-1}b_1}{2}+\frac{C_1^-C_2^-}{2}\int_0^{\frac{\pi}{4}}q(t)dt+\frac{C_1^-C_2^+}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}q(t)\\
&-\theta_0\left[\Gamma_{20}x-C_1^+C_2^-\pi-a_2C_1^-\frac{\pi}{2}\right]\Bigg\},
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
f_3(x):=\lim_{n\to\infty}\frac{2}{\Gamma_{30}}\Bigg\{&Ah+\frac{b_1C_2^+C_3^+}{2}+\frac{b_2C_1^+C_3^+}{2}+\frac{b_3C_1^+C_2^+}{2}-\frac{b_1C_2^+C_3^-}{2}\\
&+\frac{C_1^+C_2^+C_3^+}{2}\int_0^{\frac{3\pi}{4}}q(t)dt-C_1^+C_2^+C_3^-h-\frac{b_2C_2^+C_3^-}{2}+\frac{b_3C_1^+C_2^+}{2}\\
&-\frac{C_1^+C_2^+C_3^-}{2}\int_0^{\frac{3\pi}{4}}q(t)dt+Ah+\frac{b_1C_2^-C_3^+}{2}-\frac{b_2C_1^+C_3^+}{2}+\frac{b_3C_1^+C_2^-}{2}\\
&-\frac{C_1^+C_2^-C_3^+}{2}\int_0^{\frac{3\pi}{4}}q(t)dt-\frac{C_1^-a_2C_3^-}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}q(t)dt-\frac{C_1^-a_2^{-1}C_3^-}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}q(t)dt\\
&-C_1^+C_2^-C_3^-h-\frac{b_1C_2^-C_3^-}{2}+\frac{b_2C_1^+C_3^-}{2}-C_1^-a_2^{-1}C_3^+h+\frac{a_2^{-1}b_1C_3^+}{2}\\
&+b_2C_1^-C_3^++\frac{b_3C_1^+C_2^-}{2}+\frac{b_3a_2C_1^-}{2}+\frac{b_3a_2C_1^-}{2}-\frac{C_1^-C_2^-C_3^-}{2}\int_0^{\frac{3\pi}{4}}q(t)dt\\
&-\frac{C_1^-a_2C_3^+}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}q(t)dt+\frac{C_1^-a_2^{-1}C_3^+}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}q(t)dt+\frac{C_1^+C_2^-C_3^+}{2}\int_0^{\frac{\pi}{4}}q(t)dt\\
&-\theta_0\Bigg[C_1^+C_2^+C_3^+\tau_{2n}^{s_n}-\left(C_1^-a_2C_3^-+C_1^+C_2^-C_3^+\right)\left(\frac{\pi}{2}-\tau_{2n}^{s_n}\right)\\
&\qquad-\left(C_1^-a_2C_3^-+C_1^+C_2^-C_3^+\right)\left(\pi-\tau_{2n}^{s_n}\right)-C_1^+C_2^+C_3^-\left(\frac{3\pi}{2}-\tau_{2n}^{s_n}\right)\Bigg]\Bigg\}\\
=\frac{2}{\Gamma_{30}}\Bigg\{&2C_1^+C_2^+C_3^+h-C_1^+a_2C_3^-h-C_1^-a_2^{-1}C_3^+h+\frac{b_1a_2a_3^{-1}}{2}\\
&+\frac{b_1a_2^{-1}C_3^+}{2}+b_2C_1^+C_3^-+b_2C_1^-C_3^++a_1a_2b_3\\
&+\frac{C_1^+C_2^-C_3^+}{2}\int_0^{\frac{\pi}{4}}q(t)dt+\frac{C_1^-a_2^{-1}a_3^{-1}}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}q(t)dt-\frac{C_1^-a_2a_3}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}q(t)dt\\
&+\left(\frac{C_1^+C_2^+a_3^{-1}}{2}-\frac{C_1^+C_2^-C_3^+}{2}-\frac{C_1^-C_2^-C_3^-}{2}\right)\int_0^{\frac{3\pi}{4}}q(t)dt\\
&-\theta_0\Bigg[\Gamma_{30}x-\left(C_1^-a_2C_3^-+C_1^+C_2^-C_3^+\right)\frac{\pi}{2}\\
&\qquad-\left(C_1^-a_2C_3^-+C_1^+C_2^-C_3^+\right)\pi-C_1^+C_2^+C_3^-\frac{3\pi}{2}\Bigg]\Bigg\},
\end{aligned}
\tag{3.12}
$$

and

$$A(x) := \lim_{n\to\infty} 8n^2\big(x_{2n}^{s_n} - \tau_{2n}^{s_n}\big), \qquad 0 < x < \frac{\pi}{4}, \tag{3.13}$$

$$B(x) := \lim_{n\to\infty} 8n^2\big(x_{2n}^{s_n} - \tau_{2n}^{s_n}\big), \qquad \frac{\pi}{4} < x < \frac{\pi}{2}, \tag{3.14}$$

$$C(x) := \lim_{n\to\infty} \frac{8n^2 a_1 a_2\big(x_{2n}^{s_n} - \tau_{2n}^{s_n}\big)}{C_1^+ a_2^{-1} + C_1^- a_2}, \qquad \frac{\pi}{2} < x < \frac{3\pi}{4}, \tag{3.15}$$

$$D(x) := \lim_{n\to\infty} \frac{8n^2 a_1 a_2 a_3\big(x_{2n}^{s_n} - \tau_{2n}^{s_n}\big)}{C_1^+ C_2^+ a_3^{-1} - C_1^+ C_2^- a_3 - C_1^- a_2 a_3}, \qquad \frac{3\pi}{4} < x < \pi. \tag{3.16}$$

Moreover,

$$A(x) := \int_0^x q(t)dt + 2h - 2\theta_0 x, \quad 0 < x < \frac{\pi}{4}, \tag{3.17}$$

$$B(x) := \int_{\frac{\pi}{4}}^x q(t)dt + f_1(x), \qquad \frac{\pi}{4} < x < \frac{\pi}{2}, \tag{3.18}$$

$$C(x) := \int_{\frac{\pi}{2}}^x q(t)dt + f_2(x), \qquad \frac{\pi}{2} < x < \frac{3\pi}{4}, \tag{3.19}$$

$$D(x) := \int_{\frac{3\pi}{4}}^x q(t)dt + f_3(x), \qquad \frac{3\pi}{4} < x < \pi, \tag{3.20}$$

where

$$\theta_0 := \theta_{2n} = \frac{-\omega_1 + \omega_2 - \omega_3}{A\pi + B\pi/2}$$

is bounded.

Proof. Take the proofs of (3.11) and (3.14), for example. It follows from (3.8) that

$$\begin{aligned} 4n^2 a_1\big(x_{2n}^s - \tau_{2n}^s\big) = \frac{a_1}{2}\int_{\frac{\pi}{4}}^{\tau_{2n}^s} q(t)dt + C_1^+ h + b_1 - C_1^- h - C_1^- \int_0^{\frac{\pi}{4}} q(t)dt \\ + \frac{C_1^+}{2}\int_0^{\frac{\pi}{4}} q(t)dt - \theta_0\left[\Gamma_{10}\tau_{2n}^{s_n} - C_1^- \frac{\pi}{2}\right]. \end{aligned} \tag{3.21}$$

Take the limit on both sides of (3.21), then

$$\begin{aligned} B(x) &:= \lim_{n\to\infty} 8n^2\big(x_{2n}^{s_n} - \tau_{2n}^{s_n}\big) \\ &= \lim_{n\to\infty}\Bigg\{\int_{\frac{\pi}{4}}^{\tau_{2n}^s} q(t)dt + \frac{a_1}{2}\Big\{C_1^+ h + b_1 - C_1^- h - C_1^- \int_0^{\frac{\pi}{4}} q(t)dt \\ &\qquad + \frac{C_1^+}{2}\int_0^{\frac{\pi}{4}} q(t)dt - \theta_0\Big[C_1^+\tau_{2n}^{s_n} - C_1^-\left(\frac{\pi}{2} - \tau_{2n}^{s_n}\right)\Big]\Big\}\Bigg\} \\ &= \int_{\frac{\pi}{4}}^x q(t)dt + \lim_{n\to\infty}\Bigg\{\frac{a_1}{2}\Big\{C_1^+ h + b_1 - C_1^- h - C_1^- \int_0^{\frac{\pi}{4}} q(t)dt \\ &\qquad + \frac{C_1^+}{2}\int_0^{\frac{\pi}{4}} q(t)dt - \theta_0\Big[C_1^+\tau_{2n}^s - C_1^-\left(\frac{\pi}{2} - \tau_{2n}^s\right)\Big]\Big\}\Bigg\} \\ &= \int_{\frac{\pi}{4}}^x q(t)dt + \Bigg\{\frac{a_1}{2}\Big\{C_1^+ h + b_1 - C_1^- h - C_1^- \int_0^{\frac{\pi}{4}} q(t)dt \end{aligned}$$

$$+\frac{C_1^+}{2}\int_0^{\frac{\pi}{4}} q(t)dt-\theta_0\Big[C_1^+x-C_1^-\left(\frac{\pi}{2}-x\right)\Big]\Big\}\Big\}$$

$$=\int_{\frac{\pi}{4}}^{x} q(t)dt+f_1(x).$$

Similarly, (3.10), (3.12) and (3.17)-(3.20) can be obtained by the above method. □

Finally, we could formulate the uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem. Denote by X a subset of dense nodes on $(0,\pi)$ and $X\subset X_L$. Assume the a_i $(i=\overline{1,3})$ are given a priori in the next statement.

Theorem 3.2. *If $X=\tilde{X}$, then $q(x)=\tilde{q}(x)$ a.e. on $(0,\pi)$, $h=\tilde{h}$, $H=\tilde{H}$, and $b_i=\tilde{b_i}$ $(i=\overline{1,3})$. The function $q(x)$ and the parameters h, H, b_i can be constructed by the following formulae:*

$$(i)\qquad h=\frac{A(0)}{2}, \tag{3.22}$$

$$\begin{aligned}(ii)\qquad H=&\frac{1}{C_1^-a_2a_3-C_1^+a_2^{-1}C_3^++C_1^+a_2C_3^-}\\ &\times\left[B_1(\pi)+B_2(\pi)+B_3(\pi)+B_4(\pi)-\theta_0\left(A\pi+\frac{B\pi}{2}\right)\right],\\ &\big(when\quad C_1^-a_2a_3-C_1^+a_2^{-1}C_3^++C_1^+a_2C_3^-\neq 0\big),\end{aligned} \tag{3.23}$$

$$(iii)\qquad b_1=\frac{f_1(0)a_1}{2}-a_1^{-1}h+\left(C_1^--\frac{C_1^+}{2}\right)\int_0^{\frac{\pi}{4}} q(t)dt-\frac{\pi}{2}C_1^-\theta_0, \tag{3.24}$$

$$\begin{aligned}(iv)\qquad b_2=\frac{1}{C_1^-}\Bigg[&\frac{f_2(0)\big(C_1^+a_2^{-1}+C_1^-a_2\big)}{2}-C_1^+a_2h-\frac{b_1a_2}{2}+\frac{C_1^+a_2^{-1}}{2}\int_0^{\frac{\pi}{2}} q(t)dt\\ &+C_1^-a_2^{-1}h-\frac{a_2^{-1}b_1}{2}-\frac{C_1^-C_2^-}{2}\int_0^{\frac{\pi}{4}} q(t)dt-\frac{C_1^-C_2^+}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} q(t)\\ &+\theta_0\left(-\pi C_1^+C_2^--\frac{\pi}{2}a_2C_1^-\right)\Bigg],\end{aligned} \tag{3.25}$$

$$\begin{aligned}(v)\qquad b_3=\frac{1}{a_1a_2}\Bigg[&\frac{f_3(0)\big(C_1^+C_2^+a_3^{-1}-C_1^+C_2^-a_3-C_1^-a_2a_3\big)}{2}-2C_1^+C_2^+C_3^+h\\ &-2C_1^+C_2^+C_3^+h+C_1^+a_2C_3^-h+C_1^-a_2^{-1}C_3^+h-\frac{b_1a_2a_3^{-1}}{2}\\ &-\frac{b_1a_2^{-1}C_3^+}{2}-b_2C_1^+C_3^--b_2C_1^-C_3^+-\frac{C_1^+C_2^-C_3^+}{2}\int_0^{\frac{\pi}{4}} q(t)dt\\ &-\frac{C_1^-a_2^{-1}a_3^{-1}}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} q(t)dt+\frac{C_1^-a_2a_3}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} q(t)dt\\ &-\left(\frac{C_1^+C_2^+a_3^{-1}}{2}-\frac{C_1^+C_2^-C_3^+}{2}-\frac{C_1^-C_2^-C_3^-}{2}\right)\int_0^{\frac{3\pi}{4}} q(t)dt\\ &+\theta_0\Big[-\big(C_1^-a_2C_3^-+C_1^+C_2^-C_3^+\big)\frac{\pi}{2}\\ &\qquad-\big(C_1^-a_2C_3^-+C_1^+C_2^-C_3^+\big)\pi-C_1^+C_2^+C_3^-\frac{3\pi}{2}\Big]\Bigg],\end{aligned}$$

$$(vi)\qquad q(x)=A^{'}(x)+2\theta_0,\quad 0<x<\frac{\pi}{4}, \tag{3.26}$$

$$(vii) \quad q(x) = B^{'}(x) - f_1'(x), \quad \frac{\pi}{4} < x < \frac{\pi}{2}, \tag{3.27}$$

$$(viii) \quad q(x) = C^{'}(x) - f_2'(x), \quad \frac{\pi}{2} < x < \frac{3\pi}{4}, \tag{3.28}$$

$$(ix) \quad q(x) = D^{'}(x) - f_3'(x), \quad \frac{3\pi}{4} < x < \pi. \tag{3.29}$$

where

$$M = A\left(\frac{\pi}{4}\right) - A(0) + B\left(\frac{\pi}{2}\right) - B\left(\frac{\pi}{4}\right), \quad K = f_1\left(\frac{\pi}{2}\right) - f_1\left(\frac{\pi}{4}\right).$$

Proof. Putting $x = 0$ into (3.17), we have $A(0) = 2h$, i.e. $h = A(0)/2$. Substituting $x = \pi/4$ into (3.17), we obtain that

$$A\left(\frac{\pi}{4}\right) = \int_0^{\frac{\pi}{4}} q(t)dt - \frac{\pi}{2}\theta_0 + 2h.$$

Deriving (3.17), it follows that

$$q(x) = A'(x) - 2\theta_0, \quad 0 < x < \frac{\pi}{4}. \tag{3.30}$$

Putting $x = \pi/4$ into (3.18), we have $B(\pi/4) = f_1(\pi/4)$. Similarly, letting $x = \pi/2$, it yields

$$B\left(\frac{\pi}{2}\right) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} q(t)dt + f_1\left(\frac{\pi}{2}\right).$$

Deriving (3.19), we have

$$q(x) = B'(x) - f_1'(x), \quad \frac{\pi}{2} < x < \pi. \tag{3.31}$$

It follows from above formulaes that

$$A\left(\frac{\pi}{4}\right) - A(0) = \int_0^{\frac{\pi}{4}} q(t)dt - \frac{\pi}{2}\theta_0, \tag{3.32}$$

$$B\left(\frac{\pi}{2}\right) - B\left(\frac{\pi}{4}\right) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} q(t)dt + K. \tag{3.33}$$

So

$$M = A\left(\frac{\pi}{4}\right) - A(0) + B\left(\frac{\pi}{2}\right) - B\left(\frac{\pi}{4}\right) = \int_0^{\frac{\pi}{2}} q(t)dt - \frac{\pi}{2}\theta_0 + K. \tag{3.34}$$

If we take $\int_0^{\frac{\pi}{2}} q(t)dt = 0$ into account, we can obtain that

$$\theta_0 = \frac{2(K-d)}{\pi}. \tag{3.35}$$

Then, it follows from (3.30) that

$$q(x) = A'(x) + \frac{4(K-d)}{\pi}, \quad 0 < x < \frac{\pi}{4}.$$

Similarly, we can get (3.27)-(3.29). Next, we calculate $b_i, i = 1, 2, 3$.

Putting $x = 0$ into (3.10), we have

$$f_1(0) = \frac{2}{a_1}\left\{a_1^{-1}h + b_1 - C_1^{-}\int_0^{\frac{\pi}{4}} q(t)dt + \frac{C_1^{+}}{2}\int_0^{\frac{\pi}{4}} q(t)dt + \theta_0 C_1^{-}\frac{\pi}{2}\right\}. \tag{3.36}$$

Then, together with the information of $q(x)$ on $(0,\pi/4)$ and (3.36), we can construct b_1. That is

$$b_1=\frac{f_1(0)a_1}{2}-a_1^{-1}h+\left(C_1^- -\frac{C_1^+}{2}\right)\int_0^{\frac{\pi}{4}}q(t)dt-\frac{\pi}{2}C_1^-\theta_0.$$

Substituting $x=0$ into $f_2(x)$, we obtain

$$\begin{aligned}f_2(0)=\frac{2}{C_1^+a_2^{-1}+C_1^-a_2}\Bigg\{&C_1^+a_2h+\frac{b_1a_2}{2}+\frac{C_1^+a_2^{-1}}{2}\int_0^{\frac{\pi}{2}}q(t)dt-C_1^-a_2^{-1}h\\&+b_2C_1^-+\frac{a_2^{-1}b_1}{2}+\frac{C_1^-C_2^-}{2}\int_0^{\frac{\pi}{4}}q(t)dt+\frac{C_1^-C_2^+}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}q(t)\\&-\theta_0\left[-C_1^+C_2^-\pi-a_2C_1^-\frac{\pi}{2}\right]\Bigg\}.\end{aligned}\tag{3.37}$$

Combing the information of $q(x)$ on $(\pi/4,\pi/2)$ and (3.37), we can construct that

$$\begin{aligned}b_2=\frac{1}{C_1^-}\Bigg[&\frac{f_2(0)\big(C_1^+a_2^{-1}+C_1^-a_2\big)}{2}-C_1^+a_2h-\frac{b_1a_2}{2}+\frac{C_1^+a_2^{-1}}{2}\int_0^{\frac{\pi}{2}}q(t)dt\\&+C_1^-a_2^{-1}h-\frac{a_2^{-1}b_1}{2}-\frac{C_1^-C_2^-}{2}\int_0^{\frac{\pi}{4}}q(t)dt-\frac{C_1^-C_2^+}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}q(t)\\&+\theta_0\left(-\pi C_1^+C_2^- -\frac{\pi}{2}a_2C_1^-\right)\Bigg].\end{aligned}$$

Similarly, we get

$$\begin{aligned}f_3(0)=&\frac{2}{C_1^+C_2^+a_3^{-1}-C_1^+C_2^-a_3-C_1^-a_2a_3}\\&\times\Bigg\{2C_1^+C_2^+C_3^+h-C_1^+a_2C_3^-h-C_1^-a_2^{-1}C_3^+h\\&\quad+\frac{b_1a_2a_3^{-1}}{2}+\frac{b_1a_2^{-1}C_3^+}{2}+b_2C_1^+C_3^-+b_2C_1^-C_3^+\\&\quad+\frac{C_1^+C_2^-C_3^+}{2}\int_0^{\frac{\pi}{4}}q(t)dt+\frac{C_1^-a_2^{-1}a_3^{-1}}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}q(t)dt-\frac{C_1^-a_2a_3}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}q(t)dt\\&\quad+\left(\frac{C_1^+C_2^+a_3^{-1}}{2}-\frac{C_1^+C_2^-C_3^+}{2}-\frac{C_1^-C_2^-C_3^-}{2}\right)\int_0^{\frac{3\pi}{4}}q(t)dt+a_1a_2b_3\\&\quad-\theta_0\Bigg[-\big(C_1^-a_2C_3^-+C_1^+C_2^-C_3^+\big)\frac{\pi}{2}\\&\qquad-\big(C_1^-a_2C_3^-+C_1^+C_2^-C_3^+\big)\pi-C_1^+C_2^+C_3^-\frac{3\pi}{2}\Bigg]\Bigg\}.\end{aligned}\tag{3.38}$$

By virtue of the information of $q(x)$ on $(\pi/2,3\pi/4)$ and (3.38), we can construct that

$$\begin{aligned}b_3=\frac{1}{a_1a_2}\Bigg[&\frac{f_3(0)\big(C_1^+C_2^+a_3^{-1}-C_1^+C_2^-a_3-C_1^-a_2a_3\big)}{2}-2C_1^+C_2^+C_3^+h\\&-2C_1^+C_2^+C_3^+h+C_1^+a_2C_3^-h+C_1^-a_2^{-1}C_3^+h-\frac{b_1a_2a_3^{-1}}{2}\end{aligned}$$

$$
\begin{aligned}
&-\frac{b_1 a_2^{-1} C_3^+}{2} - b_2 C_1^+ C_3^- - b_2 C_1^- C_3^+ - \frac{C_1^+ C_2^- C_3^+}{2} \int_0^{\frac{\pi}{4}} q(t)dt \\
&-\frac{C_1^- a_2^{-1} a_3^{-1}}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} q(t)dt + \frac{C_1^- a_2 a_3}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} q(t)dt \\
&-\left(\frac{C_1^+ C_2^+ a_3^{-1}}{2} - \frac{C_1^+ C_2^- C_3^+}{2} - \frac{C_1^- C_2^- C_3^-}{2}\right) \int_0^{\frac{3\pi}{4}} q(t)dt \\
&+\theta_0 \Bigg[-\left(C_1^- a_2 C_3^- + C_1^+ C_2^- C_3^+\right)\frac{\pi}{2} \\
&\qquad -\left(C_1^- a_2 C_3^- + C_1^+ C_2^- C_3^+\right)\pi - C_1^+ C_2^+ C_3^- \frac{3\pi}{2} \Bigg]\Bigg]. \qquad (3.39)
\end{aligned}
$$

Finally, we construct H. It follows from (2.10)-(2.12) and (2.16) that

$$
\begin{aligned}
\left(A\pi + \frac{B\pi}{2}\right)\theta_0 &= B_1(\pi) + B_2(\pi) + B_3(\pi) + B_4(\pi) \\
&\quad - H\left(a_2 C_1^- C_3^- + C_1^+ C_2^- C_3^+ - C_1^+ C_2^+ C_3^+ + a_2 C_1^+ C_3^- + a_2 C_1^- C_3^+\right) \\
&= B_1(\pi) + B_2(\pi) + B_3(\pi) + B_4(\pi) \\
&\quad - H\left(C_1^- a_2 a_3 - C_1^+ a_2^{-1} C_3^+ + C_1^+ a_2 C_3^-\right). \qquad (3.40)
\end{aligned}
$$

So, we have

$$
\begin{aligned}
H = &\frac{1}{C_1^- a_2 a_3 - C_1^+ a_2^{-1} C_3^+ + C_1^+ a_2 C_3^-} \\
&\times \left[B_1(\pi) + B_2(\pi) + B_3(\pi) + B_4(\pi) - \theta_0\left(A\pi + \frac{B\pi}{2}\right)\right],
\end{aligned}
$$

where

$$
C_1^- a_2 a_3 - C_1^+ a_2^{-1} C_3^+ + C_1^+ a_2 C_3^- \neq 0.
$$

Considering Theorem 3.1 and deriving (3.17)-(3.20), respectively, we can obtain the formulae (3.22)-(3.29). The proof is complete. □

4. Numerical Calculation

In the section, we introduce a new method called the "Legendre wavelet method", which is used to compute the approximate solution of the potential function. By comparing reconstructed potential with the actual potential, we further verify the effectiveness of the method.

4.1. Legendre wavelet method

Definition 4.1 (Legendre Wavelet Method, [14]). *For $k \in \mathbb{N}^+$,*

$$
\varphi_{l,m}(t) = \begin{cases} \sqrt{\dfrac{2m+1}{2}}\, 2^{\frac{k}{2}} P_m(2^k t - 2l + 1), & \dfrac{l-1}{2^{k-1}} \leq t \leq \dfrac{l}{2^{k-1}}, \\ 0, & \textit{otherwise.} \end{cases}
$$

Here $\varphi_{l,m}(t)$ is called the wavelet basis function, $l = 1, 2, \ldots, 2^{k-1}, m = 0, 1, 2, \ldots, M-1$. $P_m(t)$ is Legendre polynomial defined in $(0, \pi)$ and can be determined by formula

$$\begin{aligned}
&P_0(t) = 1, \quad P_1(t) = t, \\
&P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) tP_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t), \quad m = 1, 2, 3, \ldots .
\end{aligned}$$

Definition 4.2 ([14]). *For an arbitrary function $h(t) \in L^2[0, \pi]$, $h(t) \in L^2[0, \pi]$ can be expanded in terms of Legendre wavelet series*

$$h(t) = \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} c_{l,m}\varphi_{l,m}(t),$$

here $c_{l,m}$ is coefficient and $c_{l,m} = \langle h(t) \cdot \varphi_{l,m}(t)\rangle, \langle\cdot\rangle$ represents inner product.

Applying the similar method in [12,19], we can get that the potential $q(x)$ can be approximated by Legendre wavelet method and

$$q(t) \cong \sum_{l=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{l,m}\varphi_{l,m}(t) - c_{2^{k-1},M-1}\varphi_{2^{k-1},M-1}(t) = \mathbf{C}^\top \Psi(t),$$

where

$$\begin{aligned}
&\mathbf{C}^\top = [c_{1,0}, \cdots, c_{1,M-1}, c_{2,0}, \cdots, c_{2,M-1}, \cdots, c_{2^{k-1}-1,0}, \cdots, c_{2^{k-1}-1,M-1}], \\
&\Psi^\top(t) = [\varphi_{1,0}(t), \cdots, \varphi_{1,M-1}(t), \cdots, \varphi_{2^{k-1}-1,0}(t), \cdots, \varphi_{2^{k-1}-1,M-1}(t)].
\end{aligned}$$

Let

$$q_{k,M}(t) = \sum_{l=1}^{2^{k-1}} c_{l,m}\varphi_{l,m}(t) - c_{2^{k-1},M-1}\varphi_{2^{k-1},M-1}(t).$$

According to the results of [12, 14, 19], we can obtain that

$$q(t) = \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} c_{l,m}\varphi_{l,m}(t),$$

here $c_{l,m} = \langle q(t) \cdot \varphi_{l,m}(t)\rangle_{L^2[0,1]}$. We employ the finite element solution $q_{k,M}(t)$ as an approximation to the continuum solution $q(t)$.

4.2. Numerical example

Based on the Algorithm 4.1, the approximate solution of $q(x)$ is computed using the Legendre wavelet method. Additionally, two numerical examples are provided to demonstrate the effectiveness and feasibility of the numerical computation method.

For the positive problem, we can calculate the information which we need by the Sturm-Liouville problem. In this paper, we can calculate the information of the nodes (3.6)-(3.9) by the Sturm-Liouville operator $L = L(q, h, H, a_i, b_i, d_i)$. But for the inverse problem, we can reconstruct the Sturm-Liouville operator, especially the potential by the dense nodes which are

Algorithm 4.1: Approximation Algorithm for Potential [12, 19].

1 Select k, M. Let $n = 2^{k-2}M, n \gg 1$.

2 Calculate the unknown vector $\mathbf{C}$ by using the following linear equation:

$$\mathbf{AC} = \mathbf{B},$$

here

$$\mathbf{A} = \begin{pmatrix} a^1_{1,0} & \cdots & a^1_{1,M-1} & a^1_{2,0} & \cdots & a^1_{2,M-1} & \cdots & a^1_{2^{k-1},0} & \cdots & a^1_{2^{k-1},M-1} \\ a^2_{1,0} & \cdots & a^2_{1,M-1} & a^2_{2,0} & \cdots & a^2_{2,M-1} & \cdots & a^2_{2^{k-1},0} & \cdots & a^2_{2^{k-1},M-1} \\ \vdots & & & & & & & & & \vdots \\ a^{n-1}_{1,0} & \cdots & a^{n-1}_{1,M-1} & a^1_{2,0} & \cdots & a^{n-1}_{2,M-1} & \cdots & a^{n-1}_{2^{k-1},0} & \cdots & a^{n-1}_{2^{k-1},M-1} \end{pmatrix},$$

and

$$a^j_{l,m} = \int_0^{x^j_n} \xi_{l,m}(t) \sin^2\left(\rho^0_{2n}\right) dt, \quad l = \overline{1, 2^{k-1}}, \quad m = \overline{0, M-1}, \quad j = \overline{1, n-1}.$$

Here

$$\mathbf{B}^\top = \left(\rho^0_{2n} \tan\left(\rho^0_{2n} x^1_{2n}\right), \rho^0_{2n} \tan\left(\rho^0_{2n} x^2_{2n}\right), \cdots, \rho^0_{2n} \tan\left(\rho^0_{2n} x^{2n}_{2n}\right)\right).$$

3 Using the following formula to solve the approximate value $q(t_i)$ of $q(t)$, $i = \overline{1, n-1}$:

$$[q(t_i)] = \mathbf{C\Phi},$$

where

$$t_i = \frac{2i-1}{2n}, \quad i = 1, 2, \ldots, n-1, \quad \mathbf{\Phi} = \left[\xi\left(\frac{1}{2n}\right), \xi\left(\frac{3}{2n}\right), \cdots, \xi\left(\frac{2n-3}{2n}\right)\right].$$

obtained by the positive problem. Below we provide two examples to verify the validity of the inverse node problem. Let

$$\begin{aligned} &a_1 = a_2 = a_3 = 2, \qquad a_1^{-1} = a_2^{-1} = a_3^{-1} = \frac{1}{2}, \\ &b_1 = b_2 = b_3 = 0, \\ &C_1^+ = C_2^+ = C_3^+ = \frac{5}{4}, \quad C_1^- = C_2^- = C_3^- = \frac{3}{4}, \quad h = H = 1. \end{aligned}$$

Example 4.1. Denote by $q_1(x)$ the potential of the direct problem (1.1)-(1.4), and support that $q_1(x)$ is constant function

$$\begin{aligned} q_1(x) &= 1, \quad x \in \left(0, \frac{\pi}{4}\right), \\ q_1(x) &= 0, \quad x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right), \\ q_1(x) &= -1, \quad x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right), \end{aligned}$$

$$q_1(x) = 0, \qquad x \in \left(\frac{3\pi}{4}, \pi\right).$$

Taking $n = 10$ for example, we are able to explicitly determine the zeros x_{2n}^s of the eigenfunctions $\varphi_{2n}(x)$ arising in the direct problem. Selected the approximate node value of x_{2n}^s are tabulated as in Table 4.1.

Using the zero data in Table 4.1 as input data for the inverse nodal problem, we apply the Legendre wavelet series to reconstruct the potential function. The recovered potential $q_2(x)$ from the inverse problem is then compared with the original potential $q_1(x)$ from the direct problem for both two cases: when $k = 3, M = 5$ and $k = 3, M = 7$, with the results visualized in Fig. 4.1.

Table 4.1: The approximate node value of x_{2n}^s.

$n = 10$					
S	1	2	3	4	5
x_{2n}^S	0.08124293	0.23871534	0.39607463	0.55350357	0.71102817
S	6	7	8	9	10
x_{2n}^S	0.86459560	1.02204637	1.17901475	1.33633141	1.49340687
S	11	12	13	14	15
x_{2n}^S	1.65118286	1.80841946	1.96544606	2.12265148	2.27973422
S	16	17	18	19	20
x_{2n}^S	2.43668263	2.59393379	2.75116728	2.90832095	3.06542817

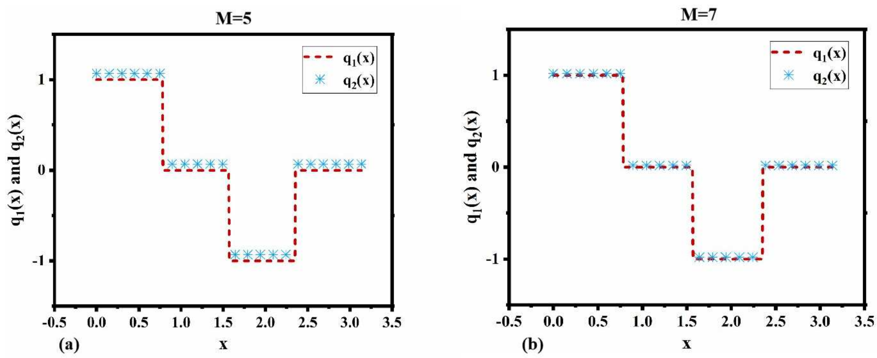

Fig. 4.1. Comparison of $q_1(x)$ and $q_2(x)$.

Example 4.2. Denote by $q_1(x)$ the potential of the direct problem, and support that $q_1(x)$ is piecewise function

$$q_1(x) = x, \qquad x \in \left(0, \frac{\pi}{4}\right),$$
$$q_1(x) = x^2, \qquad x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right),$$
$$q_1(x) = \frac{1}{2}x, \qquad x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right),$$

$$q_1(x) = -x + \frac{5}{4}\pi, \quad x \in \left(\frac{3\pi}{4}, \pi\right).$$

Similarly, when $n = 10$, we can get the zeros x_{2n}^s of the eigenfunctions $\varphi_{2n}(x)$ from the direct problem.

Choosing the approximate node value of x_{2n}^s (Table 4.2) as the input data for the inverse nodal problem and applying the Legendre wavelet series, we can get the recovered potential $q_2(x)$. Comparing the original potential $q_1(x)$ from the direct problem with the recovered potential $q_2(x)$ from the inverse problem in two cases: $k = 3, M = 5$ and $k = 3, M = 7$, respectively, we can obtain the comparison graphs of the two functions $q_1(x)$ and $q_2(x)$ in Fig. 4.2.

As clearly demonstrated in Tables 4.1 and 4.2, the node data also increases with increasing n. Figs. 4.1 and 4.2 reveal that for $n \geq 10$, the potential reconstructed from the inverse problem closely approximates the original potential, that is the error between two kinds of potential extremely small which conclusively validates the effectiveness of numerical solution method.

Table 4.2: The approximate node value of x_{2n}^s.

$n = 10$					
S	1	2	3	4	5
x_{2n}^S	0.08113253	0.23783727	0.39518429	0.55188391	0.70902501
S	6	7	8	9	10
x_{2n}^S	0.86335274	1.02053433	1.17738064	1.33485138	1.49216499
S	11	12	13	14	15
x_{2n}^S	1.65119125	1.80827969	1.96528949	2.12225722	2.27952576
S	16	17	18	19	20
x_{2n}^S	2.43550197	2.59278907	2.74946076	2.90669254	3.06385752

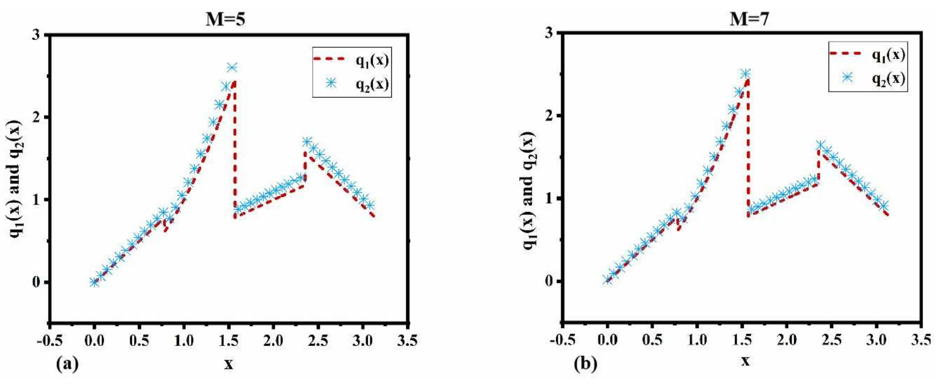

Fig. 4.2. Comparison of $q_1(x)$ and $q_2(x)$.

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