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MEAN-SQUARE CONVERGENCE OF TWO-STEP MILSTEIN METHODS FOR NONLINEAR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS*

Lijuan Peng, Lihang Zhou and Wenqiang Wang[1)]
Hunan Key Laboratory for Computation and Simulation in Science and Engineering,
School of Mathematical and Computational Sciences, Xiangtan University,
Xiangtan 411105, China
Emails: 202221511217@smail.xtu.edu.cn, 202221511271@smail.xtu.edu.cn,
wwq@xtu.edu.cn

Abstract

In this paper, a numerical method for solving nonlinear stochastic delay differential equations is proposed: two-step Milstein method. The mean square consistent and mean square convergence of the numerical method are studied. Through the relevant derivation, the conditions that the coefficients need to be satisfied when the numerical method is mean-square consistent and mean-square convergent are obtained, and it is proved that the mean-square convergence order of the numerical method is 1. Finally, the theoretical results are verified by numerical experiments.

Mathematics subject classification: 60H10, 60H35, 65L20.
Key words: Stochastic delay differential equation, Two-step Milstein method, Mean-square consistent, Mean-square convergence.

1. Introduction

Stochastic delay differential equations (SDDEs) can be used to describe problems in many fields such as biology, physics, and economy (see, e.g. [1, 8, 10]). Therefore, they have received extensive attention and research from scholars at home and abroad, and have produced many research results. However, these research results are mainly based on one-step numerical methods. For instance, Mao and Sabanis [14] applied Euler-Maruyama method to nonlinear stochastic delay differential equations whose coefficients satisfy local Lipshitz condition, and proved that the numerical method is convergent in mean square. Cao *et al.* [5] and others applied the Euler-Maruyama method to linear SDDEs with multiplicative noise, and obtained the condition of mean square stability. In the same year, Hu *et al.* [9] proposed a strong Milstein approximation method for solving SDDEs, and proved that the strong convergence order of the numerical method is 1. In addition, an infinite-dimensional Itô formula for SDDEs is proposed and a rigorous proof process is given. Subsequently, Wang and Zhang [22] discussed the mean square stability of Milstein methods for linear SDDEs and obtained the conditions of mean square stability. In addition to the research on explicit methods, many scholars have also studied semi-implicit methods. Liu *et al.* [12] proposed a semi-implicit Euler method for solving linear SDDEs, and studied the mean square convergence and mean square stability of the numerical method. It is proved that the mean square convergence order of the numerical method is 0.5, and the mean square stability condition of the method is obtained. In addition,

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1) Corresponding author

many scholars have made some innovations on the basis of Euler-Maruyama method and Milstein method, and obtained some new methods, such as θ-Maruyama method, balanced Euler method, truncated Euler-Maruyama method and truncated Milstein method. Wang *et al.* [21] applied the θ-Maruyama method to nonlinear stochastic delay differential equations (NSDDEs) under the condition of global Lipschitz, and proved that the numerical method has a mean square convergence order of 0.5 order. And for a sufficiently small step size $h > 0$, the numerical methods are exponentially mean-square stable. Cao *et al.* [4] studied the strong convergence of strong explicit numerical methods for SDDEs with superlinear growth coefficients. Under the condition of non-global Lipschitz, a class of balanced Euler methods is proposed. It is proved that the obtained numerical solution is bounded and the mean square convergence order of the numerical method is 0.5. Song *et al.* [18] put forth a class of truncated Euler-Maruyama methods for NSDDEs and investigated their strong convergence and mean-square stability. Zhang *et al.* [23] studied the strong convergence of the truncated Milstein method for highly NSDDEs without nonlinear growth conditions, and proved that the strong convergence rate of the numerical method is close to 1.

On the other hand, many scholars have gradually studied the multi-step methods of stochastic differential equations (SDEs). For example, Tocino and Senosiain [19,20] analyzed the mean square stability of the SDEs two-step Maruyama method and the mean square convergence and mean square stability of the two-step Milstein method. They derived the necessary and sufficient conditions for the mean square stability of the two-step Maruyama method and the two-step Milstein method, and proved that the mean square convergence order of the two-step Milstein method is 1. Ren and Tian [16, 17] proposed the generalized two-step Maruyama method and the two-step Milstein method for solving SDEs. The mean square convergence and mean square stability of these two numerical methods are studied, and it is proved that the mean square convergence order of the generalized two-step Maruyama method is 0.5 and the two-step Milstein method is 1. In addition, by comparing the stability regions, it is shown that the generalized two-step Maruyama method and the generalized Milstein method have better stability than the two-step Maruyama method and the two-step Milstein method. D'Ambrosio and Scalone [7] introduced the theory of the two-step Runge-Kutta method for SDEs, and studied the mean square convergence and mean square stability of the numerical method.

At present, there are few research results on the multi-step method of stochastic delay differential equations. Buckwar and Winkler [3] studied the p-order convergence and stability of the multi-step Maruyama method for SDDEs. Cao and Zhang [6] performed numerical simulations related to the two-step Maruyama method for several classes of NSDDEs. Li and Cao [11] applied an implicit two-step Maruyama method for nonlinear neutral SDDEs under the condition of one-sided Lipschitz, and proved that the numerical method can maintain the mean square stability of the solution. At present, there are few studies on multi-step methods of SDDEs, and most of them are multi-step Maruyama methods. The content of this paper is the mean square convergence of the two-step Milstein method for NSDDEs, which has certain significance for the research theory of the multi-step method for SDDEs.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space, and let the filters $\{\mathcal{F}_t\}_{t\geq 0}$ satisfy the usual conditions, namely that they are right-continuous and that each $\mathcal{F}_t$ contains all sets of zero probabilities. In this paper, we consider the following NSDDEs:

$$\begin{cases} \mathrm{d}X(t) = f\big(t, X(t), X(t-\tau)\big)\mathrm{d}t + g\big(t, X(t), X(t-\tau)\big)\mathrm{d}W(t), & t \in [0, T], \\ X(t) = \Psi(t), & t \in [-2\tau, 0], \end{cases} \tag{1.1}$$