A NOVEL TGMFECN DIMENSION REDUCTION METHOD BASED ON POD FOR THE IMPROVED NONLINEAR FOURTH-ORDER CAHN-HILLIARD EQUATIONS[*]

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Abstract

An improved nonlinear fourth-order Cahn-Hilliard (INFOCH) equation is first developed to ensure that its numerical model is symmetric, positive definite, and solvable. Then, by introducing an auxiliary function, the INFOCH equation is decomposed into the nonlinear system of equations with second-order derivatives of spatial variables. Subsequently, by using the Crank-Nicolson (CN) technique to discretize the time derivative, a new time semi-discretized mixed CN (TSDMCN) scheme with second-order accuracy is constructed, and the existence, stability, and error estimates of TSDMCN solutions are analyzed. Thenceforth, a new two-grid mixed finite element (MFE) CN (TGMFECN) method is created by using two-grid MFE method to discretize the TSDMCN scheme, and the existence, stability, and error estimates of TGMFECN solutions are discussed. Next, it is most important that by using proper orthogonal decomposition to reduce the dimension of unknown coefficient vectors of TGMFECN solutions and keep the MFE basis functions unchanged, a new TGMFECN dimensionality reduction (TGMFECNDR) method with very few unknowns, unconditional stability, and second-order time precision is created, and the existence, stability, and error estimates of TGMFECNDR solutions are proved. Finally, the superiority of TGMFECNDR method and the correctness of the obtained theoretical results are showed by two sets of numerical experiments.

Mathematics subject classification: 65N12, 65M15, 65N35.
Key words: The improved nonlinear fourth-order Cahn-Hilliard equation, Proper orthogonal decomposition, Time semi-discrete mixed Crank-Nicolson scheme, Two-grid mixed finite element Crank-Nicolson method, Two-grid mixed finite element Crank-Nicolson dimension reduction method, Existence, stability, and error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n = 1, 2, 3$) be a connected bounded region with the boundary $\partial\Omega$ and t_N be a given time upper-bound. We develop a new improved nonlinear fourth-order Cahn-Hilliard

equation with two strong nonlinear terms as follows, which is slightly different from the usual Cahn-Hilliard equation in [4, 8, 17, 29, 33].

Problem 1.1. For any $t \in [0, t_N)$, find $\psi : [0, t_N) \to C^4(\bar{\Omega})$, satisfying

$$
\begin{cases}
\partial_t \psi(t, \boldsymbol{x}) + \epsilon^2 \nabla \cdot \big(G\big(\psi(t, \boldsymbol{x})\big) \nabla \big(\Delta \psi(t, \boldsymbol{x})\big)\big) \\
\quad - \nabla \cdot \big[\tilde{f}\big(\psi(t, \boldsymbol{x})\big) \nabla \psi(t, \boldsymbol{x}) \big] = 0, & \forall (t, \boldsymbol{x}) \in (0, t_N) \times \Omega, \\
\psi(t, \boldsymbol{x}) = \Delta \psi(t, \boldsymbol{x}) = 0, & \forall (t, \boldsymbol{x}) \in (0, t_N) \times \partial\Omega, \\
\psi(0, \boldsymbol{x}) = \psi^0(\boldsymbol{x}), & \forall \boldsymbol{x} \in \Omega,
\end{cases}
\tag{1.1}
$$

where the unknown function $\psi(t, \boldsymbol{x})$ indicates the phase field or order parameter, $C^4(\bar{\Omega})$ represents a space formed with functions with fourth-order consecutive derivatives, $\partial_t = \partial/\partial t$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, $\epsilon > 0$ is a constant and indicates the interface width, $G(\psi(t, \boldsymbol{x}))$ indicates a positive mobility, $\tilde{f}(\psi(t, \boldsymbol{x})) > 0$ is a nonlinear source function, and $\psi^0(\boldsymbol{x})$ is an adequately smooth known initial value function.

Remark 1.1. There are two explanation for Problem 1.1.

(1) The improvement of this paper is to take the source function $\tilde{f}(\psi(t, \boldsymbol{x})) > 0$, which can ensure that the time semi-discrete mixed Crank-Nicolson scheme, and finite element (FE) and two-grid mixed FE CN methods have unique solutions. If $F(\psi(t, \boldsymbol{x})) = (\psi^2(t, \boldsymbol{x}) - 1)^2/4$ stands for the free energy density in mixture and $f(\psi(t, \boldsymbol{x})) = F'(\psi(t, \boldsymbol{x})) = \psi^3(t, \boldsymbol{x}) - \psi(t, \boldsymbol{x})$, then we can take $\tilde{f}(\psi(t, \boldsymbol{x})) = f'_+(\psi(t, \boldsymbol{x})) + f'_-(\psi(t, \boldsymbol{x}))$, where $f'_+(\psi)$ and $f'_-(\psi)$ are the positive part and negative part of $f'(\psi)$, respectively, as defined below

$$
f'_+(\psi) = \begin{cases}
3\psi^2 - 1, & \psi \in [-1, -1/\sqrt{3}] \cup [1/\sqrt{3}, 1], \\
0, & \psi \in [-1/\sqrt{3}, 1/\sqrt{3}],
\end{cases}
$$

$$
f'_-(\psi) = \begin{cases}
0, & \psi \in [-1, -1/\sqrt{3}] \cup [1/\sqrt{3}, 1], \\
1 - 3\psi^2, & \psi \in [-1/\sqrt{3}, 1/\sqrt{3}].
\end{cases}
$$

Thus, $\tilde{f}(\psi) = f'_+(\psi) + f'_-(\psi) > 0$ almost everywhere. Two numerical examples in Section 4 show that the INFOCH equation with $\tilde{f}(\psi) = f'_+(\psi) + f'_-(\psi)$ is very effective and reasonable. Of course, the nonlinear positive function $\tilde{f}(\psi(t, \boldsymbol{x}))$ is also taken as other functions. For example, it can be taken as $\tilde{f}(\psi(t, \boldsymbol{x})) = \cos(\psi(t, \boldsymbol{x}))$ $(-1 \le \psi(t, \boldsymbol{x}) \le 1)$. Thus, it satisfies $\tilde{f}(\psi(t, \boldsymbol{x})) > 0$. Hence, the INFOCH equation has wider adaptability and versatility than the usual Cahn-Hilliard equation in [4, 8, 17, 29, 33].

(2) The unknown function $\psi(t, \boldsymbol{x})$ in Problem 1.1 usually represents the difference between the two components in the concentrated binary mixture, namely $\psi(t, \boldsymbol{x}) = g_a(t, \boldsymbol{x}) - g_b(t, \boldsymbol{x})$ satisfying $0 \le g_a(t, \boldsymbol{x}), g_b(t, \boldsymbol{x}) \le 1$ and $g_a(t, \boldsymbol{x}) + g_b(t, \boldsymbol{x}) = 1$. So, there holds $-1 \le \psi(t, \boldsymbol{x}) \le 1$, here g_a and g_b are the mole fractions of phases a and b. In some application of Problem 1.1, for instance, it is used to characterize the diseases transmission, ones take

$$
G\big(\psi(t, \boldsymbol{x})\big) = G_0\big(1 - \psi^2(t, \boldsymbol{x})\big),
$$

here $G_0 > 0$ is a real number.

The improved INFOCH equation with two strong nonlinear terms is an important mathematical physics model. It can not only describe the phase separation process in the binary alloy

mixture, like the usual Cahn-Hilliard equation in [5,6], but can also applied to many fields mentioned in [2,3,5,7,10–12,15,16,18,28,37,38], such as the dynamics spin decomposition, diblock copolymer, image restoration, multi-phase fluid flows, elastic inhomogeneity micro-structure, phyma development simulation, and topological optimization.

However, because the INFOCH equation has two strong nonlinear terms, it is difficultly to solve analytically. Using numerical methods to find the numerical solutions of the INFOCH equation is the most choice. Fortunately, the numerical methods for the INFOCH equation play a important role in the numerical calculations of the dynamics spin decomposition, diblock copolymer generation, image restoration, multi-phase fluid flows, elastic inhomogeneity micro-structure, phyma development process, and optimizing of topological structure. Therefore, it is very meaningful to study the INFOCH equation by numerical methods.

Two-grid FE algorithm is one of the best numerical methods for solving nonlinear partial differential equations (PDEs). It consists of a nonlinear FE system of equations on coarser grids and a linear FE system of equations on finer grids with sufficient precision. Hence, it can simplify calculation and improve calculating efficiency. It was originally used to solve quasi-linear elliptic equations (see [34]). More recently, it has been used by Liu's and Shi's teams to solve some more complicated nonlinear PDEs (see [25,30,31]). However, to our knowledge, at the moment, the INFOCH equation has not been solved by the TGMFECN method. Therefore, the first task of this paper is to create a new TGMFECN method with second-order time precision and unconditional stability for the INFOCH equation. The TGMFECN method has at least the following three advantages. First, the TGMFECN method has unconditional stability and time second order precision. Second, the TGMFECN method is also composed of a system of nonlinear MFE equations on a set of coarser meshes and a system of linear MFE equations on a set of finer meshes with adequately high precision, which can greatly simplify the calculation and enhance the calculation efficiency. Third, by introducing an ancillary function $\varpi = -\Delta\psi$, the INFOCH equation is decomposed into two system of second-order equations that can be easily settled by lower-order FEs, such as linear or quadratic FEs. While the standard FE methods including stabilization methods for the fourth-order equations since they need at least to use the piecewise fifth-degree interpolation polynomials to construct conforming FE space on triangulation or the piecewise third-degree interpolation polynomials to construct conforming FE space on rectangular subdivision (see [26, Chapter 1]). The TGMFECN method is better than the standard FE methods including stabilization methods for the fourth-order equations.

However, the TGMFECN method is still very complicated and contains lots of unknowns. In particular, when the TGMFECN method is applied to practical engineering calculations, it will have tens of millions of unknowns. Therefore, the second task of this paper is to create a new TGMFECN dimension reduction method by using the proper orthogonal decomposition (POD) to reduce the dimension of unknown coefficient vectors in the TGMFECN solutions, so as to vastly reduce calculating burden, delay the cumulation of computation errors, and improve calculation efficiency.

A large number of numerical examples (see, e.g., [1,9,13,19,20,22,27,32]) have shown that the POD method has a very powerful action for reducing the unknowns in numerical methods. But these examples belong to the category of reduced-dimension methods of FE space, namely the vectors formed with FE basis functions for the Galerkin and FE methods, which are thoroughly different from the TGMFECNDR in this paper. This is explained in detail below.

In theory, the unknown FE solutions ψ_h^k ($1 \leq k \leq K$) at moment t_k are represented by the linear combination of the known FE basis functions $\zeta_j(\boldsymbol{x})$ and the unknown solution coefficients

ψ_j^k $(1 \le j \le M_h)$ as follows:

$$\psi_h^k = \sum_{j=1}^{M_h} \zeta_j(\boldsymbol{x}) \psi_j^k = \boldsymbol{\zeta} \cdot \boldsymbol{\Psi}^k, \quad 1 \le k \le K, \qquad (1.2)$$

where h is the maximum diameter of all FEs, $\boldsymbol{\zeta} = (\zeta_1(\boldsymbol{x}), \zeta_2(\boldsymbol{x}), \cdots, \zeta_{M_h}(\boldsymbol{x}))^\top$ is a vector formed with the FE basis functions $\zeta_j(\boldsymbol{x})$ $(1 \le j \le M_h)$, whose existence is proved in [36, Proposition 1.6.21], and $\boldsymbol{\Psi}^k = (\psi_1^k, \psi_2^k, \cdots, \psi_{M_h}^k)^\top$ $(1 \le k \le K)$ are K vectors formed by the unknown FE solution coefficients ψ_j^k, whose dimension M_h is equal to the number of FE grid nodes. Hence, to find the unknown FE solutions of PDE is just to find the unknown FE solution coefficient vectors $\boldsymbol{\Psi}^k$ $(1 \le k \le K)$ after substituting (1.2) into the variational formulation of PDE.

In the algorithm, the FE model is written as a total stiff matrix and the values of FE solutions at FE grid nodes are solved on a computer. In essence, it is to calculate the unknown FE solution coefficient vectors $\boldsymbol{\Psi}^k$ $(1 \le k \le K)$, which are exactly made up of the values of FE solutions at the FE grid nodes.

It follows that both theoretically and algorithmically, finding the unknown FE solutions of PDE boils down to finding the unknown FE solution coefficient vectors $\boldsymbol{\Psi}^k$ $(1 \le k \le K)$.

(a) The dimension reduction of the FE space is actually to reduce the dimension of FE basis function vector in (1.2). If the FE space $\mathbb{U}_h$ is spanned by the FE basis functions $\zeta_1(\boldsymbol{x}), \zeta_2(\boldsymbol{x}), \cdots, \zeta_{M_h}(\boldsymbol{x})$, then it is denoted as follows:

$$\mathbb{U}_h = \mathrm{span}\{\zeta_1(\boldsymbol{x}), \zeta_2(\boldsymbol{x}), \cdots, \zeta_{M_h}(\boldsymbol{x})\}. \qquad (1.3)$$

Thus, the previous reduced-dimension methods in [1, 9, 13, 19, 20, 22, 27, 32] are established by replacing the FE basis functions $\zeta_1(\boldsymbol{x}), \zeta_2(\boldsymbol{x}), \cdots, \zeta_{M_h}(\boldsymbol{x})$ with the most main d continuous POD basis functions $\tilde{\phi}_1(\boldsymbol{x}), \tilde{\phi}_2(\boldsymbol{x}), \cdots, \tilde{\phi}_d(\boldsymbol{x})$ $(d \ll M_h)$ formed with some continuous FE solutions obtained from classical FE method, namely by replacing the FE space $\mathbb{U}_h$ in (1.3) with the sub-space $\mathbb{U}_d = \mathrm{span}\{\tilde{\phi}_1(\boldsymbol{x}), \tilde{\phi}_2(\boldsymbol{x}), \cdots, \tilde{\phi}_d(\boldsymbol{x})\}$ generated by the most main d continuous POD basis functions $\tilde{\phi}_1(\boldsymbol{x}), \tilde{\phi}_2(\boldsymbol{x}), \cdots, \tilde{\phi}_d(\boldsymbol{x})$. Thus, the construction of continuous POD basis functions and the theoretical analysis of existence, stability, and error estimates for the reduced-dimension solutions of the FE space reduced-dimension method need very abstract functional analysis (see [26, Section 4.1]), which is very complex and not easily understood by engineers with a weak mathematical foundation. Moreover, the precision of reduced-dimension solutions is also influenced by dimensionality reduction of the FE space.

(b) The dimension reduction of unknown FE solution coefficient vectors of this paper is started from another direction, that is, the FE basis functions $\zeta_1(\boldsymbol{x}), \zeta_2(\boldsymbol{x}), \cdots, \zeta_{M_h}(\boldsymbol{x})$ in (1.2) are kept unchanged, that is, the FE space $\mathbb{U}_h$ in (1.3) is kept unchanged, and the dimensionality reduction method is built by replacing the unknown FE solution coefficient vectors $\boldsymbol{\Psi}^k$ with the linear combination $\boldsymbol{\Psi}_d^k = \boldsymbol{\Phi}_h \boldsymbol{\beta}^k$ of the most main d POD basis vectors $\boldsymbol{\Phi}_h = (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_d)$ formed with some FE solution coefficient vectors obtained from classical FE method. Thus, the obtained reduced-dimension solutions ψ_d^k $(1 \le k \le K)$ of unknown FE solution coefficient vectors can be expressed as follows:

$$\psi_d^k = \boldsymbol{\zeta} \cdot \boldsymbol{\Psi}_d^k = \boldsymbol{\zeta} \cdot \left(\boldsymbol{\Phi}_h \boldsymbol{\beta}^k\right), \quad 1 \le k \le K, \qquad (1.4)$$

where $\boldsymbol{\beta}^k = (\beta_1^k, \beta_2^k, \cdots, \beta_d^k)^\top$ are the unknown vectors with only d unknown components $(d \ll M_h)$. By comparing (1.4) with (1.2), it is easy to see that reduced-dimension solutions ψ_d^k

of unknown FE solution coefficient vectors have the same FE basis functions $\boldsymbol{\zeta} = (\zeta_1(\boldsymbol{x}), \zeta_2(\boldsymbol{x}), \cdots, \zeta_{M_h}(\boldsymbol{x}))^\top$ and precision as the usual FE solutions ψ_h^k. In particular, in Section 3, we will see that the construction of POD basis vectors and the theory analysis of existence, stability, and error estimates for the reduced-dimension solutions of unknown FE solution coefficient vectors need only matrix analysis, rather than very abstract functional analysis, which is very straightforward and easily accepted by engineers with a weak mathematical background. Hence, the reduced-dimension method of unknown FE solution coefficient vectors of this paper is distinctly better than the existing reduced-dimension method of FE space, including those in [1, 9, 13, 19, 20, 22, 27, 32].

Although the TGMFECNDR methods for the unsaturated soil water flow problem, the nonlinear wave equation, the fourth-order extended Fisher-Kolmogorov equation, and the nonlinear fourth-order reaction diffusion equation have been posed in [14, 23, 24, 35], respectively, the INFOCH equation with two strong nonlinear terms and fourth-order spatial derivatives herein are much more complex than the above four equations. Hence, the creation of TGMFECNDR algorithm of the INFOCH equation and the theory analysis of the existence, stability, and error estimates of the TGMFECNDR solutions are confronted with more difficulties and require more techniques than all the previous works. But, just as mentioned above, the TGMFECNDR method for the INFOCH equation has very important applications. Therefore, it is very valuable to study the TGMFECNDR method for the INFOCH equation.

The rest of this article consists of the following four sections. First, in Section 2, a new TSDMCN scheme with second-order time accuracy and a new TGMFECN method with unconditional stability and second-order time accuracy for the INFOCH equation are created, the existence and stability together with error estimates of the TSDMCN and TGMFECN solutions are proved. Thenceforth, in Section 3, a new TGMFECNDR method with unconditional stability and second-order time accuracy is developed by using the POD method only to reduce the dimensionality of unknown coefficient vectors in the TGMFECN solutions and keep the FE basis functions unchanged, and the existence and stability as well as error estimates of the TGMFECNDR solutions are proven by matrix analysis. Next, in Section 4, the validity of TGMFECNDR method and the correctness of theoretical results are verified by two numerical examples. Finally, in Section 5, the main conclusions of this paper are summarized and some expectations in the future research are given.

2. The TSDMCN and TGMFECN Methods of the INFOCH Equation

2.1. The TSDMCN scheme

The Sobolev spaces and norms presented in context are classical (see [21, 26]). Let $\mathbb{U} = H_0^1(\Omega)$. By introducing auxiliary function $\varpi = -\Delta\psi$ and using the Green formula, a mixed variational formulation for Problem 1.1 can be built as follows.

Problem 2.1. For any $t \in (0, t_N)$, find $(\psi, \varpi) \in \mathbb{U} \times \mathbb{U}$, satisfying

$$\begin{cases} (\partial_t \psi, \vartheta) + \epsilon^2 \big(G(\psi)\nabla\varpi, \nabla\vartheta\big) + \big(\tilde{f}(\psi)\nabla\psi, \nabla\vartheta\big) = 0, & \forall\, \vartheta \in \mathbb{U}, \\ (\varpi, \upsilon) = (\nabla\psi, \nabla\upsilon), & \forall\, \upsilon \in \mathbb{U}, \\ \psi(0, \boldsymbol{x}) = \psi^0(\boldsymbol{x}), \quad \varpi(0, \boldsymbol{x}) = -\Delta\psi^0(\boldsymbol{x}), & \forall\, \boldsymbol{x} \in \Omega, \end{cases} \quad (2.1)$$

where $(\cdot\,, \cdot)$ denotes the L^2 inner product, i.e., $(\vartheta, \upsilon) = \int_\Omega \vartheta(\boldsymbol{x}) \cdot \upsilon(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$.

The existence and stability of the weak solution of Problem 2.1 can be proved by using the same technique as proving Theorem 2.1.

It is obvious that $G(\psi)$, $\tilde{f}(\psi)$, $G'(\psi) =: \mathrm{d}G(\psi)/\mathrm{d}\psi$, and $\tilde{f}'(\psi) =: \mathrm{d}\tilde{f}(\psi)/\mathrm{d}\psi$ are bounded, namely there are two positive constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \le G(\psi), \tilde{f}(\psi), |G'(\psi)|, |\tilde{f}'(\psi)| \le c_2. \tag{2.2}$$

In addition, by Poincaré's inequality (see [26, Corollary 1.1.19]), there is a constant $c_0 > 0$ such that $\|\vartheta\|_1 \le c_0 \|\nabla\vartheta\|_0$ for any $\vartheta \in \mathbb{U}$.

In order to construct the TGMFECN method, we firstly design a new TSDMCN method. Toward this purpose, we assume that $K > 0$ is an integer, $\Delta t = t_e/K$ indicates the time step, and ψ^k and ϖ^k ($0 \le k \le K$) indicate the approximations to $\psi(\boldsymbol{x}, t)$ and $\varpi(\boldsymbol{x}, t)$ at time node $t_k = k\Delta t$, respectively.

Discretizing the first equation of (2.1) by the implicit scheme with first-order time precision yields

$$\frac{1}{\Delta t}(\psi^k - \psi^{k-1}, \upsilon) + \epsilon^2\big(G(\psi^k)\nabla\varpi^k, \nabla\vartheta\big) + \big(\tilde{f}(\psi^k)\nabla\psi^k, \nabla\vartheta\big) = 0, \quad \forall\vartheta \in \mathbb{U}. \tag{2.3}$$

Discretizing the first equation of (2.1) by the explicit scheme with first-order time precision yields

$$\frac{1}{\Delta t}(\psi^k - \psi^{k-1}, \vartheta) + \epsilon^2\big(G(\psi^{k-1})\nabla\varpi^{k-1}, \nabla\vartheta\big) + \big(\tilde{f}(\psi^{k-1})\nabla\psi^{k-1}, \nabla\vartheta\big) = 0, \quad \forall\vartheta \in \mathbb{U}. \tag{2.4}$$

Adding (2.3) and (2.4) yields the following new TSDMCN scheme with second-order time precision, which completely differs from all the previous time semi-discrete schemes of the usual Cahn-Hilliard equation.

Problem 2.2. Find $\{(\psi^k, \varpi^k)\}_{k=1}^K \subset \mathbb{U} \times \mathbb{U}$, satisfying

$$\frac{1}{\Delta t}(\psi^k - \psi^{k-1}, \vartheta) + \frac{\epsilon^2}{2}\big(G(\psi^k)\nabla\varpi^k + G(\psi^{k-1})\nabla\varpi^{k-1}, \nabla\vartheta\big)$$
$$+ \frac{1}{2}\big(\tilde{f}(\psi^k)\nabla\psi^k + \tilde{f}(\psi^{k-1})\nabla\psi^{k-1}, \nabla\vartheta\big) = 0, \quad \forall\vartheta \in \mathbb{U}, \quad 1 \le k \le K, \tag{2.5}$$
$$(\varpi^k, \upsilon) = (\nabla\psi^k, \nabla\upsilon), \qquad\qquad\qquad \forall\upsilon \in \mathbb{U}, \quad 1 \le k \le K, \tag{2.6}$$
$$\psi^0 = \psi^0(\boldsymbol{x}), \quad \varpi^0 = -\Delta\psi^0(\boldsymbol{x}), \qquad\qquad \boldsymbol{x} \in \Omega. \tag{2.7}$$

For the TSDMCN scheme (Problem 2.2), we obtain the following results.

Theorem 2.1. *Problem 2.2 has a unique series of solutions* $\{(\psi^k, \varpi^k)\}_{k=1}^K \subset \mathbb{U} \times \mathbb{U}$, *satisfying the following boundedness, i.e., stability:*

$$\|\psi^k\|_1 + \|\varpi^k\|_1 \le c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big), \quad 1 \le k \le K, \tag{2.8}$$

here and thereafter c indicates a generic positive constant independent of Δt. Furthermore, when the solution $(\psi(t, \boldsymbol{x}), \varpi(t, \boldsymbol{x}))$ for Problem 2.1 is adequately smooth, the series of TSDMCN solutions $\{(\psi^k, \varpi^k)\}_{k=1}^K$ to Problem 2.2 meets the error estimates

$$\big|\psi(t_k) - \psi^k\big|_1 + \big|\varpi(t_k) - \varpi^k\big|_1 \le c\Delta t^2, \quad 1 \le k \le K, \tag{2.9}$$

in which $(\psi(t_k), \varpi(t_k)) = (\psi(t_k, \boldsymbol{x}), \varpi(t_k, \boldsymbol{x}))$ $(1 \le k \le K)$.

Proof. (1) The proof of existence and uniqueness for the TSDMCN solutions.

(i) The proof of existence for the TSDMCN solutions.

From (2.6) in Problem 2.2 we obtain

$$(\varpi^k - \varpi^{k-1}, \psi^k - \psi^{k-1}) = (\nabla(\psi^k - \psi^{k-1}), \nabla(\psi^k - \psi^{k-1})), \quad 1 \le k \le K, \tag{2.10}$$

$$(\varpi^k \pm \varpi^{k-1}, \varpi^k - \varpi^{k-1}) = (\nabla(\psi^k \pm \psi^{k-1}), \nabla(\varpi^k - \varpi^{k-1})), \quad 1 \le k \le K. \tag{2.11}$$

Taking $\vartheta = \varpi^k - \varpi^{k-1}$ in Problem 2.2, and using Lagrangian's differential middle value (LDMV) formula, the Hölder and Cauchy inequalities, (2.10), and (2.11), we obtain

$$\|\nabla(\psi^k - \psi^{k-1})\|_0^2 + \frac{c_1\epsilon^2\Delta t}{2}\big(\|\nabla\varpi^k\|_0^2 - \|\nabla\varpi^{k-1}\|_0^2\big) + \frac{c_1\Delta t}{2}\big(\|\varpi^k\|_0^2 - \|\varpi^{k-1}\|_0^2\big)$$

$$\le \left| (\nabla(\psi^k - \psi^{k-1}), \nabla(\psi^k - \psi^{k-1})) + \frac{\epsilon^2\Delta t}{2}\big(G(\psi^k)\nabla(\varpi^k + \varpi^{k-1}), \nabla(\varpi^k - \varpi^{k-1})\big) \right.$$

$$\left. + \frac{\Delta t}{2}\big(\tilde{f}(\psi^k)\nabla(\psi^k + \psi^{k-1}), \nabla(\varpi^k - \varpi^{k-1})\big) \right|$$

$$= \left| (\varpi^k - \varpi^{k-1}, \psi^k - \psi^{k-1}) + \frac{\epsilon^2\Delta t}{2}\big(G(\psi^k)\nabla(\varpi^k + \varpi^{k-1}), \nabla(\varpi^k - \varpi^{k-1})\big) \right.$$

$$\left. + \frac{\Delta t}{2}\big(\tilde{f}(\psi^k)\nabla(\psi^k + \psi^{k-1}), \nabla(\varpi^k - \varpi^{k-1})\big) \right|$$

$$= \left| \frac{\epsilon^2\Delta t}{2}\big((G(\psi^k) - G(\psi^{k-1}))\nabla\varpi^{k-1}, \nabla(\varpi^k - \varpi^{k-1})\big) \right.$$

$$\left. - \frac{\Delta t}{2}\big((\tilde{f}(\psi^k) - \tilde{f}(\psi^{k-1}))\nabla\psi^{k-1}, \nabla(\varpi^k - \varpi^{k-1})\big) \right|$$

$$\le \frac{\epsilon^2\Delta t}{2}\max_{\varsigma\in\mathbb{U}}|G'(\varsigma)|\,\|\nabla\varpi^{k-1}\|_{0,\infty}\|\psi^k - \psi^{k-1}\|_0\|\nabla(\varpi^k - \varpi^{k-1})\|_0$$

$$+ \frac{\Delta t}{2}\max_{\varsigma\in\mathbb{U}}|\tilde{f}'(\varsigma)|\,\|\nabla\psi^{k-1}\|_{0,\infty}\|\psi^k - \psi^{k-1}\|_0\|\nabla(\varpi^k - \varpi^{k-1})\|_0$$

$$\le c\Delta t^2\|\nabla(\varpi^k - \varpi^{k-1})\|_0^2 + \frac{1}{2}\|\nabla(\psi^k - \psi^{k-1})\|_0, \quad 1 \le k \le K. \tag{2.12}$$

Simplifying (2.12) yields

$$c_1\epsilon^2\big(\|\nabla\varpi^k\|_0^2 - \|\nabla\varpi^{k-1}\|_0^2\big) + c_1\big(\|\varpi^k\|_0^2 - \|\varpi^{k-1}\|_0^2\big)$$

$$\le c\Delta t\big(\|\nabla\psi^{k-1}\|_{0,\infty}^2 + \|\nabla\varpi^{k-1}\|_{0,\infty}^2\big)\big(\|\nabla\varpi^k\|_0^2 + \|\nabla\varpi^{k-1}\|_0^2\big)$$

$$\le c\Delta t\big(\|\nabla\varpi^k\|_0^2 + \|\nabla\varpi^{k-1}\|_0^2\big), \quad 1 \le k \le K. \tag{2.13}$$

Summating (2.13) from 1 to k, when Δt is fully small, satisfying $c\Delta t \le 1/2$, by Gronwall's inequality (see [26, Lemma 3.1.9]), we get

$$\|\varpi^k\|_1^2 \le \|\varpi^0\|_1^2 + c\Delta t\sum_{i=1}^{k-1}\|\varpi^i\|_1^2 \le |\varpi^0|_1^2\exp(ck\Delta t) \le c\|\varpi^0\|_1^2, \quad 1 \le k \le K. \tag{2.14}$$

Noting that, when $\psi^0 = 0$, we obtain $\varpi^0 = 0$. Thus, by (2.14), we obtain $\|\varpi^k\|_1 = 0$ so as to get $\varpi^k = 0$ ($0 \le k \le K$).

Further, taking $\upsilon = \psi^k$ in Problem 2.2, by the Hölder and Poicaré inequalities, we obtain

$$\|\nabla\psi^k\|_0^2 = (\nabla\psi^k, \nabla\psi^k) = (\varpi^k, \psi^k) \le \|\varpi^k\|_0\|\psi^k\|_0$$
$$\le \|\varpi^k\|_0\|\psi^k\|_1 \le c\|\varpi^k\|_0\|\nabla\psi^k\|_0, \quad 1 \le k \le K. \tag{2.15}$$

Simplifying (2.15) yields

$$\|\nabla\psi^k\|_0 \le c\|\varpi^k\|_0, \quad 1 \le k \le K. \tag{2.16}$$

Thereupon, when $\psi^0 = \psi^0(\boldsymbol{x}) = 0$, by $\varpi^k = 0$ and (2.16), we can get $\|\nabla\psi^k\|_0 = 0$ so that $\psi^k = 0$ $(1 \le k \le K)$. Hence, Problem 2.2 has least a series of TSMCN solutions $\{(\psi^k, \varpi^k)\}_{k=1}^K \subset \mathbb{U}\times\mathbb{U}$.

(ii) The proof of uniqueness for the TSDMCN solutions.

If Problem 2.2 has another series of TSDMCN solutions $\{(\tilde{\psi}^k, \tilde{\varpi}^k)\}_{k=1}^K \subset \mathbb{U} \times \mathbb{U}$, then it should meet $\|\nabla\tilde{\psi}^k\|_0^2 \le c\|\nabla\psi^0\|_0^2$ and $\|\nabla\tilde{\varpi}^k\|_0^2 \le c\|\nabla\psi^0\|_0^2$ as well as the following equations:

$$\frac{1}{\Delta t}(\tilde{\psi}^k - \tilde{\psi}^{k-1}, \upsilon) + \frac{\epsilon^2}{2}\big(G(\tilde{\psi}^k)\nabla\tilde{\varpi}^k + G(\tilde{\psi}^{k-1})\nabla\tilde{\varpi}^{k-1}, \nabla\vartheta\big)$$
$$+ \frac{1}{2}\big(f(\tilde{\psi}^k)\nabla\tilde{\psi}^k + f(\tilde{\psi}^{k-1})\nabla\tilde{\psi}^{k-1}, \nabla\upsilon\big) = 0, \quad \forall\vartheta \in \mathbb{U}, \quad 1 \le k \le K, \tag{2.17}$$

$$(\tilde{\varpi}^k, \upsilon) = (\nabla\tilde{\psi}^k, \nabla\upsilon), \qquad\qquad \forall\upsilon \in \mathbb{U}, \quad 1 \le k \le K, \tag{2.18}$$

$$\tilde{\psi}^0 = \psi^0(\boldsymbol{x}), \quad \tilde{\varpi}^0 = -\Delta\psi^0, \qquad\qquad \boldsymbol{x}\in\Omega. \tag{2.19}$$

Let $\theta^k = \psi^k - \tilde{\psi}^k$ and $\Theta^k = \varpi^k - \tilde{\varpi}^k$. Subtracting (2.17)-(2.19) from Problem 2.2 yields

$$\frac{1}{\Delta t}(\theta^k - \theta^{k-1}, \upsilon) + \frac{\epsilon^2}{2}\big(G(\psi^k)\nabla(\Theta^k + \Theta^{k-1}) + \big(G(\psi^k) - G(\tilde{\psi}^k)\big)\nabla\tilde{\varpi}^k, \nabla\vartheta\big)$$
$$+ \frac{\epsilon^2}{2}\big(\big(G(\psi^{k-1}) - G(\psi^k)\big)\nabla\Theta^{k-1} + \big(G(\psi^{k-1}) - G(\tilde{\psi}^{k-1})\big)\nabla\tilde{\varpi}^{k-1}, \nabla\vartheta\big)$$
$$+ \frac{1}{2}\big(\tilde{f}(\psi^k)\nabla(\theta^k + \theta^{k-1}) + \big(\tilde{f}(\psi^k) - \tilde{f}(\tilde{\psi}^k)\big)\nabla\tilde{\psi}^k, \nabla\vartheta\big)$$
$$+ \frac{1}{2}\big(\big(\tilde{f}(\psi^{k-1}) - \tilde{f}(\psi^k)\big)\nabla\theta^{k-1} + \big(\tilde{f}(\psi^{k-1}) - \tilde{f}(\tilde{\psi}^{k-1})\big)\nabla\tilde{\psi}^{k-1}, \nabla\vartheta\big) = 0, \quad \forall\vartheta \in \mathbb{U}, \tag{2.20}$$

$$(\Theta^k, \upsilon) = (\nabla\theta^k, \nabla\upsilon), \quad \forall\upsilon \in \mathbb{U}, \quad 1 \le k \le K, \tag{2.21}$$

$$\theta^0(\boldsymbol{x}) = \omega^0(\boldsymbol{x}) = 0, \qquad \boldsymbol{x}\in\Omega. \tag{2.22}$$

From (2.21) we obtain

$$(\Theta^k - \Theta^{k-1}, \theta^k - \theta^{k-1}) = \big(\nabla(\theta^k - \theta^{k-1}), \nabla(\theta^k - \theta^{k-1})\big), \qquad 1 \le k \le K, \tag{2.23}$$
$$(\Theta^k + \Theta^{k-1}, \Theta^k - \Theta^{k-1}) = \big(\nabla(\theta^k + \theta^{k-1}), \nabla(\Theta^k - \Theta^{k-1})\big), \quad 1 \le k \le K. \tag{2.24}$$

By (2.20), (2.23), (2.24), the LDMV formula, the Hölder and Cauchy inequalities, we obtain

$$\frac{1}{\Delta t}\|\nabla(\theta^k - \theta^{k-1})\|_0^2 + \frac{c_1\epsilon^2}{2}\big(\|\nabla\Theta^k\|_0^2 - \|\nabla\Theta^{k-1}\|_0^2\big) + \frac{c_1}{2}\big(\|\Theta^k\|_0^2 - \|\Theta^{k-1}\|_0^2\big)$$
$$\le \left| \frac{1}{\Delta t}\big(\nabla(\theta^k - \theta^{k-1}), \nabla(\theta^k - \theta^{k-1}) + \frac{1}{2}\big(\tilde{f}(\psi^k)\nabla(\theta^k + \theta^{k-1}), \nabla(\Theta^k - \Theta^{k-1}))\big)\right.$$
$$\left. + \frac{\epsilon^2}{2}\big(G(\psi^k)\nabla(\Theta^k + \Theta^{k-1}), \nabla(\Theta^k - \Theta^{k-1})\big)\right|$$

$$= \left| \frac{1}{\Delta t}(\theta^k - \theta^{k-1}, \Theta^k - \Theta^{k-1}) + \frac{\epsilon^2}{2}\big(G(\psi^k)\nabla(\Theta^k + \Theta^{k-1}), \nabla(\Theta^k - \Theta^{k-1})\big) \right.$$
$$\left. + \frac{1}{2}\big(\tilde{f}(\psi^k)\nabla(\theta^k + \theta^{k-1}), \nabla(\Theta^k - \Theta^{k-1})\big) \right|$$

$$= \left| \frac{\epsilon^2}{2}\big((G(\psi^k) - G(\tilde{\psi}^k) - (G(\psi^{k-1}) - G(\tilde{\psi}^{k-1})))\nabla\tilde{\varpi}^k, \nabla(\Theta^k - \Theta^{k-1})\big) \right.$$
$$+ \frac{\epsilon^2}{2}\big((G(\psi^{k-1}) - G(\psi^k))\nabla\Theta^{k-1}, \nabla(\Theta^k - \Theta^{k-1})\big)$$
$$+ \frac{\epsilon^2}{2}\big((G(\psi^{k-1}) - G(\tilde{\psi}^{k-1}))\nabla(\tilde{\varpi}^k - \tilde{\varpi}^{k-1}), \nabla(\Theta^k - \Theta^{k-1})\big)$$
$$+ \frac{1}{2}\big(\tilde{f}(\psi^k) - \tilde{f}(\tilde{\psi}^k) - (\tilde{f}(\psi^{k-1}) - \tilde{f}(\tilde{\psi}^{k-1}))\nabla\tilde{\psi}^k, \nabla(\Theta^k - \Theta^{k-1})\big)$$
$$+ \frac{1}{2}\big((\tilde{f}(\psi^{k-1}) - \tilde{f}(\psi^k))\nabla\theta^{k-1}, \nabla(\Theta^k - \Theta^{k-1})\big)$$
$$\left. + \frac{1}{2}\big((\tilde{f}(\psi^{k-1}) - \tilde{f}(\tilde{\psi}^{k-1}))\nabla(\tilde{\psi}^k - \tilde{\psi}^{k-1}), \nabla(\Theta^k - \Theta^{k-1})\big) \right|$$

$$\leq c\|\theta^k - \theta^{k-1}\|_0\|\nabla(\Theta^k - \Theta^{k-1})\|_0 + c\Delta t\|\nabla\Theta^{k-1}\|_0\|\nabla(\Theta^k - \Theta^{k-1})\|_0$$
$$+ c\Delta t\|\nabla\theta^{k-1}\|_0\|\nabla(\Theta^k - \Theta^{k-1})\|_0 + c\Delta t\|\nabla\theta^{k-1}\|_0\|\nabla(\Theta^k - \Theta^{k-1})\|_0$$

$$\leq \frac{1}{2\Delta t}\|\theta^k - \theta^{k-1}\|_0^2 + c\Delta t\big(\|\nabla\theta^{k-1}\|_0^2 + \|\nabla\Theta^{k-1}\|_0^2 + \|\nabla\Theta^k\|_0^2\big), \quad 1 \leq k \leq K. \tag{2.25}$$

Simplifying (2.25) yields

$$\epsilon^2\big(\|\nabla\Theta^k\|_0^2 - \|\nabla\Theta^{k-1}\|_0^2\big) + \big(\|\Theta^k\|_0^2 - \|\Theta^{k-1}\|_0^2\big)$$
$$\leq c\Delta t\big(\|\nabla\theta^{k-1}\|_0^2 + \|\Theta^{k-1}\|_1^2 + \|\Theta^k\|_1^2\big), \quad 1 \leq k \leq K. \tag{2.26}$$

Summating (2.26) from 1 to k, using $\Theta^0 = \theta^0 = 0$, when Δt is fully small, satisfying $c\Delta t \leq 1/2$, we obtain

$$\|\Theta^k\|_1^2 \leq c\Delta t \sum_{i=0}^{k-1}\big(\|\nabla\theta^i\|_0^2 + \|\Theta^i\|_1^2\big), \quad 1 \leq k \leq K. \tag{2.27}$$

By applying Gronwall's inequality to (2.27), we get

$$\|\Theta^k\|_1^2 \leq c\Delta t \sum_{i=0}^{k-1}\|\nabla\theta^i\|_0^2 \exp(ck\Delta t^2) \leq c\Delta t \sum_{i=0}^{k-1}\|\nabla\theta^i\|_0^2, \quad 1 \leq k \leq K. \tag{2.28}$$

Further, by (2.21) and the Hölder and Poicaré inequalities, we get

$$\|\nabla\theta^k\|_0^2 \leq \|\theta^k\|_0\|\Theta^k\|_0 \leq c\|\nabla\theta^k\|_0\|\Theta^k\|_1, \quad 1 \leq k \leq K. \tag{2.29}$$

By (2.29), we get

$$\|\nabla\theta^k\|_0 \leq c\|\Theta^k\|_1, \quad 1 \leq k \leq K. \tag{2.30}$$

Substituting (2.30) into (2.28) gets

$$\|\Theta^k\|_1^2 \leq c\Delta t \sum_{i=0}^{k-1}\|\Theta^i\|_1^2, \quad 1 \leq k \leq K. \tag{2.31}$$

Once again applying Gronwall's inequality to (2.31) yields

$$\|\Theta^k\|_1^2 \le 0 \times \exp(ck\Delta t) = 0, \quad 1 \le k \le K.$$

It follows that $\Theta^k = 0$, i.e., $\tilde{\varpi}^k = \varpi^k$ $(1 \le k \le K)$. Further (2.30) we get $\|\nabla\theta^k\|_0 = 0$ so as to get $\theta^k = 0$, namely $\tilde{\psi}^k = \psi^k$ $(1 \le k \le K)$. Hence, the TSDMCN solutions of Problem 2.2 is unique.

(2) The proof of boundedness about the TSDMCN solutions.

When Problem 2.2 has a unique series of TSDMCN solutions, from the proof of the existence of the TSDMCN solutions, the boundedness of TSDMCN solutions is gotten.

(3) The error analysis of the TSDMCN solutions.

With the Taylor expansion, we obtain

$$\psi(t_k) = \psi(t_{k-\frac{1}{2}}) + \frac{\Delta t}{2}\psi'(t_{k-\frac{1}{2}}) + \frac{\Delta t^2}{8}\psi''(t_{k-\frac{1}{2}}) + \frac{\Delta t^3}{48}\psi'''(\varsigma_{1k}), \tag{2.32}$$

$$\psi(t_{k-1}) = \psi(t_{k-\frac{1}{2}}) - \frac{\Delta t}{2}\psi'(t_{k-\frac{1}{2}}) + \frac{\Delta t^2}{8}\psi''(t_{k-\frac{1}{2}}) - \frac{\Delta t^3}{48}\psi'''(\varsigma_{2k}), \tag{2.33}$$

$$g\big(\psi(t_k)\big) = g\big(\psi(t_{k-\frac{1}{2}})\big) + [\psi(t_k) - \psi(t_{k-\frac{1}{2}})]g'\big(\psi(t_{k-\frac{1}{2}})\big) + \mathcal{O}\big(\psi(t_k) - \psi(t_{k-\frac{1}{2}})\big)^2, \tag{2.34}$$

$$g\big(\psi(t_{k-1})\big) = g\big(\psi(t_{k-\frac{1}{2}})\big) + [\psi(t_{k-1}) - \psi(t_{k-\frac{1}{2}})]g'\big(\psi(t_{k-\frac{1}{2}})\big) + \mathcal{O}\big(\psi(t_{k-1}) - \psi(t_{k-\frac{1}{2}})\big)^2, \tag{2.35}$$

where $t_{k-1/2} \le \varsigma_{1k} \le t_k$ and $t_{k-1} \le \varsigma_{2k} \le t_{k-1/2}$. From (2.32)-(2.35) we obtain

$$\psi'(t_{k-\frac{1}{2}}) = \frac{\psi(t_k) - U(t_{k-1})}{\Delta t} - \frac{\Delta t^2}{24}\psi'''(\hat{\xi}_k), \quad t_{k-1} \le \hat{\xi}_k \le t_{k+1}, \tag{2.36}$$

$$\psi(t_{k-\frac{1}{2}}) = \frac{\psi(t_k) + \psi(t_{k-1})}{2} - \frac{\Delta t^2}{16}\psi''(\hat{\varsigma}_k), \quad t_{k-1} \le \hat{\varsigma}_k \le t_{k-\frac{1}{2}}, \tag{2.37}$$

$$g\big(U(t_{k-\frac{1}{2}})\big) = \frac{g\big(U(t_k)\big) + g\big(U(t_{k-1})\big)}{2} - \Delta t^2 R_g(\boldsymbol{x}, t), \tag{2.38}$$

in which $R_g(\boldsymbol{x}, t)$ is a bounded remainder function obtained by (2.32)-(2.35) and $g(\cdot) = G(\cdot)$ or $\tilde{f}(\cdot)$.

Subtracting Problem 2.2 from Problem 2.1 after taking $t = t_{k-1/2}$, and setting

$$u^k = \psi(t_k, \boldsymbol{x}) - \psi^k, \quad w^k = \varpi(t_k, \boldsymbol{x}) - \varpi^k,$$

we obtain the following error system of equations:

$$\frac{1}{\Delta t}(u^k - u^{k-1}, \vartheta) + \frac{\epsilon^2}{2}\big(G\big(\psi(t_{k-\frac{1}{2}})\big)\nabla\big(\varpi(t_k) + \varpi(t_{k-1})\big), \nabla\vartheta\big)$$

$$- \frac{\epsilon^2}{2}\big((G(\psi^k)\nabla\varpi^k + G(\psi^{k-1})\nabla\varpi^{k-1}), \nabla\vartheta\big)$$

$$+ \frac{1}{2}\big(\tilde{f}\big(\psi(t_{k-\frac{1}{2}})\big)\nabla\big(\psi(t_k) + \psi(t_{k-1})\big) - \tilde{f}(\psi^k)\nabla\psi^k - \tilde{f}(\psi^{k-1})\nabla\psi^{k-1}, \nabla\vartheta\big)$$

$$= \Delta t^2(\tilde{R}, \nabla\vartheta), \qquad \forall\, \vartheta \in \mathbb{U}, \quad 1 \le k \le K, \tag{2.39}$$

$$(w^k, \upsilon) = (\nabla u^k, \nabla\upsilon), \quad \forall\, \upsilon \in \mathbb{U}, \quad 1 \le k \le K, \tag{2.40}$$

$$u^0 = w^0 = 0, \qquad \boldsymbol{x} \in \Omega, \tag{2.41}$$

where $\tilde{R}$ is also a bounded remainder function determined by subtracting Problem 2.2 from Problem 2.1.

By (2.40), we obtain

$$(w^k \pm w^{k-1}, u^k - u^{k-1}) = \left(\nabla(u^k \pm u^{k-1}), \nabla(u^k - u^{k-1})\right), \quad 1 \le k \le K, \qquad (2.42)$$
$$(w^k \pm w^{k-1}, w^k - w^{k-1}) = \left(\nabla(u^k \pm u^{k-1}), \nabla(w^k - w^{k-1})\right), \quad 1 \le k \le K. \qquad (2.43)$$

Taking $\vartheta = w^k - w^{k-1}$ in (2.39), by (2.42), (2.43), the LDMV formula, the Hölder and Cauchy inequalities, Green's formula, and Taylor's expansion, we obtain

$$2\|\nabla(u^k - u^{k-1})\|_0^2 + c_1\epsilon^2\Delta t\left(\|\nabla w^k\|_0^2 - \|\nabla w^{k-1}\|_0^2\right) + c_1\Delta t\left(\|w^k\|_0^2 - \|w^{k-1}\|_0^2\right)$$
$$\le \left|2\left(\nabla(u^k - u^{k-1}), \nabla(u^k - u^{k-1})\right) + \Delta t(w^k + w^{k-1}, w^k - w^{k-1})\right.$$
$$\left. + \epsilon^2\Delta t\left(G\left(\psi(t_{k-\frac{1}{2}})\right)\nabla(w^k + w^{k-1}), \nabla(w^k - w^{k-1})\right) + \Delta t^3(\tilde{R}, w^k - w^{k-1})\right|$$
$$\le \left|2\left(\nabla(u^k - u^{k-1}), \nabla(u^k - u^{k-1})\right) + \Delta t^3(\tilde{R}, w^k - w^{k-1})\right.$$
$$\left. + \epsilon^2\Delta t\left(G\left(\psi(t_{k-\frac{1}{2}})\right)\nabla(w^k + w^{k-1}), \nabla(w^k - w^{k-1})\right)\right.$$
$$\left. + \Delta t\left(\tilde{f}\left(\psi(t_{k-\frac{1}{2}})\right)\nabla(u^k + u^{k-1}), \nabla(w^k - w^{k-1})\right)\right|$$
$$= \left|2(w^k - w^{k-1}, u^k - u^{k-1}) + \Delta t^3(\tilde{R}, w^k - w^{k-1})\right.$$
$$+ \frac{\epsilon^2\Delta t}{2}\left(\left(G(\psi^k) + G(\psi^{k-1})\nabla(\varpi^k - \varpi^{k-1}), \nabla(w^k - w^{k-1})\right)\right.$$
$$- \frac{\epsilon^2\Delta t}{2}\left(\left(G\left(\psi(t_k)\right) + G\left(\psi(t_{k-1})\right)\nabla(\varpi^k - \varpi^{k-1}), \nabla(w^k - w^{k-1})\right)\right.$$
$$+ \frac{\epsilon^2\Delta t}{2}\left(\left(G(\psi^k) - G(\psi^{k-1})\right)\nabla(\varpi^k - \varpi^{k-1}), \nabla(w^k - w^{k-1})\right)$$
$$+ \frac{\Delta t}{2}\left(\left(\tilde{f}(\psi^k) + \tilde{f}(\psi^{k-1})\right)\nabla(\psi^k - \psi^{k-1}), \nabla(w^k - w^{k-1})\right)$$
$$- \frac{\Delta t}{2}\left(\left(\tilde{f}\left(\psi(t_k)\right) + \tilde{f}\left(\psi(t_{k-1})\right)\right)\nabla(\psi^k - \psi^{k-1}), \nabla(w^k - w^{k-1})\right)$$
$$\left. + \frac{\Delta t}{2}\left(\left(\tilde{f}(\psi^k) - \tilde{f}(\psi^{k-1})\right)\nabla(\psi^k - \psi^{k-1}), \nabla(w^k - w^{k-1})\right)\right|$$
$$\le c\|\nabla\partial_t\varpi\|_{0,\infty}\max_{v\in\mathbb{U}}|G'(v)|\Delta t^2\left(\|\nabla w^k\|_0 + \|\nabla w^{k-1}\|_0\right)\|w^k - w^{k-1}\|_0$$
$$+ c\Delta t^3\|\nabla\partial_t\varpi\|_{0,\infty}\max_{v\in\mathbb{U}}|G'(v)|\|w^k - w^{k-1}\|_0 + c\Delta t^3\|w^k - w^{k-1}\|_0$$
$$+ c\|\nabla\partial_t\psi\|_{0,\infty}\max_{v\in\mathbb{U}}|\tilde{f}'(v)|\Delta t^2\|\nabla(u^k - u^{k-1})\|_0\|w^k - w^{k-1}\|_0$$
$$+ c\Delta t^3\|\nabla\partial_t\psi\|_{0,\infty}\max_{v\in\mathbb{U}}|\tilde{f}'(v)|\|w^k - w^{k-1}\|_0$$
$$\le \|\nabla(u^k - u^{k-1})\|_0^2 + c\Delta t^6 + c\Delta t^2\left(\|\nabla w^k\|_0^2 + \|\nabla w^{k-1}\|_0^2\right), \quad 1 \le k \le K. \qquad (2.44)$$

By simplifying (2.44), summating from 1 to k, and noting that $w^0 = 0$, when Δt is adequately small, satisfying $c\Delta t \le 1/2$, we obtain

$$\|w^k\|_1^2 \le ck\Delta t^5 + c\Delta t\sum_{j=1}^{k-1}\|w^j\|_1^2, \quad 1 \le k \le K. \qquad (2.45)$$

Applying Gronwall's inequality to (2.45) yields

$$\|w^k\|_1 \le \sqrt{ck\Delta t^5\exp(ck\Delta t)} \le c\Delta t^2, \quad 1 \le k \le K. \qquad (2.46)$$

Further, by (2.40) we obtain

$$\|\nabla u^k\|_0^2 \leq \|u^k\|_0 \|w^k\|_0 \leq c\|\nabla u^k\|_0 \|w^k\|_1, \quad 1 \leq k \leq K. \tag{2.47}$$

Simplifying (2.47) and using (2.46) yield

$$\|\nabla u^k\|_0 \leq c\|w^k\|_1 \leq c\Delta t^2, \quad 1 \leq k \leq K. \tag{2.48}$$

To combine (2.46) with (2.48) yields the error estimates of TSDMCN solutions. The proof of Theorem 2.1 ends. $\qquad\square$

Remark 2.1. Theorem 2.1 shows that the TSDMCN solutions to Problem 2.2 are uniquely existing and stable, and achieve second-order time precision.

2.2. The TGMFECN method for the INFOCH equation

To create the TGMFECN method, we need to use the two-grid MFE method to further discretize the spatial variables in Problem 2.2. For this end, we assume that $\mathfrak{J}_H$ is a quasi-uniform coarse grid division on $\bar{\Omega}$ and H indicates the max-diameter for all FEs in $\mathfrak{J}_H$. Then, the FE space on $\mathfrak{J}_H$ is defined as

$$\mathbb{U}_H = \{v_H \in C(\bar{\Omega}) \cap \mathbb{U} : v_H|_E \in \mathbb{P}_l(E), \ \forall E \in \mathfrak{J}_H\},$$

where $\mathbb{P}_l(E)$ ($l \geq 1$) indicates the space of polynomials with degree less than or equal to l defined on $E \in \mathfrak{J}_H$.

Further, we assume that $\mathfrak{J}_h$ indicates a quasi-uniform fine grid division on $\bar{\Omega}$ and h indicates the max-diameter for all FEs in $\mathfrak{J}_h$ ($h \ll H$). Likewise, the FE space defined on $\mathfrak{J}_h$ is taken as

$$\mathbb{U}_h = \{v_h \in C(\bar{\Omega}) \cap \mathbb{U} : v_h|_e \in \mathbb{P}_l(e), \ \forall e \in \mathfrak{J}_h\}.$$

Define two Ritz projections $R_\kappa : \mathbb{U} \to \mathbb{U}_\kappa$ ($\kappa = H, h$), i.e., for $u \in \mathbb{U}$, there are two unique $R_\kappa u \in \mathbb{U}_\kappa$, satisfying

$$\big(\nabla(u - R_\kappa u), \nabla v_\kappa\big) = 0, \quad \forall v_\kappa \in \mathbb{U}_\kappa, \quad \kappa = H, h, \tag{2.49}$$

and the error estimates

$$|u - R_\kappa u|_r \leq c\kappa^{l+1-r}, \quad \forall u \in \mathbb{U} \cap H^{l+1}(\Omega), \quad r = 0, 1, \quad \kappa = h, H, \tag{2.50}$$

where c presented in the context indicates a generical positive constant independent of $\Delta t, h$, and H, and is different at different places. Thereupon, a new TGMFECN method can be created as follows.

Problem 2.3. Step 1. Find $\{(\psi_H^k, \varpi_H^k)\}_{k=1}^K \subset \mathbb{U}_H \times \mathbb{U}_H$ defined on the coarse grid division $\mathfrak{J}_H$, satisfying the following nonlinear system of equations:

$$\frac{1}{\Delta t}\big(\psi_H^k - \psi_H^{k-1}, \vartheta_H\big) + \frac{\epsilon^2}{2}\big(G(\psi_H^k)\nabla\varpi_H^k + G(\psi_H^{k-1})\nabla\varpi_H^{k-1}, \nabla\vartheta_H\big)$$

$$+ \frac{1}{2}\big(\tilde{f}(\psi_H^k)\nabla\psi_H^k + \tilde{f}(\psi_H^{k-1})\nabla\psi_H^{k-1}, \nabla\vartheta_H\big) = 0, \quad \forall\vartheta_H \in \mathbb{U}_H, \quad 1 \leq k \leq K, \tag{2.51}$$

$$\big(\varpi_H^k, \upsilon_H\big) = \big(\nabla\psi_H^k, \nabla\upsilon_H\big), \qquad\qquad \forall\upsilon_H \in \mathbb{U}_H, \quad 1 \leq k \leq K, \tag{2.52}$$

$$\psi_H^0 = R_H\psi^0(\boldsymbol{x}), \quad \varpi_H^0 = R_H\varpi^0(\boldsymbol{x}), \qquad\qquad \boldsymbol{x} \in \Omega. \tag{2.53}$$

Step 2. Find $\{(\psi_h^k, \varpi_h^k)\}_{k=1}^K \subset \mathbb{U}_h \times \mathbb{U}_h$ defined on the fine grid division $\mathfrak{J}_h$, satisfying the following linear system of equations:

$$\frac{1}{\Delta t}\big(\psi_h^k - \psi_h^{k-1}, \vartheta_h\big) + \frac{\epsilon^2}{2}\big(G\big(\psi_H^k\big)\nabla\varpi_h^k + G\big(\psi_h^{k-1}\big)\nabla\varpi_h^{k-1}, \nabla\vartheta_h\big)$$

$$+ \frac{1}{2}\big(\tilde{f}\big(\psi_H^k\big)\nabla\psi_h^k + \tilde{f}\big(\psi_h^{k-1}\big)\nabla\psi_h^{k-1}, \nabla\vartheta_h\big) = 0, \quad \forall\,\vartheta_h \in \mathbb{U}_h, \quad 1 \le k \le K, \tag{2.54}$$

$$\big(\varpi_h^k, \upsilon_h\big) = \big(\nabla\psi_h^k, \nabla\upsilon_h\big), \qquad\qquad\qquad \forall\,\upsilon_h \in \mathbb{U}_h, \quad 1 \le k \le K, \tag{2.55}$$

$$\psi_h^0 = R_h\psi^0(\boldsymbol{x}), \quad \varpi_h^0 = R_h\varpi^0(\boldsymbol{x}), \qquad\qquad \boldsymbol{x}\in\Omega. \tag{2.56}$$

Remark 2.2. Problem 2.2 may be simplified by other form two-grid FE methods, but Problem 2.3 is the simplest and most convenient.

For Problem 2.3, we obtain the following results.

Theorem 2.2. *Under the same conditions as Theorem* 2.1, *Problem* 2.3 *has a unique series of solutions* $\{(\psi_H^k, \varpi_H^k)\}_{k=1}^K \subset \mathbb{U}_H \times \mathbb{U}_H$ *defined on the coarse grid division* $\mathfrak{J}_H$ *and a unique series of solutions* $\{(\psi_h^k, \varpi_h^k)\}_{k=1}^K \subset \mathbb{U}_h \times \mathbb{U}_h$ *defined on the fine grid division* $\mathfrak{J}_h$, *respectively, satisfying the following unconditional boundedness, i.e., unconditional stability:*

$$\|\varpi_H^k\|_1 + \|\varpi_h^k\|_1 + \|\psi_H^k\|_1 + \|\psi_h^k\|_1 \le c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big), \quad 1 \le k \le K. \tag{2.57}$$

When $\Delta t = \mathcal{O}(h)$ *and* $h = \mathcal{O}(H^{1+1/l})$, *the series of solutions* $\{(\psi_H^k, \varpi_H^k)\}_{k=1}^K$ *and* $\{(\psi_h^k, \varpi_h^k)\}_{k=1}^K$ *have the following error estimations:*

$$\big\|\psi(t_k) - \psi_H^k\big\|_0 + H\big|\psi(t_k) - \psi_H^k\big|_1 + \big\|\varpi(t_k) - \varpi_H^k\big\|_0 + H\big|\varpi(t_k) - \varpi_H^k\big|_1$$
$$\le c(\Delta t^2 + H^{l+1}), \quad 1 \le k \le K, \tag{2.58}$$

$$\big\|\psi(t_k) - \psi_h^k\big\|_0 + h\big|\psi(t_k) - \psi_h^k\big|_1 + \big\|\varpi(t_k) - \varpi_h^k\big\|_0 + h\big|\varpi(t_k) - \varpi_h^k\big|_1$$
$$\le c(\Delta t^2 + h^{l+1}), \quad 1 \le k \le K, \tag{2.59}$$

where $\psi(t_k) = \psi(t_k, \boldsymbol{x})$ *and* $\varpi(t_k) = \varpi(t_k, \boldsymbol{x})$ $(1 \le k \le K)$.

Proof. The demonstration of Theorem 2.2 constitutes the following two steps.

(1) Prove the existence and unconditional stability of the TGMFECN solutions.

(i) Prove the existence as well as unconditional stability for the solutions of Problem 2.3 defined on the coarse grid division $\mathfrak{J}_H$.

Because the system of Eqs. (2.51)-(2.53) has the same structure as the system of Eqs. (2.5)-(2.7), by using the same approach as demonstrating Theorem 2.2, we can demonstrate that the system of Eqs. (2.51)-(2.53) has a unique series of solutions $\{(\psi_H^k, \varpi_H^k)\}_{k=1}^K \subset \mathbb{U}_H \times \mathbb{U}_H$, satisfying

$$\big\|\psi_H^k\big\|_1 + \big\|\varpi_H^k\big\|_1 \le c\big(\|\psi^0\|_1 + \|\omega^0\|_1\big), \quad 1 \le k \le K. \tag{2.60}$$

(ii) Prove the existence along with unconditional stability for the solutions of Problem 2.3 defined on the fine grid division $\mathfrak{J}_h$.

The linear system of Eqs. (2.54)-(2.56) is rewritten into the following problem.

Find $\{(\psi_h^k, \varpi_h^k)\}_{k=1}^K \subset \mathbb{U}_h \times \mathbb{U}_h$, satisfying the following linear system of equations:

$$\begin{cases} \tilde{A}\big((\psi_h^k, \varpi_h^k), (\upsilon_h, \vartheta_h)\big) = \tilde{F}(\upsilon_h, \vartheta_h), & \forall\,(\upsilon_h, \vartheta_h) \in \mathbb{U}_h \times \mathbb{U}_h, \\ \psi_h^0 = R_h\psi^0(\boldsymbol{x}), \quad \varpi_h^0 = R_h\varpi^0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \end{cases} \tag{2.61}$$

where

$$
\tilde{A}\big((\psi_h^k,\varpi_h^k),(\upsilon_h,\vartheta_h)\big) = (\nabla\psi_h^k,\nabla\upsilon_h) - (\varpi_h^k,\upsilon_h) + \frac{\epsilon^2\Delta t}{2}\big(G(\psi_H^k)\nabla\varpi_h^k,\nabla\vartheta_h\big) + (\psi_h^k,\vartheta_h)
$$

$$
+ \frac{1}{2}\big(\tilde{f}(\psi_H^k)\nabla\psi_h^k,\nabla\vartheta_h\big),
$$

$$
\tilde{F}(\upsilon_h,\vartheta_h) = -\frac{\epsilon^2\Delta t}{2}\big(G(\psi_h^{k-1})\nabla\varpi_h^{k-1},\nabla\vartheta_h\big) - \frac{\Delta t}{2}\big(\tilde{f}(\psi_h^{k-1})\nabla\psi_h^{k-1},\nabla\vartheta_h\big) + (\psi_h^{k-1},\vartheta_h).
$$

Using the Hölder and Cauchy inequalities, we obtain

$$
\begin{aligned}
|\tilde{F}(\upsilon_h,\vartheta_h)| &\le \big\|\psi_h^{k-1}\big\|_0\|\vartheta_h\|_1 + \frac{c_2\epsilon^2\Delta t}{2}\big\|\nabla\varpi_h^{k-1}\big\|_0\|\vartheta_h\|_1 + \frac{c_2\Delta t}{2}\big\|\nabla\psi_h^{k-1}\big\|_0\|\vartheta_h\|_1 \\
&\le M_0\big(\|\vartheta_h\|_1^2 + \|\upsilon_h\|_1^2\big)^{\frac{1}{2}}, \quad \forall\,(\upsilon_h,\vartheta_h)\in\mathbb{U}_h\times\mathbb{U}_h, \tag{2.62}
\end{aligned}
$$

$$
\begin{aligned}
\big|\tilde{A}\big((\psi_h^k,\varpi_h^k),(\upsilon_h,\vartheta_h)\big)\big| &= \big|(\psi_h^k,\vartheta_h) - (\varpi_h^k,\upsilon_h) + (\nabla\psi_h^k,\nabla\upsilon_h) \\
&\quad + \frac{\epsilon^2\Delta t}{2}\big(G(\psi_H^k)\nabla\varpi_h^k,\nabla\vartheta_h\big) + \frac{\Delta t}{2}\big(\tilde{f}(\psi_H^k)\nabla\psi_h^k,\nabla\vartheta_h\big)\big| \\
&\le \big\|\psi_h^k\big\|_1\|\vartheta_h\|_1 + \big\|\varpi_h^k\big\|_1\|\upsilon_h\|_1 + \big\|\psi_h^k\big\|_1\|\upsilon_h\|_1 \\
&\quad + \frac{c_2\epsilon^2\Delta t}{2}\big\|\varpi_h^k\big\|_1\|\vartheta_h\|_1 + \frac{c_2\Delta t}{2}\big\|\psi_h^k\big\|_1\|\vartheta_h\|_1 \\
&\le \max\left\{1 + \frac{c_2\epsilon^2\Delta t}{2}, 1 + \frac{c_2\Delta t}{2}\right\}\big(\big\|\varpi_h^k\big\|_1 + \big\|\psi_h^k\big\|_1\big)\big(\|\vartheta_h\|_1 + \|\nabla\upsilon_h\|_1\big) \\
&\le M_1\left[\big(\big\|\psi_h^k\big\|_1^2 + \big\|\varpi_h^k\big\|_1^2\big)^{1/2}\big(\|\upsilon_h\|_1^2 + \|\vartheta_h\|_1^2\big)^{\frac{1}{2}}\right], \\
&\qquad\qquad \forall\,(\psi_h^k,\varpi_h^k),(\upsilon_h,\vartheta_h)\in\mathbb{U}_h\times\mathbb{U}_h, \tag{2.63}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{A}\big((\psi_h^k,\varpi_h^k),(\psi_h^k,\varpi_h^k)\big) &= (\psi_h^k,\varpi_h^k) - (\varpi_h^k,\psi_h^k) + (\nabla\psi_h^k,\nabla\psi_h^k) \\
&\quad + \frac{\epsilon^2\Delta t}{2}\big(G(\psi_H^k)\nabla\varpi_h^k,\nabla\varpi_h^k\big) + \frac{\Delta t}{2}\big(\tilde{f}(\psi_H^k)\nabla\psi_h^k,\nabla\varpi_h^k\big) \\
&\ge \big\|\nabla\psi_h^k\big\|_0^2 + \frac{c_1\epsilon^2\Delta t}{2}\big\|\nabla\varpi_h^k\big\|_0^2 - \frac{1}{2}\big\|\nabla\psi_h^k\big\|_0^2 - \frac{c_2^2\Delta t^2}{2}\big\|\nabla\varpi_h^k\big\|_0^2 \\
&\ge \alpha_0\left(\big\|\nabla\psi_h^k\big\|_0^2 + \big\|\nabla\varpi_h^k\big\|_0^2\right), \quad \forall\,(\psi_h^k,\varpi_h^k)\in\mathbb{U}_h\times\mathbb{U}_h, \tag{2.64}
\end{aligned}
$$

when

$$
M_0 = \big\|\psi_h^{k-1}\big\|_0 + \frac{c_2\epsilon^2\Delta t}{2}\big\|\nabla\varpi_h^{k-1}\big\|_0 + \frac{c_2\Delta t}{2}\big\|\nabla\psi_h^{k-1}\big\|_0,
$$

$$
M_1 = \sqrt{2}\max\left\{1 + \frac{c_2\epsilon^2\Delta t}{2}, 1 + \frac{c_2\Delta t}{2}\right\},
$$

$$
\alpha_0 = \min\left\{\frac{1}{2}, \frac{c_1\epsilon^2\Delta t}{2} - \frac{c_2^2\Delta t^2}{2}\right\} > 0,
$$

when Δt is adequately small, satisfying $\Delta t < c_1/c_2^2$. Hence, $\tilde{F}(\cdot,\cdot)$ is a bounded linear functional and $\tilde{A}((\cdot,\cdot),(\cdot,\cdot))$ is a bounded and positive definite bilinear functional when $\psi_H^k,\varpi_H^k,\psi_h^{k-1}$, and ϖ_h^{k-1} are given. Thus, by Lax-Milgram's theorem (see [26, Theorem 1.2.1]), we assert that (2.61), i.e., Step 2 of Problem 2.3 has a unique set of solutions $\{(\psi_h^k,\varpi_h^k)\}_{k=1}^K\subset\mathbb{U}_h\times\mathbb{U}_h$. And by (2.62), (2.64), and the inverse estimation theorem (see [26, Corollary 1.3.2]), when Δt and h are sufficiently small such that $c\max\{\Delta t,h\}\le 1$, we obtain

$$
\big\|\nabla\psi_h^k\big\|_0 + \big\|\nabla\varpi_h^k\big\|_0 \le c\max\{\Delta t,h\}\big(\big\|\nabla\varpi_h^{k-1}\big\|_0 + \big\|\nabla\psi_h^{k-1}\big\|_0\big)
$$

$$\leq \left\|\nabla\varpi_h^{k-1}\right\|_0 + \left\|\nabla\psi_h^{k-1}\right\|_0 \leq \cdots$$

$$\leq \left\|\nabla\varpi_h^0\right\|_0 + \left\|\nabla\psi_h^0\right\|_0 \leq \left\|\nabla\varpi^0\right\|_0 + \left\|\nabla\psi^0\right\|_0, \quad 1 \leq k \leq K. \tag{2.65}$$

Combining (2.60) with (2.65) yields (2.57).

(2) Prove the error estimations for the TGMFECN solutions.

Let

$$w_\kappa^k = \varpi^k - \varpi_\kappa^k, \quad \sigma_\kappa^k = \varpi^k - R_\kappa\varpi^k, \quad \theta_\kappa^k = R_\kappa\varpi^k - \varpi_\kappa^k,$$

$$\tilde{w}_\kappa^k = \psi^k - \psi_\kappa^k, \quad \tilde{\sigma}_\kappa^k = \psi^k - R_\kappa\psi^k, \quad \tilde{\theta}_\kappa^k = R_\kappa\psi^k - \psi_\kappa^k, \quad \kappa = H, h.$$

By (2.6), (2.52), (2.55), (2.2), and (2.49), we obtain

$$\left(\left\|\nabla\tilde{w}_\kappa^k\right\|_0^2 - \left\|\nabla\tilde{w}_\kappa^{k-1}\right\|_0^2\right) = \left(\nabla\left(\tilde{w}_\kappa^k + \tilde{w}_\kappa^{k-1}\right), \nabla\left(\tilde{w}_\kappa^k - \tilde{w}_\kappa^{k-1}\right)\right)$$

$$= \left(\nabla\left(\tilde{w}_\kappa^k + \tilde{w}_\kappa^{k-1}\right), \nabla\left(\tilde{\sigma}_\kappa^k - \tilde{\sigma}_\kappa^{k-1}\right)\right) + \left(\nabla\left(\tilde{w}_\kappa^k + \tilde{w}_\kappa^{k-1}\right), \nabla\left(\tilde{\theta}_\kappa^k - \tilde{\theta}_\kappa^{k-1}\right)\right)$$

$$= \left(\nabla\left(\tilde{\sigma}_\kappa^k + \tilde{\sigma}_\kappa^{k-1}\right), \nabla\left(\tilde{\sigma}_\kappa^k - \tilde{\sigma}_\kappa^{k-1}\right)\right) + \left(w_\kappa^k + w_\kappa^{k-1}, \tilde{\theta}_\kappa^k - \tilde{\theta}_\kappa^{k-1}\right)$$

$$= \left(\nabla\left(\tilde{\sigma}_\kappa^k + \tilde{\sigma}_\kappa^{k-1}\right), \nabla\left(\tilde{\sigma}_\kappa^k - \tilde{\sigma}_\kappa^{k-1}\right)\right) + \left(w_\kappa^k + w_\kappa^{k-1}, \tilde{\sigma}_\kappa^k - \tilde{\sigma}_\kappa^{k-1}\right)$$

$$+ \left(\sigma_\kappa^k + \sigma_\kappa^{k-1}, \tilde{w}_\kappa^k - \tilde{w}_\kappa^{k-1}\right) + \left(\tilde{w}_\kappa^k - \tilde{w}_\kappa^{k-1}, \theta_\kappa^k + \theta_\kappa^{k-1}\right), \tag{2.66}$$

$$c_1\left(\left|w_\kappa^k\right|_1^2 - \left|w_\kappa^{k-1}\right|_1^2\right) \leq \left(G(\psi^k)\nabla\left(w_\kappa^k + w_\kappa^{k-1}\right), \nabla\left(\sigma_\kappa^k - \sigma_\kappa^{k-1}\right)\right)$$

$$= \left(G(\psi^k)\nabla\left(w_\kappa^k + w_\kappa^{k-1}\right), \nabla\left(\sigma_\kappa^k - \sigma_\kappa^{k-1}\right)\right) + \left(G(\psi^k)\nabla\left(w_\kappa^k + w_\kappa^{k-1}\right), \nabla\left(\theta_\kappa^k - \theta_\kappa^{k-1}\right)\right)$$

$$= \left(G(\psi^k)\nabla\left(w_\kappa^k + w_\kappa^{k-1}\right), \nabla\left(\sigma_\kappa^k - \sigma_\kappa^{k-1}\right)\right) + \left(G(\psi^k)\nabla\varpi^k + G(\psi^{k-1})\nabla\varpi^{k-1}, \nabla\left(\theta_\kappa^k - \theta_\kappa^{k-1}\right)\right)$$

$$- \left(G(\psi_\kappa^k)\nabla\varpi_\kappa^k + G(\psi_\kappa^{k-1})\nabla\varpi_\kappa^{k-1}, \nabla\left(\theta_\kappa^k - \theta_\kappa^{k-1}\right)\right)$$

$$+ \left((G(\psi_\kappa^{k-1}) - G(\psi^k))\nabla\varpi_\kappa^{k-1}, \nabla\left(\theta_\kappa^k - \theta_\kappa^{k-1}\right)\right)$$

$$+ \left((G(\psi^k) - G(\psi^{k-1}))\nabla\varpi^{k-1} + (G(\psi_\kappa^k) - G(\psi^k))\nabla\varpi_\kappa^k, \nabla\left(\theta_\kappa^k - \theta_\kappa^{k-1}\right)\right), \tag{2.67}$$

$$c_1\left(\left\|w_\kappa^k\right\|_0^2 - \left\|w_\kappa^{k-1}\right\|_0^2\right) = c_1\left(\nabla\left(\tilde{w}_\kappa^k + \tilde{w}_\kappa^{k-1}\right), \nabla\left(w_\kappa^k - w_\kappa^{k-1}\right)\right)$$

$$\leq \left(\tilde{f}(\psi^k)\nabla\left(\tilde{w}_H^k + \tilde{w}_\kappa^{k-1}\right), \nabla\left(\sigma_\kappa^k - \sigma_\kappa^{k-1}\right)\right) + \left(\tilde{f}(\psi^k)\nabla\left(\tilde{w}_\kappa^k + \tilde{w}_\kappa^{k-1}\right), \nabla\left(\theta_\kappa^k - \theta_\kappa^{k-1}\right)\right)$$

$$= \left(\tilde{f}(\psi^k)\nabla\left(\tilde{w}_\kappa^k + \tilde{w}_\kappa^{k-1}\right), \nabla\left(\sigma_\kappa^k - \sigma_\kappa^{k-1}\right)\right) + \left(\tilde{f}(\psi^k)\nabla\psi^k + \tilde{f}(\psi^{k-1})\nabla\psi^{k-1}, \nabla\left(\theta_\kappa^k - \theta_\kappa^{k-1}\right)\right)$$

$$- \left(\tilde{f}(\psi_H^k)\nabla\psi_H^k + \tilde{f}(\psi_H^{k-1})\nabla\psi_H^{k-1}, \nabla\left(\theta_H^k - \theta_H^{k-1}\right)\right)$$

$$+ \left((\tilde{f}(\psi_\kappa^{k-1}) - \tilde{f}(\psi^k))\nabla\psi_\kappa^{k-1}, \nabla\left(\theta_\kappa^k - \theta_\kappa^{k-1}\right)\right)$$

$$+ \left((\tilde{f}(\psi^k) - \tilde{f}(\psi^{k-1}))\nabla\psi^{k-1} + (\tilde{f}(\psi_\kappa^k) - \tilde{f}(\psi^k))\nabla\psi_\kappa^k, \nabla\left(\theta_\kappa^k - \theta_\kappa^{k-1}\right)\right), \tag{2.68}$$

$$\left\|\nabla\tilde{w}_\kappa^k\right\|_0 \leq c\left\|w_\kappa^k\right\|_0 \leq c\left\|\nabla w_\kappa^k\right\|_0, \quad \kappa = H, h, \quad 1 \leq k \leq K. \tag{2.69}$$

(a) Prove the error estimates of solutions $\{(\psi_H^k, \varpi_H^k)\}_{k=1}^K$ of Problem 2.3 defined on $\mathfrak{J}_H$.

Subtracting the system of Eqs. (2.51)-(2.53) from the system of Eqs. (2.5)-(2.7) taking $\vartheta = \vartheta_H$ and $\upsilon = \upsilon_H$, we obtain

$$\frac{1}{\Delta t}\left(\tilde{w}_H^k - \tilde{w}_H^{k-1}, \vartheta_H\right) + \frac{\epsilon^2}{2}\left(G(\psi^k)\nabla\varpi^k - G(\psi_H^k)\nabla\varpi_H^k, \nabla\vartheta_H\right)$$

$$+ \frac{\epsilon^2}{2}\left(G(\psi^{k-1})\nabla\varpi^{k-1} - G(\psi_H^{k-1})\nabla\varpi_H^{k-1}, \nabla\vartheta_H\right) + \frac{1}{2}\left(\tilde{f}(\psi^k)\nabla\psi^k - \tilde{f}(\psi_H^k)\nabla\psi_H^k, \nabla\vartheta_H\right)$$

$$+ \frac{1}{2}\left(\tilde{f}(\psi^{k-1})\nabla\psi^{k-1} - \tilde{f}(\psi_H^{k-1})\nabla\psi_H^{k-1}, \nabla\vartheta_H\right) = 0, \quad \forall\vartheta_H \in \mathbb{U}_H, \quad 1 \leq k \leq K, \tag{2.70}$$

$$\left(w_H^k, v_H\right) = \left(\nabla \tilde{w}_H^k, \nabla v_H\right), \qquad\qquad \forall\, v_H \in \mathbb{U}_H, \quad 1 \le k \le K, \qquad (2.71)$$

$$\tilde{w}_H^0 = \psi^0(\boldsymbol{x}) - R_H \psi^0(\boldsymbol{x}), \quad w_H^0 = \varpi^0(\boldsymbol{x}) - R_H \varpi^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \qquad (2.72)$$

By (2.2), (2.66)-(2.70), the LDMV formula, and the Hölder and Cauchy inequalities, when $\Delta t = \mathcal{O}(H^2)$, noting that

$$\left|\sigma_H^k - \sigma_H^k\right|_s = \left|(\varpi^k - \varpi^{k-1}) - R_H(\varpi^k - \varpi^{k-1})\right|_s \le c\Delta t H^{l+1-s}, \quad s = 0, 1,$$

$$\left|\tilde{\sigma}_H^k - \tilde{\sigma}_H^k\right|_s = \left|(\psi^k - \psi^{k-1}) - R_H(\psi^k - \psi^{k-1})\right|_s \le c\Delta t H^{l+1-s}, \quad\ \ s = 0, 1,$$

we obtain

$$\frac{1}{\Delta t}\left|\tilde{w}_H^k - \tilde{w}_H^{k-1}\right|_1^2 + \frac{c_1 \epsilon^2}{2}\left(\left|w_H^k\right|_1^2 - \left|w_H^{k-1}\right|_1^2\right) + \frac{c_1}{2}\left(\left\|w_H^k\right\|_0^2 - \left\|w_H^{k-1}\right\|_0^2\right)$$

$$\le \frac{1}{\Delta t}\left(\nabla\left(\tilde{\sigma}_H^k + \tilde{\sigma}_H^{k-1}\right), \nabla\left(\tilde{\sigma}_H^k - \tilde{\sigma}_H^{k-1}\right)\right) + \frac{1}{\Delta t}\left(w_H^k + w_H^{k-1}, \tilde{\sigma}_H^k - \tilde{\sigma}_H^{k-1}\right)$$

$$+ \frac{1}{\Delta t}\left(\sigma_H^k + \sigma_H^{k-1}, \tilde{w}_H^k - \tilde{w}_H^{k-1}\right) + \frac{\epsilon^2}{2}\left(G(\psi^k)\nabla\left(w_H^k + w_H^{k-1}\right), \nabla\left(\sigma_H^k - \sigma_H^{k-1}\right)\right)$$

$$+ \frac{\epsilon^2}{2}\left(\left(G(\psi^k) - G(\psi^{k-1})\right)\nabla w_H^{k-1} + \left(G(\psi^k) - G(\psi_H^k)\right)\nabla\varpi_H^k, \nabla\left(\theta_H^k - \theta_H^{k-1}\right)\right)$$

$$+ \frac{\epsilon^2}{2}\left(\left(G(\psi^{k-1}) - G(\psi_H^{k-1})\right)\nabla\varpi_H^{k-1}, \nabla\left(w_H^k - w_H^{k-1} - \sigma_H^k + \sigma_H^{k-1}\right)\right)$$

$$+ \frac{1}{2}\left(\left(\tilde{f}(\psi^k) - \tilde{f}(\psi^{k-1})\right)\nabla\tilde{w}_H^{k-1} + \left(\tilde{f}(\psi^k) - \tilde{f}(\psi_H^k)\right)\nabla\psi_H^k, \nabla\left(\theta_H^k - \theta_H^{k-1}\right)\right)$$

$$+ \frac{1}{2}\left(\left(\tilde{f}(\psi^{k-1}) - \tilde{f}(\psi_H^{k-1})\right)\nabla\psi_H^{k-1}, \nabla\left(w_H^k - w_H^{k-1} - \sigma_H^k + \sigma_H^{k-1}\right)\right)$$

$$\le \frac{1}{\Delta t}\left\|\tilde{w}_H^k - \tilde{w}_H^{k-1}\right\|_0^2 + cH^{2l} + c\Delta t\left\|w_H^k + w_H^{k-1}\right\|_0^2$$

$$+ \epsilon^2 \Delta t \max_{v_H \in \mathbb{U}_H}\left|G'(v_H)\right|\left|w_H^{k-1}\right|_1\left|w_H^k - w_H^{k-1} - \sigma_H^k + \sigma_H^{k-1}\right|_1$$

$$+ \epsilon^2 \max_{v_H \in \mathbb{U}_H}\left|G'(v_H)\right|\left|\psi_H^{k-1}\right|_{1,\infty}\left|w_H^k\right|_1\left\|w_H^k - w_H^{k-1} - \sigma_H^k + \sigma_H^{k-1}\right\|_0$$

$$+ \epsilon^2 \max_{v_H \in \mathbb{U}_H}\left|G'(v_H)\right|\left|\varpi_H^{k-1}\right|_{1,\infty}\left|w_H^{k-1}\right|_1\left\|w_H^k - w_H^{k-1} - \sigma_H^k + \sigma_H^{k-1}\right\|_0$$

$$+ \Delta t \max_{v_H \in \mathbb{U}_H}\left|\tilde{f}'(v_H)\right|\left|\tilde{w}_H^{k-1}\right|_1\left|w_H^k - w_H^{k-1} - \sigma_H^k + \sigma_H^{k-1}\right|_1$$

$$+ \max_{v_H \in \mathbb{U}_H}\left|\tilde{f}'(v_H)\right|\left|\psi_H^{k-1}\right|_{1,\infty}\left|\tilde{w}_H^k\right|_1\left\|w_H^k - w_H^{k-1} - \sigma_H^k + \sigma_H^{k-1}\right\|_0$$

$$+ \max_{v_H \in \mathbb{U}_H}\left|\tilde{f}'(v_H)\right|\left|\psi_H^{k-1}\right|_{1,\infty}\left|\tilde{w}_H^{k-1}\right|_1\left\|w_H^k - w_H^{k-1} - \sigma_H^k + \sigma_H^{k-1}\right\|_0$$

$$\le \frac{1}{2\Delta t}\left\|\tilde{w}_H^k - \tilde{w}_H^{k-1}\right\|_0^2 + c\Delta t\left(\left|w_H^k\right|_1^2 + \left|w_H^{k-1}\right|_1^2\right) + c\Delta t H^{2l}, \quad 1 \le k \le K. \qquad (2.73)$$

Simplifying and summating (2.73) from 1 to k, using (2.50), we obtain

$$\left\|w_H^k\right\|_1^2 \le c\Delta t \sum_{j=0}^{k}\left\|w_H^j\right\|_1^2 + ck\Delta t H^{2l}, \quad 1 \le k \le K. \qquad (2.74)$$

When Δt is adequately small, satisfying $c\Delta t \le 1/2$, by (2.74), we get

$$\left\|w_H^k\right\|_1^2 \le c\Delta t \sum_{j=0}^{k-1}\left\|w_H^j\right\|_1^2 + cH^{2l}, \quad 1 \le k \le K. \qquad (2.75)$$

By applying the Gronwall inequality to (2.75), we obtain

$$\left\|w_H^k\right\|_1 \le c\sqrt{H^{2l}\exp(ck\Delta t)} \le cH^l, \quad 1 \le k \le K. \tag{2.76}$$

Combining (2.69) with (2.76) yields

$$\left\|\tilde{w}_H^k\right\|_1 + \left\|w_H^k\right\|_1 \le 2\left\|w_H^k\right\|_1 \le cH^l, \quad 1 \le k \le K. \tag{2.77}$$

Further, using the Nitsche technique (see [26, Theorem 1.9.3]) and (2.77), we can directly obtain the following error estimations:

$$\left\|\tilde{w}_H^k\right\|_0 + \left\|w_H^k\right\|_0 + H\left\|\tilde{w}_H^k\right\|_1 + H\left\|w_H^k\right\|_1 \le cH^{l+1}, \quad 1 \le k \le K. \tag{2.78}$$

Combining (2.78) with Theorem 2.1 yields (2.58).

(b) Prove the error estimates of solutions $\{(\psi_h^k, \varpi_h^k)\}_{k=1}^K$ of Problem 2.3 defined on $\mathfrak{J}_h$.

Subtracting the system of Eqs. (2.54)-(2.56) from the system of Eqs. (2.5)-(2.7) and taking $\vartheta = \vartheta_h$ and $\upsilon = \upsilon_h$, respectively, we obtain

$$\frac{1}{\Delta t}\big(\tilde{w}_h^k - \tilde{w}_h^{k-1}, \vartheta_h\big) + \frac{\epsilon^2}{2}\big(G(\psi^k)\nabla\varpi^k - G\big(\psi_H^k\big)\nabla\varpi_h^k, \nabla\vartheta_h\big)$$

$$+ \frac{\epsilon^2}{2}\big(G(\psi^{k-1})\nabla\varpi^{k-1} - G\big(\psi_h^{k-1}\big)\nabla\varpi_h^{k-1}, \nabla\vartheta_h\big)$$

$$+ \frac{1}{2}\big(\tilde{f}(\psi^{k-1})\nabla\psi^{k-1} - \tilde{f}(\psi_h^{k-1})\nabla\psi_h^{k-1}, \nabla\vartheta_H\big)$$

$$+ \frac{1}{2}\big(\tilde{f}(\psi^k)\nabla\psi^k - \tilde{f}(\psi_H^k)\nabla\psi_h^k, \nabla\vartheta_h\big) = 0, \qquad \forall\,\vartheta_h \in \mathbb{U}_h, \quad 1 \le k \le K, \tag{2.79}$$

$$\big(w_h^k, \upsilon_h\big) = \big(\nabla\tilde{w}_h^k, \nabla\upsilon_h\big), \qquad\qquad \forall\,\upsilon_H \in \mathbb{U}_H, \quad 1 \le k \le K, \tag{2.80}$$

$$\tilde{w}_h^0 = \psi^0(\boldsymbol{x}) - R_h\psi^0(\boldsymbol{x}), \quad w_h^0 = \varpi^0(\boldsymbol{x}) - R_h\varpi^0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega. \tag{2.81}$$

By (2.2), (2.66)-(2.69), (2.79), the LDMV formula, and the Hölder and Cauchy inequalities, when $\Delta t = \mathcal{O}(H^2)$, noting that

$$\big|\sigma_h^k - \sigma_h^k\big|_s = \big|(\varpi^k - \varpi^{k-1}) - R_h(\varpi^k - \varpi^{k-1})\big|_s \le c\Delta t h^{l+1-s},$$

$$\big|\tilde{\sigma}_h^k - \tilde{\sigma}_h^k\big|_s = \big|(\psi^k - \psi^{k-1}) - R_h(\psi^k - \psi^{k-1})\big|_s \le c\Delta t h^{l+1-s}, \quad s = 0, 1,$$

and $\psi_h^k, \psi_H^k, \varpi_h^k$ and ϖ_H^k are bounded, we obtain

$$\frac{1}{\Delta t}\big|\tilde{w}_h^k - \tilde{w}_h^{k-1}\big|_1^2 + \frac{c_1\epsilon^2}{2}\left(\big|w_h^k\big|_1^2 - \big|w_h^{k-1}\big|_1^2\right) + \frac{c_1}{2}\left(\big\|w_h^k\big\|_0^2 - \big\|w_h^{k-1}\big\|_0^2\right)$$

$$\le \frac{1}{\Delta t}\big(\nabla\big(\tilde{\sigma}_h^k + \tilde{\sigma}_h^{k-1}\big), \nabla\big(\tilde{\sigma}_h^k - \tilde{\sigma}_h^{k-1}\big)\big) + \frac{1}{\Delta t}\big(w_h^k + w_h^{k-1}, \tilde{\sigma}_h^k - \tilde{\sigma}_h^{k-1}\big)$$

$$+ \frac{1}{\Delta t}\big(\sigma_h^k + \sigma_h^{k-1}, \tilde{w}_h^k - \tilde{w}_h^{k-1}\big) + \frac{\epsilon^2}{2}\big(G(\psi^k)\nabla(w_h^k + w_h^{k-1}), \nabla(\sigma_h^k - \sigma_h^{k-1})\big)$$

$$+ \frac{\epsilon^2}{2}\big((G(\psi^k) - G(\psi^{k-1}))\nabla w_h^{k-1}, \nabla(\theta_h^k - \theta_h^{k-1})\big)$$

$$- \frac{\epsilon^2}{2}\big((G(\psi^k) - G(\psi_H^k))\nabla\varpi_h^k + (G(\psi^{k-1}) - G(\psi_h^{k-1}))\nabla\varpi_h^{k-1}, \nabla(\theta_h^k - \theta_h^{k-1})\big)$$

$$+ \frac{1}{2}\big(\tilde{f}(\psi^k)\nabla(\tilde{w}_h^k + \tilde{w}_h^{k-1}), \nabla(\sigma_h^k - \sigma_h^{k-1})\big) + \frac{1}{2}\big((\tilde{f}(\psi^k) - \tilde{f}(\psi^{k-1}))\nabla\tilde{w}_h^{k-1}, \nabla(\theta_h^k - \theta_h^{k-1})\big)$$

$$- \frac{1}{2} \big((\tilde{f}(\psi^k) - \tilde{f}(\psi_H^k)) \nabla \psi_h^k + (\tilde{f}(\psi^{k-1}) - \tilde{f}(\psi_h^{k-1})) \nabla \psi_h^{k-1}, \nabla(\theta_h^k - \theta_h^{k-1}) \big)$$

$$\leq \frac{1}{\Delta t} \big| \tilde{w}_h^k - \tilde{w}_h^{k-1} \big|_1^2 + ch^{2l} + c\Delta t \| w_h^k - w_h^{k-1} \|_0^2$$

$$+ c\Delta t \| G' \|_{0,\infty} \left(\big| w_h^{k-1} \big|_1^2 + \big| w_h^k - w_h^{k-1} \big|_1^2 + \big| \sigma_h^k - \sigma_h^{k-1} \big|_1^2 \right)$$

$$+ c \| G' \|_{0,\infty} \max \big\{ \big| \varpi_h^k \big|_{1,\infty}, \big| \varpi_h^{k-1} \big|_{1,\infty} \big\} \big(\big| w_H^k \big|_1 + \big| w_h^{k-1} \big|_1 \big) \| \theta_h^k - \theta_h^{k-1} \|_0$$

$$+ c\Delta t \| \tilde{f}' \|_{0,\infty} \left(\big| \tilde{w}_h^{k-1} \big|_1^2 + \big| w_h^k - w_h^{k-1} \big|_1^2 + \big| \sigma_h^k - \sigma_h^{k-1} \big|_1^2 \right)$$

$$+ c \| \tilde{f}' \|_{0,\infty} \max \big\{ \big| \psi_h^k \big|_{1,\infty}, \big| \psi_h^{k-1} \big|_{1,\infty} \big\} \big(\big| \tilde{w}_H^k \big|_1 + \big| \tilde{w}_h^{k-1} \big|_1 \big) \| \theta_h^k - \theta_h^{k-1} \|_0$$

$$\leq \frac{1}{\Delta t} \big| \tilde{w}_h^k - \tilde{w}_h^{k-1} \big|_1^2 + c\Delta t \left(\big| w_h^k \big|_1^2 + \big| w_h^{k-1} \big|_1^2 \right) + c\Delta t H^{2l+2} + c\Delta t h^{2l}, \quad 1 \leq k \leq K. \tag{2.82}$$

By simplifying and summating (2.82) from 1 to k, when Δt is adequately small, satisfying $c\Delta t \leq 1/2$, we obtain

$$\big\| w_h^k \big\|_1^2 \leq c\Delta t \sum_{i=0}^{k-1} \big\| w_h^i \big\|_1^2 + ck\Delta t H^{2l+2} + ck\Delta t h^{2l}, \quad 1 \leq k \leq K. \tag{2.83}$$

With Gronwall's inequality, from (2.83) we obtain

$$\big\| w_h^k \big\|_1^2 \leq c(H^{2l+2} + h^{2l}) \exp(ck\Delta t), \quad 1 \leq k \leq K. \tag{2.84}$$

By (2.84) and (2.69), we obtain

$$\big\| \tilde{w}_h^k \big\|_1 + \big\| w_h^k \big\|_1 \leq 2 \big\| w_h^k \big\|_1 \leq c(H^{l+1} + h^l), \quad 1 \leq k \leq K. \tag{2.85}$$

Further, using the Nitsche technique and (2.85), when $h = \mathcal{O}(H^{1+1/l})$, we can directly obtain

$$\big\| \tilde{w}_h^k \big\|_0 + \big\| w_h^k \big\|_0 + h \big\| \tilde{w}_h^k \big\|_1 + h \big\| w_h^k \big\|_1 \leq ch^{l+1}, \quad 1 \leq k \leq K. \tag{2.86}$$

Combining (2.86) with Theorem 2.1 yields (2.59). This finishes the proof of Theorem 2.2. □

Remark 2.3. Theorem 2.2 implies that the TGMFECN solutions to Problem 2.2 are unconditionally stable and achieve optimal order error estimates when the coarse and fine grids satisfy the relationship $h = \mathcal{O}(H^{1+1/l})$. However, when Ω is a non convex regions or regions with poor smoothness, the generalized solution for the INFOCH equation has at most third-order derivatives, in this case, l can at most take as 2.

2.3. The matrix form of TGMFECN model for the INFOCH equation

The most important step in constructing the TGMFECNDR method is to rewrite the TGMFECN method in vector form. Toward this purpose, we set that $\{\zeta_\ell\}_{\ell=1}^{M_H} \subset \mathbb{U}_H$ and $\{\tilde{\zeta}_m\}_{m=1}^{M_h} \subset \mathbb{U}_h$ are two sets of normalized FE basis functions under the L^2 inner product, which may be generated by the normalized approach in [36, Section 1.6.3] and whose existence has been provided in [36, Proposition 1.6.21]. Whereupon, the FE spaces $\mathbb{U}_H$ and $\mathbb{U}_h$ can be, respectively, indicated by

$$\mathbb{U}_H = \{ v_H \in C(\bar{\Omega}) \cap \mathbb{U} : v_H|_E \in \mathbb{P}_l(E), \, \forall E \in \mathfrak{J}_H \} = \mathrm{span}\{ \zeta_1, \zeta_2, \cdots, \zeta_{M_H} \},$$

$$\mathbb{U}_h = \{v_h \in C(\bar{\Omega}) \cap \mathbb{U} : v_h|_e \in \mathbb{P}_l(e), \ \forall e \in \mathfrak{J}_h\} = \mathrm{span}\{\tilde{\zeta}_1, \tilde{\zeta}_2, \cdots, \tilde{\zeta}_{M_h}\},$$

in which M_H and M_h indicate the dimensions of $\mathbb{U}_H$ and $\mathbb{U}_h$, respectively.

Thus, the TGMFECN solutions ψ_H^k and ψ_h^k can be indicated in vectors as follows:

$$\psi_H^k = \sum_{i=1}^{M_H} \zeta_i V_i^k = (\zeta_1, \zeta_2, \cdots, \zeta_{M_H})^\top \cdot \left(V_1^k, V_2^k, \cdots, V_{M_H}^k\right)^\top = \boldsymbol{\zeta} \cdot \boldsymbol{V}^k, \qquad (2.87)$$

$$\psi_h^k = \sum_{j=1}^{M_h} \tilde{\zeta}_j w_j^k = (\tilde{\zeta}_1, \tilde{\zeta}_2, \cdots, \tilde{\zeta}_{M_h})^\top \cdot \left(v_1^k, v_2^k, \cdots, v_{M_h}^k\right)^\top = \tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{v}^k, \qquad (2.88)$$

where the vectors

$$\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \cdots, \zeta_{M_H})^\top, \quad \tilde{\boldsymbol{\zeta}} = (\tilde{\zeta}_1, \tilde{\zeta}_2, \cdots, \tilde{\zeta}_{M_h})^\top$$

are, respectively, formed with the FE basis functions $\zeta_1, \zeta_2, \cdots, \zeta_{M_H}$ and $\tilde{\zeta}_1, \tilde{\zeta}_2, \cdots, \tilde{\zeta}_{M_h}$, and the vectors

$$\boldsymbol{V}^k = \left(V_1^k, V_2^k, \cdots, V_{M_H}^k\right)^\top, \quad \boldsymbol{v}^k = \left(v_1^k, v_2^k, \cdots, v_{M_h}^k\right)^\top$$

are, respectively, formed by the unknown coefficients in the TGMFECN solutions ψ_H^k defined on the coarse grid division $\mathfrak{J}_H$ and ψ_h^k defined on the fine grid division $\mathfrak{J}_h$.

If we set that $\boldsymbol{W}_H^k$ and $\boldsymbol{w}_h^k$ are, respectively, the unknown solution coefficient vectors of ϖ_H^k and ϖ_h^k, then by (2.52) and (2.55), we immediately obtain

$$\boldsymbol{W}_H^k = \boldsymbol{B}_H \boldsymbol{V}_H^k, \quad \boldsymbol{w}_h^k = \boldsymbol{B}_h \boldsymbol{v}_h^k, \quad \varpi_H^k = \boldsymbol{\zeta} \cdot \boldsymbol{B}_H \boldsymbol{V}_H^k, \quad \varpi_H^k = \tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{B}_h \boldsymbol{v}_h^k,$$

where

$$\boldsymbol{B}_H = \left((\nabla \zeta_i, \nabla \zeta_j)\right)_{M_H \times M_H}, \quad \boldsymbol{B}_h = \left((\nabla \tilde{\zeta}_i, \nabla \tilde{\zeta}_j)\right)_{M_h \times M_h}.$$

Accordingly, the functional TGMFECN method can be reexpressed equivalently in the following matrix form, which is independent of the TGMFECN solutions ϖ_H and ϖ_h of the auxiliary function ϖ, and easy to solve.

Problem 2.4. Step 1. Find $\{\boldsymbol{V}^k\}_{k=1}^K \subset \mathbb{R}^{M_H}$ and $\{\psi_H^k\}_{k=1}^K \subset \mathbb{U}_H$ such that

$$\begin{cases} \boldsymbol{V}^0 = \left((R_H \psi^0, \zeta_1), (R_H \psi^0, \zeta_2), \cdots, (R_H \psi^0, \zeta_{M_H})\right)^T, \\ \boldsymbol{V}^k = \boldsymbol{V}^{k-1} - 0.5\Delta t\left(\boldsymbol{D}_H^k \boldsymbol{B}_H \boldsymbol{V}^k + \boldsymbol{D}_H^{k-1} \boldsymbol{B}_H \boldsymbol{V}^{k-1}\right) \\ \qquad\quad -0.5\Delta t\left(\boldsymbol{F}_H^k \boldsymbol{V}^k + \boldsymbol{F}_H^{k-1} \boldsymbol{V}^{k-1}\right), \quad 1 \le k \le K, \\ \psi_H^k = \boldsymbol{\zeta} \cdot \boldsymbol{V}^k, \qquad\qquad\qquad\qquad\quad 1 \le k \le K, \end{cases} \qquad (2.89)$$

where

$$\boldsymbol{D}_H^m = \epsilon^2 \left((G(\boldsymbol{\zeta} \cdot \boldsymbol{V}^m) \nabla \zeta_i, \nabla \zeta_j)\right)_{M_H \times M_H},$$
$$\boldsymbol{F}_H^m = \left((\tilde{f}(\boldsymbol{\zeta} \cdot \boldsymbol{V}^m) \nabla \zeta_i, \nabla \zeta_j)\right)_{M_H \times M_H}, \quad m = k, k-1.$$

Step 2. Find $\{\boldsymbol{v}^k\}_{k=1}^K \subset \mathbb{R}^{M_h}$ and $\{\psi_h^k,\}_{k=1}^K \subset \mathbb{U}_h$ such that

$$\begin{cases} \boldsymbol{v}^0 = \left((R_h \psi^0, \tilde{\zeta}_1), (R_h \tilde{\psi}^0, \tilde{\zeta}_2), \cdots, (R_h \psi^0, \tilde{\zeta}_{M_h})\right)^T, \\ \boldsymbol{v}^k = \boldsymbol{v}^{k-1} - 0.5\Delta t\left(\tilde{\boldsymbol{D}}_h^k \boldsymbol{B}_h \boldsymbol{v}^k + \boldsymbol{D}_h^{k-1} \boldsymbol{B}_h \boldsymbol{v}^{k-1}\right) \\ \qquad\quad -0.5\Delta t\left(\tilde{\boldsymbol{F}}_H^k \boldsymbol{v}^k + \boldsymbol{F}_h^{k-1} \boldsymbol{v}^{k-1}\right), \quad 1 \le k \le K, \\ \psi_h^k = \tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{v}^k, \qquad\qquad\qquad\qquad\quad 1 \le k \le K, \end{cases} \qquad (2.90)$$

where

$$\tilde{\boldsymbol{D}}_h^k = \epsilon^2\big(\big(G\big(\psi_H^k\big)\nabla\tilde\zeta_j, \nabla\tilde\zeta_j\big)\big)_{M_h\times M_h}, \qquad \tilde{\boldsymbol{F}}_H^k = \big(\big(\tilde f\big(\psi_H^k\big)\nabla\tilde\zeta_i, \nabla\tilde\zeta_j\big)\big)_{M_h\times M_h},$$

$$\boldsymbol{F}_h^{k-1} = \big(\big(\tilde f(\tilde\zeta\cdot\boldsymbol{v}^{k-1})\nabla\tilde\zeta_i, \nabla\tilde\zeta_j\big)\big)_{M_h\times M_h}, \quad \boldsymbol{D}_h^{k-1} = \epsilon^2\big(\big(G(\tilde\zeta\cdot\boldsymbol{v}^{k-1})\nabla\tilde\zeta_i, \nabla\tilde\zeta_j\big)\big)_{M_h\times M_h}.$$

For the TGMFECN solution vectors of Problem 2.4, we obtain the following results.

Theorem 2.3. *Under the same conditions as Theorem 2.2, Problem 2.4 has two unique series of TGMFECN solution vectors $\{\boldsymbol{V}^k\}_{k=1}^K \subset \mathbb{R}^{M_H}$ and $\{\boldsymbol{v}^k\}_{k=1}^K \subset \mathbb{R}^{M_h}$, satisfying the following unconditional boundedness, i.e., unconditional stability:*

$$\|\boldsymbol{V}^k\| + \|\boldsymbol{v}^k\| \le c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big), \quad 1 \le k \le K,$$

in which $\|\boldsymbol{v}\|$ indicates the Euclidean norm of vector $\boldsymbol{v}$.

Proof. Severally, multiplying (2.87) and (2.88) by the FE basis vectors $\boldsymbol{\zeta}$ and $\tilde{\boldsymbol{\zeta}}$ yields

$$\boldsymbol{V}^k = \frac{\psi_H^k\boldsymbol{\zeta}}{\|\boldsymbol{\zeta}\|^2}, \quad \boldsymbol{v}^k = \frac{\psi_h^k\tilde{\boldsymbol{\zeta}}}{\|\tilde{\boldsymbol{\zeta}}\|^2}, \quad 1 \le k \le K.$$

Thus, by Theorem 2.2, we can assert that Problem 2.4 has two unique series of coefficient vectors of TGMFECN solutions $\{\boldsymbol{V}^k\}_{k=1}^K \subset \mathbb{R}^{M_H}$ and $\{\boldsymbol{v}^k\}_{k=1}^K \subset \mathbb{R}^{M_h}$.

Further, using the inverse estimate inequality and (2.57) in Theorem 2.2, we obtain

$$\|\boldsymbol{V}^k\| + \|\boldsymbol{v}^k\| = \big|\psi_H^k\big| + \big|\psi_h^k\big| \le c\big(\|\psi_H^k\|_{0,\infty} + \|\psi_h^k\|_{0,\infty}\big)$$
$$\le c\big(\big|\psi_H^k\big|_1 + \big|\psi_h^k\big|_1\big) \le c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big), \quad 1 \le k \le K.$$

Hence, the TGMFECN solution vectors $\{\boldsymbol{V}^k\}_{k=1}^K$ and $\{\boldsymbol{v}^k\}_{k=1}^K$ to Problem 2.4 are unconditionally bounded, i.e., unconditionally stable. This finishes the proof for Theorem 2.3. $\square$

Remark 2.4. If the time step increment Δt, the coarse and fine division parameters H and h, the parameter ϵ, the initial function $\psi^0(\boldsymbol{x})$, and the nonlinear functions $G(\cdot)$ and $\tilde f(\cdot)$ are provided, a series of TGMFECN solutions $\{\psi_h^k\}_{k=1}^K \subset \mathbb{U}_h$ can be obtained by solving Problem 2.4. It is not necessary to find the TGMFECN solutions ϖ_H and ϖ_h of the auxiliary function ϖ. However, when Problem 2.4 is applied to real engineering problems, it usually involves many unknowns (usually exceeds millions) so that its computational effort is too much for a common computer. Therefore, it is highly necessary to use the POD technique to reduce the dimension of unknown TGMFECN solution vectors in the TGMFECN method and build a new TGMFECNDR method.

3. The TGMFECNDR Method for the INFOCH Equation

3.1. Generation for the POD basis vectors

The POD basis vectors can be generated according to the following flowchart.

(1) Obtain two series of coefficient vectors of TGMFECN solutions $\{\boldsymbol{\tau}^k\}_{k=1}^L$ by solving the first and second equations of (2.89) and (2.90) in Problem 2.4 at the initial L time steps and constitute two matrices $\boldsymbol{A}_\tau = (\boldsymbol{\tau}^1, \boldsymbol{\tau}^2, \cdots, \boldsymbol{\tau}^L)$ $(\boldsymbol{\tau} = \boldsymbol{V}, \boldsymbol{v})$.

(2) Find two sets of normalized eigenvectors $\hat{\boldsymbol{\varphi}}_{\tau i}$ $(1 \leq i \leq r_\tau =: \mathrm{rank}(\boldsymbol{A}_\tau))$ of matrices $\boldsymbol{A}_\tau^\top \boldsymbol{A}_\tau$ associated with two sets of positive eigenvalues $\lambda_{\tau 1} \geq \lambda_{\tau 2} \geq \cdots \geq \lambda_{\tau r_\tau} > 0$ $(\tau = V, v)$.

(3) Find two sets of d_τ most important orthogonal vectors $\{\boldsymbol{\varphi}_{\tau 1}, \boldsymbol{\varphi}_{\tau 2}, \cdots, \boldsymbol{\varphi}_{\tau d_\tau}\}$ of matrices $\boldsymbol{A}_\tau \boldsymbol{A}_\tau^\top$ by the formulas $\boldsymbol{\varphi}_{\tau i} = \boldsymbol{A}_\tau \hat{\boldsymbol{\varphi}}_{\tau i} / \sqrt{\lambda_{\tau i}}$ and construct two matrices $\boldsymbol{\Phi}_\tau = (\boldsymbol{\varphi}_{\tau 1}, \boldsymbol{\varphi}_{\tau 2}, \cdots, \boldsymbol{\varphi}_{\tau d_\tau})$ $(d_\tau \leq r_\tau$ and $\tau = V, v)$, which are known as two sets of POD basis vectors.

It has been demonstrated in [26, Section 5.1] that $\boldsymbol{\Phi}_\tau$ $(\tau = V, v)$ meet the following properties:

$$\left\| \boldsymbol{A}_\tau - \boldsymbol{\Phi}_\tau \boldsymbol{\Phi}_\tau^\top \boldsymbol{A}_\tau \right\|_{2,2} = \sqrt{\lambda_{\tau(d_\tau+1)}}, \quad \tau = V, v, \tag{3.1}$$

in which $\|\boldsymbol{A}_\tau\|_{2,2} = \sup_{\boldsymbol{u} \in \mathbb{R}^L} \|\boldsymbol{A}_\tau \boldsymbol{u}\| / \|\boldsymbol{u}\|$ $(\tau = V, v)$ and $\|\boldsymbol{u}\|$ still indicates the Euclidean norm of vector $\boldsymbol{u}$.

By (3.1), we obtain

$$\left\| \boldsymbol{\tau}^k - \boldsymbol{\Phi}_\tau \boldsymbol{\Phi}_\tau^\top \boldsymbol{\tau}^k \right\| = \left\| (\boldsymbol{A}_\tau - \boldsymbol{\Phi}_\tau \boldsymbol{\Phi}_\tau^\top \boldsymbol{A}_\tau) \boldsymbol{e}^k \right\| \leq \left\| \boldsymbol{A}_\tau - \boldsymbol{\Phi}_\tau \boldsymbol{\Phi}_\tau^\top \boldsymbol{A}_\tau \right\|_{2,2} \|\boldsymbol{e}^k\|$$

$$\leq \sqrt{\lambda_{\tau(d_\tau+1)}}, \quad \tau = V, v, \quad 1 \leq k \leq L, \tag{3.2}$$

in which $\boldsymbol{e}_k$ $(1 \leq k \leq L)$ indicate the L-dimension orthonormal vectors with k-th element 1.

3.2. Establishment of the TGMFECNDR method

If we set that

$$\boldsymbol{\tau}_d^k = \left(\tau_{1d}^k, \tau_{2d}^k, \cdots, \tau_{M_\kappa d}^k \right)^\top, \quad \boldsymbol{\alpha}_\tau^k = \left(\alpha_{\tau 1}^k, \alpha_{\tau 2}^k, \cdots, \alpha_{\tau d_\tau}^k \right)^\top, \quad \kappa = h, H, \quad \tau = V, v,$$

and

$$\psi_V^k = \boldsymbol{\zeta} \cdot \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^k, \quad \psi_v^k = \tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{\Phi}_v \boldsymbol{\alpha}_v^k, \quad 1 \leq k \leq K$$

indicate the TGMFECNDR solutions, then two series of first L coefficient vectors of TGMFECNDR solutions

$$\boldsymbol{V}_d^k = \boldsymbol{\Phi}_V \boldsymbol{\Phi}_V^\top \boldsymbol{V}^k =: \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^k, \quad \boldsymbol{v}_d^k = \boldsymbol{\Phi}_v \boldsymbol{\Phi}_v^\top \boldsymbol{v}^k =: \boldsymbol{\Phi}_v \boldsymbol{\alpha}_v^k, \quad 1 \leq k \leq L$$

are immediately obtained by Section 3.1.

Thereupon, by replacing $\boldsymbol{\tau}^k$ in Problem 2.4 with $\boldsymbol{\tau}_d^k = \boldsymbol{\Phi}_\tau \boldsymbol{\alpha}_\tau^k$ $(\tau = V, v$ and $L+1 \leq k \leq K)$ and using the orthogonality of vectors in $\boldsymbol{\Phi}_\tau$, respectively, the TGMFECNDR method can be created in the following.

Problem 3.1. Step 1. Find $\{\boldsymbol{\alpha}_V^k\}_{k=1}^K \subset \mathbb{R}^{d_V}$ and $\{\psi_V^k\}_{k=1}^K \subset \mathbb{U}_H$ defined on the coarse grid division $\mathfrak{J}_H$, satisfying the following nonlinear system of equations:

$$\begin{cases} \boldsymbol{\alpha}_V^k = \boldsymbol{\Phi}_V^\top \boldsymbol{V}^k, & 1 \leq k \leq L, \\ \boldsymbol{\alpha}_V^k = \boldsymbol{\alpha}_V^{k-1} - 0.5\Delta t \boldsymbol{\Phi}_V^\top \left(\boldsymbol{D}_V^k \boldsymbol{B}_H \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^k + \boldsymbol{D}_V^{k-1} \boldsymbol{B}_H \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^{k-1} \right) & \\ \qquad -0.5\Delta t \boldsymbol{\Phi}_V^\top \left(\boldsymbol{F}_V^k \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^k + \boldsymbol{F}_V^{k-1} \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^{k-1} \right), & L+1 \leq k \leq K, \\ \psi_V^k = \boldsymbol{\zeta} \cdot \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^k, & 1 \leq k \leq K, \end{cases} \tag{3.3}$$

where $\boldsymbol{V}^k$ $(1 \leq k \leq L)$ indicate the initial L solution vectors of Problem 2.4 defined on the coarse grid division $\mathfrak{J}_H$,

$$\boldsymbol{D}_V^m = \epsilon^2 \left(\left(G\left(\boldsymbol{\zeta} \cdot \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^m \right) \nabla \zeta_i, \nabla \zeta_j \right) \right)_{M_H \times M_H}, \quad m = k, k-1,$$

$$\boldsymbol{F}_V^m = \left(\left(\tilde{f}\left(\boldsymbol{\zeta} \cdot \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^m \right) \nabla \zeta_i, \nabla \zeta_j \right) \right)_{M_H \times M_H}, \quad m = k, k-1,$$

$$\boldsymbol{B}_H = \left(\left(\nabla \zeta_i, \nabla \zeta_j \right) \right)_{M_H \times M_H}.$$

Step 2. Find $\{\boldsymbol{\alpha}_v^k\}_{k=1}^K \subset \mathbb{R}^{d_v}$ and $\{\psi_v^k\}_{k=1}^K \subset \mathbb{U}_h$ defined on the fine grid division $\mathfrak{J}_h$, satisfying the following linear system of equations:

$$
\begin{cases}
\boldsymbol{\alpha}_v^k = \boldsymbol{\Phi}_v^\top \boldsymbol{v}^k, & 1 \le k \le L, \\
\boldsymbol{\alpha}_v^k = \boldsymbol{\alpha}_v^{k-1} - 0.5\Delta t \boldsymbol{\Phi}_v^\top \big(\tilde{\boldsymbol{D}}_v^k \boldsymbol{B}_h \boldsymbol{\Phi}_v \boldsymbol{\alpha}_v^k + \boldsymbol{D}_v^{k-1} \boldsymbol{B}_h \boldsymbol{\Phi}_v \boldsymbol{\alpha}_v^{k-1}\big) & \\
\quad\quad -0.5\Delta t \boldsymbol{\Phi}_v^\top \big(\tilde{\boldsymbol{F}}_V^k \boldsymbol{\Phi}_v \boldsymbol{\alpha}_v^k + \boldsymbol{F}_v^{k-1} \boldsymbol{\Phi}_v \boldsymbol{\alpha}_v^{k-1}\big), & L+1 \le k \le K, \\
\psi_v^k = \tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{\Phi}_v \boldsymbol{\alpha}_v^k, & 1 \le k \le K,
\end{cases}
\tag{3.4}
$$

where $\boldsymbol{v}^k$ $(1 \le k \le L)$ are the initial L solution vectors of Problem 2.4 defined on the fine grid $\mathfrak{J}_h$,

$$
\begin{aligned}
\tilde{\boldsymbol{D}}_v^k &= \epsilon^2 \big((G(\boldsymbol{\zeta} \cdot \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^k)\nabla\tilde{\zeta}_j, \nabla\tilde{\zeta}_j)\big)_{M_h \times M_h}, \\
\boldsymbol{D}_v^{k-1} &= \epsilon^2 \big((G(\tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{\Phi}_v \boldsymbol{\alpha}_v^{k-1})\nabla\tilde{\zeta}_i, \nabla\tilde{\zeta}_j)\big)_{M_h \times M_h}, \\
\tilde{\boldsymbol{F}}_V^k &= \big((\tilde{f}(\boldsymbol{\zeta} \cdot \boldsymbol{\Phi}_V \boldsymbol{\alpha}_V^k)\nabla\tilde{\zeta}_i, \nabla\tilde{\zeta}_j)\big)_{M_h \times M_h}, \\
\boldsymbol{F}_v^{k-1} &= \big((\tilde{f}(\tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{\Phi}_v \boldsymbol{\alpha}_v^{k-1})\nabla\tilde{\zeta}_i, \nabla\tilde{\zeta}_j)\big)_{M_h \times M_h}, \\
\boldsymbol{B}_h &= \big((\nabla\tilde{\zeta}_i, \nabla\tilde{\zeta}_j)\big)_{M_h \times M_h}.
\end{aligned}
$$

Remark 3.1. It is easy to see that Problem 2.4 in per time step has $(M_H + M_h)$ unknowns, while Problem 3.1 in per time step has only $(d_V + d_v)$ unknowns $(d_V \ll M_H, d_v \ll M_h)$. Therefore, the unknowns in the TGMFECNDR method are much less than those in the classical TGMFECN method, but the TGMFECNDR method has the same FE basis function vectors $\boldsymbol{\zeta}$ and $\tilde{\boldsymbol{\zeta}}$ as the TGMFECN method so as to keep precision unchanged. That is, even if the unknowns of the TGMFECNDR method are greatly reduced, it keeps the same precision as the TGMFECN method. Therefore, the TGMFECNDR method is markedly superior to the TGMFECN method.

3.3. The theoretical analysis of the TGMFECNDR solutions

To analyze the TGMFECNDR solutions requires the following Kellogg's inequalities (see [26, Lemmas 5.1.2 and 5.1.3]).

Lemma 3.1 (Kellogg's inequalities). *If $\boldsymbol{I}_\kappa$ $(\kappa = h, H)$ indicate two $M_\kappa \times M_\kappa$ identity matrix and $\boldsymbol{B}_\kappa$ $(\kappa = h, H)$ are two positive semidefinite $M_\kappa \times M_\kappa$ matrices , then we obtain the following estimates:*

$$
\big\|(\boldsymbol{I}_\kappa + \gamma\boldsymbol{B}_\kappa)^{-1}\big\|_{2,2} \le 1, \quad \big\|(\boldsymbol{I}_\kappa + \gamma\boldsymbol{B}_\kappa)^{-1}(\boldsymbol{I}_\kappa - \gamma\boldsymbol{B}_\kappa)\big\|_{2,2} \le 1, \quad \forall \gamma \ge 0, \quad \kappa = h, H.
$$

The theoretical analysis of the TGMFECNDR solutions in Problem 3.1 still needs the following lemma, which can be obtained directly by [26, Lemma 1.3.5].

Lemma 3.2. *Let $(\cdot, \cdot)$ indicate the L^2 inner product, and ζ_j, $(1 \le j \le M_H)$ and $\tilde{\zeta}_j$, $(1 \le j \le M_h)$ be respectively the FE basis functions defined on the grid divisions $\mathfrak{J}_H$ and $\mathfrak{J}_h$. Then, $\boldsymbol{B}_H := ((\nabla\zeta_i, \nabla\zeta_j))_{M_H \times M_H}$ and $\boldsymbol{B}_h := ((\nabla\tilde{\zeta}_i, \nabla\tilde{\zeta}_j))_{M_h \times M_h}$ meet the following estimates:*

$$
\|\boldsymbol{B}_\kappa\|_{2,2} \le c\kappa^{\frac{n-2}{2}}, \quad \big\|\boldsymbol{B}_\kappa^{-1}\big\|_{2,2} \le c\kappa^{\frac{2-n}{2}}, \quad \kappa = h, H.
$$

For Problem 3.1, we obtain the following conclusions.

Theorem 3.1. *Under the identical hypotheses of Theorem 2.2, Problem 3.1 has two unique series of TGMFECNDR solutions $\{\psi_V^k\}_{k=1}^K \subset \mathbb{U}_H$ and $\{\psi_v^k\}_{k=1}^K \subset \mathbb{U}_h$, satisfying the following unconditional boundedness, i.e., unconditional stability:*

$$\left\| \psi_V^k \right\|_1 + \left\| \psi_v^k \right\|_1 \leq c \left(\|\psi^0\|_1 + \|\varpi^0\|_1 \right), \quad 1 \leq k \leq K. \tag{3.5}$$

Furthermore, when the solution ψ of Problem 2.1 is adequately smooth and $h = \mathcal{O}(H^{1+1/l})$, $\{\psi_V^k\}_{k=1}^K$ and $\{\psi_v^k\}_{k=1}^K$ meet the following error estimations:

$$\left\| \psi(t_k) - \psi_V^k \right\|_0 \leq c \left(\Delta t^2 + H^{l+1} + \sqrt{\lambda_{V(d_V+1)}} \right), \qquad 1 \leq k \leq K, \tag{3.6}$$

$$\left\| \psi(t_k) - \psi_v^k \right\|_0 \leq c \left(\Delta t^2 + h^{l+1} + \sqrt{\lambda_{v(d_v+1)}} + \sqrt{\lambda_{V(d_V+1)}} \right), \quad 1 \leq k \leq K. \tag{3.7}$$

Proof. The proof of Theorem 2.3 constitutes the following three steps.

(1) Demonstrate the existence of TGMFECNDR solutions.

(i) When $1 \leq k \leq L$, it is obvious that the first subsystems of equations in (3.3) and (3.4) have, respectively, a unique sets of solution vectors $\{\boldsymbol{\alpha}_\tau^k\}_{k=1}^L$ ($\tau = V, v$), further to obtain two sets of TGMFECNDR solutions $\{\psi_V^k\}_{k=1}^L \subset \mathbb{U}_H$ and $\{\psi_v^k\}_{k=1}^L \subset \mathbb{U}_h$ by the third subsystem of equations in (3.3) and (3.4).

(ii) When $L + 1 \leq k \leq K$, with $\boldsymbol{\tau}_d^k = \boldsymbol{\Phi}_\tau \boldsymbol{\alpha}_\tau^k$ ($\tau = V, v$), the second and third subsystems of equations in (3.3) and (3.4) may be restored into the following two equations:

$$\begin{cases} \boldsymbol{V}_d^k = \boldsymbol{V}_d^{k-1} - 0.5\Delta t \left(\boldsymbol{D}_V^k \boldsymbol{B}_H \boldsymbol{V}_d^k + \boldsymbol{D}_V^{k-1} \boldsymbol{B}_H \boldsymbol{V}_d^{k-1} \right) \\ \qquad -0.5\Delta t \boldsymbol{F}_V^k + 0.5\Delta t \boldsymbol{F}_V^{k-1}, \quad k = L+1, L+2, \ldots, K, \\ \psi_V^k = \boldsymbol{\zeta} \cdot \boldsymbol{V}_d^k, \qquad\qquad\qquad\qquad k = L+1, L+2, \ldots, K. \end{cases} \tag{3.8}$$

$$\begin{cases} \boldsymbol{v}_d^k = \boldsymbol{v}_d^{k-1} - 0.5\Delta t \left(\tilde{\boldsymbol{D}}_v^k \boldsymbol{B}_h \boldsymbol{v}_d^k + \boldsymbol{D}_v^{k-1} \boldsymbol{B}_h \boldsymbol{v}_d^{k-1} \right) \\ \qquad -0.5\Delta t \tilde{\boldsymbol{F}}_V^k + 0.5\Delta t \boldsymbol{F}_v^{k-1}, \quad k = L+1, L+2, \ldots, K, \\ \psi_v^k = \tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{v}_d^k, \qquad\qquad\qquad\qquad k = L+1, L+2, \ldots, K. \end{cases} \tag{3.9}$$

Because (3.8) and (3.9) have the same form as (2.89) and (2.90), by the same proof as Theorems 2.3 and 2.2, we can prove that (3.8) and (3.9) have, respectively, a unique series of solutions $\{\psi_V^k\}_{k=L+1}^K \subset \mathbb{U}_H$ and $\{\psi_v^k\}_{k=L+1}^K \subset \mathbb{U}_h$.

By combining (i) and (ii), we affirm that Problem 3.1 has two unique series of TGMFECNDR solutions $\{\psi_V^k\}_{k=1}^K \subset \mathbb{U}_H$ and $\{\psi_v^k\}_{k=1}^K \subset \mathbb{U}_h$.

(2) Demonstrate the stability of TGMFECNDR solutions.

(i) When $1 \leq k \leq L$, with the boundedness and orthonormality of the POD bases vectors $\boldsymbol{\Phi}_\tau$ ($\tau = V, v$), and the unconditional stability of TGMFECN solution vectors $\boldsymbol{\tau}^k$ ($\tau = V, v$) in Theorem 2.3, we obtain

$$\left\| \boldsymbol{\tau}_d^k \right\| = \left\| \boldsymbol{\Phi}_\tau \boldsymbol{\Phi}_\tau^\top \boldsymbol{\tau}^k \right\| \leq \left\| \boldsymbol{\Phi}_\tau \boldsymbol{\Phi}_\tau^\top \right\|_{2,2} \|\boldsymbol{\tau}^k\| \leq c \|\boldsymbol{\tau}^k\| \leq c \left(\|\psi^0\|_1 + \|\varpi^0\|_1 \right), \quad \tau = V, v. \tag{3.10}$$

Thus, using the boundedness of $\boldsymbol{\zeta}$ and $\tilde{\boldsymbol{\zeta}}$, namely $\|\boldsymbol{\zeta}\|_1 \leq c$ and $\|\tilde{\boldsymbol{\zeta}}\|_1 \leq c$, (3.10), and Lemma 3.2, we obtain

$$\begin{aligned} \left\| \psi_V^k \right\|_1 + \left\| \psi_v^k \right\|_1 &= \left\| \boldsymbol{\zeta} \cdot \boldsymbol{V}_d^k \right\|_1 + \left\| \tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{v}_d^k \right\|_1 \leq \|\boldsymbol{\zeta}\|_1 \|\boldsymbol{V}_d^k\| + \|\tilde{\boldsymbol{\zeta}}\|_1 \|\boldsymbol{v}_d^k\| \\ &\leq c \left(\|\boldsymbol{V}_d^k\| + \|\boldsymbol{v}_d^k\| \right) \leq c \left(\|\psi^0\|_1 + \|\varpi^0\|_1 \right), \quad 1 \leq k \leq L. \end{aligned} \tag{3.11}$$

(ii) When $L + 1 \leq k \leq K$, with the LDMV formula, the boundedness of $\boldsymbol{\zeta}$ and $\tilde{\boldsymbol{\zeta}}$, namely $\|\boldsymbol{\zeta}\|_1 \leq c$ and $\|\tilde{\boldsymbol{\zeta}}\|_1 \leq c$, and Lemmas 3.1 and 3.2, from the first subsystem of equations (3.8) and (3.9), we obtain

$$
\begin{aligned}
\|\boldsymbol{V}_d^k\| &\leq \|\boldsymbol{V}_d^{k-1}\| + 0.5\Delta t \big\|\big(\boldsymbol{I} + 0.5\Delta t \boldsymbol{D}_V^k \boldsymbol{B}_H\big)^{-1}\big\|_{2,2}\big(\|\boldsymbol{F}_V^k\| + \|\boldsymbol{F}_V^{k-1}\|\big) \\
&\quad + \Delta t \big\|\big(\boldsymbol{I} + 0.5\Delta t \boldsymbol{D}_V^k \boldsymbol{B}_H\big)^{-1}\big\|_{2,2}\big(\|\boldsymbol{D}_V^k\|_{2,2} + \|\boldsymbol{D}_V^{k-1}\|_{2,2}\big)\|\boldsymbol{B}_H\|_{2,2}\|\boldsymbol{V}_d^{k-1}\| \\
&\leq \|\boldsymbol{V}_d^{k-1}\| + c\Delta t\big(\|\boldsymbol{V}_d^k\| + \|\boldsymbol{V}_d^{k-1}\|\big),
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
\|\boldsymbol{v}_d^k\| &\leq \|\boldsymbol{v}_d^{k-1}\| + 0.5\Delta t \big\|\big(\boldsymbol{I} + 0.5\Delta t \tilde{\boldsymbol{D}}_v^k \boldsymbol{B}_h\big)^{-1}\big\|_{2,2}\big(\|\tilde{\boldsymbol{F}}_V^k\| + \|\boldsymbol{F}_v^{k-1}\|\big) \\
&\quad + 0.5\Delta t \big\|\big(\boldsymbol{I} + 0.5\Delta t \tilde{\boldsymbol{D}}_v^k \boldsymbol{B}_h\big)^{-1}\big\|_{2,2}\big(\|\tilde{\boldsymbol{D}}_v^k\|_{2,2} + \|\boldsymbol{D}_v^{k-1}\|_{2,2}\big)\|\boldsymbol{B}_h\|_{2,2}\|\boldsymbol{v}_d^{k-1}\| \\
&\leq \|\boldsymbol{v}_d^{k-1}\| + c\Delta t\big(\|\boldsymbol{V}_d^k\| + \|\boldsymbol{v}_d^{k-1}\| + \|\boldsymbol{v}_d^k\|\big).
\end{aligned}
\tag{3.13}
$$

Summating (3.12) and (3.13) from $L + 1$ to k ($k \leq K$), respectively, we obtain

$$
\big\|\boldsymbol{V}_d^k\big\| \leq \big\|\boldsymbol{V}_d^L\big\| + c\Delta t \sum_{i=L}^{k} \big\|\boldsymbol{V}_d^i\big\|, \qquad L + 1 \leq k \leq K,
\tag{3.14}
$$

$$
\big\|\boldsymbol{v}_d^k\big\| \leq \big\|\boldsymbol{v}_d^L\big\|_2 + c\Delta t \sum_{i=L}^{k} \big(\big\|\boldsymbol{V}_d^i\big\| + \big\|\boldsymbol{v}_d^i\big\|\big), \quad L + 1 \leq k \leq K.
\tag{3.15}
$$

When Δt is adequately small, satisfying $\Delta t \leq 1/(2c)$, by (3.10), (3.14), (3.15), Gronwall's lemma, we obtain

$$
\begin{aligned}
\big\|\boldsymbol{V}_d^k\big\| &\leq c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big) + c\Delta t \sum_{i=L}^{k-1} \big\|\boldsymbol{V}_d^i\big\| \leq c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big) \exp\big(c\Delta t(k-L)\big) \\
&\leq c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big), \quad L + 1 \leq k \leq K,
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
\big\|\boldsymbol{v}_d^k\big\| &\leq c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big) + c\Delta t \sum_{i=L}^{k-1} \big\|\boldsymbol{v}_d^i\big\| \leq c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big) \exp\big(c(k-L)\Delta t\big) \\
&\leq c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big), \quad L + 1 \leq k \leq K.
\end{aligned}
\tag{3.17}
$$

By using the boundedness of $\boldsymbol{\zeta}$ and $\tilde{\boldsymbol{\zeta}}$, namely $\|\boldsymbol{\zeta}\|_1 \leq c$ and $\|\tilde{\boldsymbol{\zeta}}\|_1 \leq c$, (3.16), and (3.17), we obtain

$$
\begin{aligned}
\big\|\psi_V^k\big\|_1 + \big\|\psi_v^k\big\|_1 &\leq \big\|\boldsymbol{\zeta} \cdot \boldsymbol{V}_d^k\big\|_1 + \big\|\tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{v}_d^k\big\|_1 \leq \|\boldsymbol{\zeta}\|_1 \cdot \big\|\boldsymbol{V}_d^k\big\| + \|\tilde{\boldsymbol{\zeta}}\|_1 \cdot \big\|\boldsymbol{v}_d^k\big\| \\
&\leq c\big(\|\psi^0\|_1 + \|\varpi^0\|_1\big), \quad L + 1 \leq k \leq K.
\end{aligned}
\tag{3.18}
$$

Combining (3.11) with (3.18) yields the inequality (3.5).

(3) Demonstrate the error estimates of TGMFECNDR solutions.

(a) When $1 \leq k \leq L$, by (3.1), the first equations in (3.3) and (3.4), and $\|\boldsymbol{\zeta}\|_0 \leq c$ and $\|\tilde{\boldsymbol{\zeta}}\|_0 \leq c$, we obtain

$$
\big\|\psi_H^k - \psi_V^k\big\|_0 = \big\|\boldsymbol{\zeta} \cdot \big(\boldsymbol{V}^k - \boldsymbol{\Phi}_V \boldsymbol{\Phi}_V^\top \boldsymbol{V}^k\big)\big\|_0 \leq \|\boldsymbol{\zeta}\|_0 \big\|\boldsymbol{V}^k - \boldsymbol{\Phi}_V \boldsymbol{\Phi}_V^\top \boldsymbol{V}^k\big\| \leq c\sqrt{\lambda_{V(d_v+1)}},
\tag{3.19}
$$

$$
\big\|\psi_h^k - \psi_v^k\big\|_0 = \big\|\tilde{\boldsymbol{\zeta}} \cdot \big(\boldsymbol{v}^k - \boldsymbol{\Phi}_v \boldsymbol{\Phi}_v^\top \boldsymbol{v}^k\big)\big\|_0 \leq \|\tilde{\boldsymbol{\zeta}}\|_0 \big\|\boldsymbol{v}^k - \boldsymbol{\Phi}_v \boldsymbol{\Phi}_v^\top \boldsymbol{v}^k\big\| \leq c\sqrt{\lambda_{v(d_v+1)}}.
\tag{3.20}
$$

(b) When $L + 1 \leq k \leq K$, using the first equations in (3.8) and (3.9), the second equations in (2.89) and (2.90), the LDMV formula, and Lemmas 3.1 and 3.2, we obtain

$$\left\|\boldsymbol{V}^k - \boldsymbol{V}_d^k\right\| \leq \left\|\boldsymbol{V}^{k-1} - \boldsymbol{V}_d^{k-1}\right\| + c\Delta t\left(\left\|\boldsymbol{V}^{k-1} - \boldsymbol{V}_d^{k-1}\right\| + \left\|\boldsymbol{V}^k - \boldsymbol{V}_d^k\right\|\right), \tag{3.21}$$

$$\left\|\boldsymbol{v}^k - \boldsymbol{v}_d^k\right\| \leq \left\|\boldsymbol{v}^{k-1} - \boldsymbol{v}_d^{k-1}\right\| + c\Delta t\left(\left\|\boldsymbol{v}^k - \boldsymbol{v}_d^k\right\| + \left\|\boldsymbol{v}^{k-1} - \boldsymbol{v}_d^{k-1}\right\| + \left\|\boldsymbol{V}^k - \boldsymbol{V}_d^k\right\|\right). \tag{3.22}$$

Summating (3.21) and (3.22) from $L + 1$ to k $(k \leq K)$, when Δt is adequately small, satisfying $\Delta t \leq 1/(2c)$, by (3.19) and (3.20), we get

$$\left\|\boldsymbol{V}^k - \boldsymbol{V}_d^k\right\| \leq c\left\|\boldsymbol{V}^L - \boldsymbol{V}_d^L\right\| + c\Delta t\sum_{i=L}^{k-1}\left\|\boldsymbol{V}^i - \boldsymbol{V}_d^i\right\|$$

$$\leq c\sqrt{\lambda_{V(d_V+1)}} + c\Delta t\sum_{i=L}^{k-1}\left\|\boldsymbol{V}^i - \boldsymbol{V}_d^i\right\|, \quad L + 1 \leq k \leq K, \tag{3.23}$$

$$\left\|\boldsymbol{v}^k - \boldsymbol{v}_d^k\right\| \leq c\left\|\boldsymbol{v}^L - \boldsymbol{v}_d^L\right\| + c\Delta t\sum_{i=L}^{k-1}\left\|\boldsymbol{v}^i - \boldsymbol{v}_d^i\right\|$$

$$\leq c\sqrt{\lambda_{v(d_v+1)}} + c\Delta t\sum_{i=L}^{k-1}\left\|\boldsymbol{v}^i - \boldsymbol{v}_d^i\right\|$$

$$+ c\Delta t\sum_{i=L}^{k}\left\|\boldsymbol{V}^i - \boldsymbol{V}_d^i\right\|, \quad L + 1 \leq k \leq K. \tag{3.24}$$

Applying Gronwall's lemma to (3.23) and (3.24), we obtain

$$\left\|\boldsymbol{V}^k - \boldsymbol{V}_d^k\right\| \leq c\sqrt{\lambda_{V(d_V+1)}}\exp\left(c\Delta t(k - L - 1)\right)$$

$$\leq c\sqrt{\lambda_{V(d_V+1)}}, \quad L + 1 \leq k \leq K, \tag{3.25}$$

$$\left\|\boldsymbol{v}^k - \boldsymbol{v}_d^k\right\| \leq c\left(\sqrt{\lambda_{V(d_V+1)}} + \sqrt{\lambda_{v(d_v+1)}}\right)\exp\left(c\Delta t(k - L - 1)\right)$$

$$\leq c\left(\sqrt{\lambda_{V(d_V+1)}} + \sqrt{\lambda_{v(d_v+1)}}\right), \quad L + 1 \leq k \leq K. \tag{3.26}$$

By (3.25), (3.26), the boundedness of $\boldsymbol{\zeta}$ and $\tilde{\boldsymbol{\zeta}}$, and Lemma 3.2, we obtain

$$\left\|\psi_H^k - \psi_V^k\right\|_0 = \left\|\boldsymbol{\zeta} \cdot \left(\boldsymbol{V}^k - \boldsymbol{V}_d^k\right)\right\|_0 \leq \|\boldsymbol{\zeta}\|_0\left\|\boldsymbol{V}^k - \boldsymbol{V}_d^k\right\| \leq c\sqrt{\lambda_{V(d_V+1)}}, \tag{3.27}$$

$$\left\|\psi_h^k - \psi_v^k\right\|_0 = \left\|\tilde{\boldsymbol{\zeta}} \cdot \left(\boldsymbol{v}^k - \boldsymbol{v}_d^k\right)\right\|_0 \leq \|\tilde{\boldsymbol{\zeta}}\|_0\left\|\boldsymbol{v}^k - \boldsymbol{v}_d^k\right\| \leq c\left(\sqrt{\lambda_{v(d_v+1)}} + \sqrt{\lambda_{V(d_V+1)}}\right), \tag{3.28}$$

where $L + 1 \leq k \leq K$. Combining Theorem 2.2 with (3.19) and (3.20) as well as (3.27) and (3.28) yields (3.6) and (3.7), respectively. This completes the proof of Theorem 3.1. $\qquad\square$

Remark 3.2. Though the error estimates in Theorem 3.1 have more two terms $\sqrt{\lambda_{\tau(d_\tau+1)}}$ $(\tau = V, v)$ than those in Theorem 2.2, these increased terms can be used to determine the needed number of POD basis vectors for building the TGMFECNDR method. In fact, as long as d_τ is elected to meet $\sqrt{\lambda_{\tau(d_\tau+1)}} \leq \Delta t^2 + h^{l+1}$ $(\tau = V, v)$, the accuracy of TGM-FECNDR solutions is not influenced by dimension reduction. Lots of numerical tests (see [1, 9, 13, 19, 20, 22, 27, 32]) have demonstrated that the eigenvalues $\sqrt{\lambda_{\tau(d_\tau+1)}}$ $(\tau = V, v)$ will

soon fall to 0. Generally, when $d_\tau = 5$ or 6, $\sqrt{\lambda_{\tau(d_\tau+1)}}$ ($\tau = V, v$) are already very small to meet that $\sqrt{\lambda_{\tau(d_\tau+1)}} \le \Delta t^2 + h^{l+1}$ ($\tau = V, v$). It is more important that if the TGM-FECNDR solutions $\psi_V^{k_0+1}$ and $\psi_v^{k_0+1}$ at t_{k_0+1} fail to meet the specified precision, while ψ_V^k and ψ_v^k at time $t_k \le t_{k_0}$ still reach the specified precision, then we can retake two sets of TGMFECNDR solution vectors $\boldsymbol{\tau}_d^{k_0-L+1}, \boldsymbol{\tau}_d^{k_0-L+2}, \cdots, \boldsymbol{\tau}_d^{k_0}$ ($\tau = V, v$) to constitute two new matrices $\boldsymbol{A}_\tau = (\boldsymbol{\tau}_d^{k_0-L+1}, \boldsymbol{\tau}_d^{k_0-L+2}, \cdots, \boldsymbol{\tau}_d^{k_0})$ ($\tau = V, v$) and to produce two new sets of POD bases $\boldsymbol{\Phi}_\tau$ ($\tau = V, v$), and create a new TGMFECNDR method to calculate the TGMFECNDR solutions satisfying the specified precision. This is unmatched by the traditional TGMFECN method.

4. Two Sets of Numerical Experiments

Here, two sets of numerical experiments are given to confirm the correctness of the obtained theoretical results and show the advantage of the TGMFECNDR method for the INFOCH equation.

4.1. Two-dimensional numerical experiments

In the two-dimensional INFOCH equation, we take $\bar{\Omega} = [0, 1] \times [0, 1] \subset \mathbb{R}^2, \epsilon = 0.1, G_0 = 1$, $\tilde{f}(\psi) = f_+(\psi) + f_-(\psi), f(\psi) = \psi^3 - \psi$, and the initial function $\psi^0(\boldsymbol{x}) = 0.5 + 0.01 \operatorname{rand}(\boldsymbol{x})$. The fine grid division $\mathfrak{J}_h$ consists of the equal squares with length of side $h = 1/100$. When $l = 1$, in order to meet the condition $h = O(H^2)$, the coarse grid division $\mathfrak{J}_H$ consists of the equal squares with length of side $H = 1/10$. When time step $\Delta t = 1/100$ and $\sqrt{\lambda_{V(d_V+1)}} \le \sqrt{\lambda_{v(d_v+1)}} + \sqrt{\lambda_{V(d_V+1)}} \le 10^{-4}$, according to Theorems 2.2 and 3.1, the L^2 norm error estimates for the TGMFECN solutions ψ_h^k and the TGMFECNDR solutions ψ_v^k of the INFOCH equations can theoretically achieve $O(10^{-4})$.

The TGMFECNDR solutions can be calculated by the following flow chart:

(1) Experientially, find two sets of first 20 TGMFECN solution coefficient vectors $\boldsymbol{V}^1, \boldsymbol{V}^2, \cdots, \boldsymbol{V}^{20}$ and $\boldsymbol{v}^1, \boldsymbol{v}^2, \cdots, \boldsymbol{v}^{20}$ to constitute two matrices $\boldsymbol{A}_V = (\boldsymbol{V}^1, \boldsymbol{V}^2, \cdots, \boldsymbol{V}^{20})$ and $\boldsymbol{A}_v = (\boldsymbol{v}^1, \boldsymbol{v}^2, \cdots, \boldsymbol{v}^{20})$.

(2) According to the technique in Section 3.1, find two sets of normalized eigenvectors $\hat{\boldsymbol{\varphi}}_{\tau i}$ ($1 \le i \le 20$) corresponding to two sets of eigenvalues $\lambda_{\tau 1} \ge \lambda_{\tau 2} \ge \cdots \ge \lambda_{\tau 20} \ge 0$ for matrices $\boldsymbol{A}_\tau^\top \boldsymbol{A}_\tau$ ($\tau = V, v$).

(3) By reckoning, obtain the result that $\sqrt{\lambda_{V7}} \le \sqrt{\lambda_{V7}} + \sqrt{\lambda_{v7}} \le 2.25 \times 10^{-4}$. Therefore, it is necessary to take two sets of the first six normalized eigenvectors $\hat{\boldsymbol{\varphi}}_{\tau i}$ ($1 \le i \le 6$) to constitute two sets of POD bases $\boldsymbol{\Phi}_\tau = (\boldsymbol{\varphi}_{\tau 1}, \boldsymbol{\varphi}_{\tau 2}, \cdots, \boldsymbol{\varphi}_{\tau 6})$ by formulas $\boldsymbol{\varphi}_{\tau i} = \boldsymbol{A}_\tau \hat{\boldsymbol{\varphi}}_{\tau i}/\sqrt{\lambda_{\tau i}}$ ($\tau = V, v$ and $1 \le i \le 6$).

(4) Substitute the POD bases $\boldsymbol{\Phi}_\tau$ ($\tau = V, v$) and the above specified data into Problem 3.1 and find the TGMFECNDR solutions ψ_v^k at $t = 0, 50, 100$, and 200, as shown in Figs. 4.1(a)-4.4(a). Here, we only focus on the TGMFECNDR solutions ψ_v^k of the original unknown function ψ, the TGMFECNDR solutions of auxiliary function ϖ have not been calculated and the transition process solutions ψ_V^k are also unnecessary to show.

In order to show that the TGMFECNDR method is superior to the TGMFECN method, we also find the TGMFECN solutions ψ_h^k of the INFOCH equation at $t = 0, 50, 100$, and 200, as shown in Figs. 4.1(b)-4.4(b).
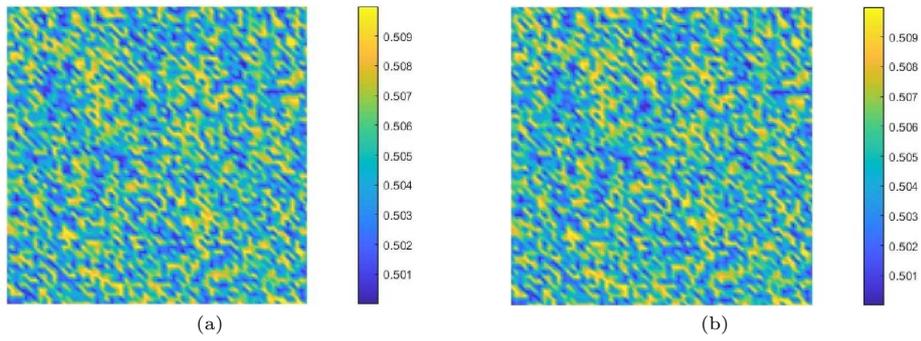
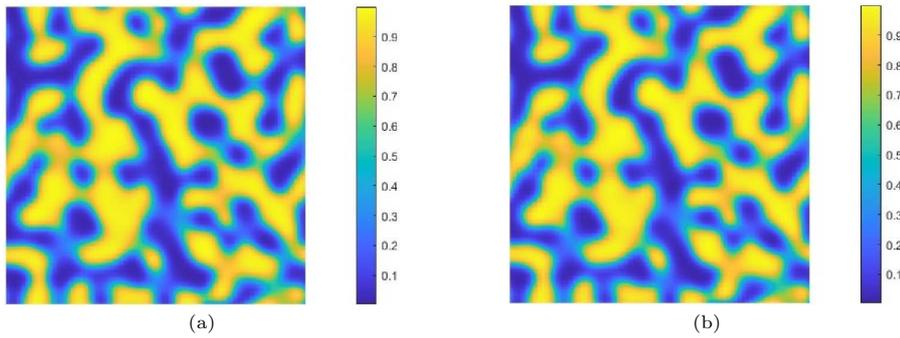Fig. 4.1. (a) The TGMFECNDR solution of ψ at $t = 0$. (b) The TGMFECN solution of ψ at $t = 0$.

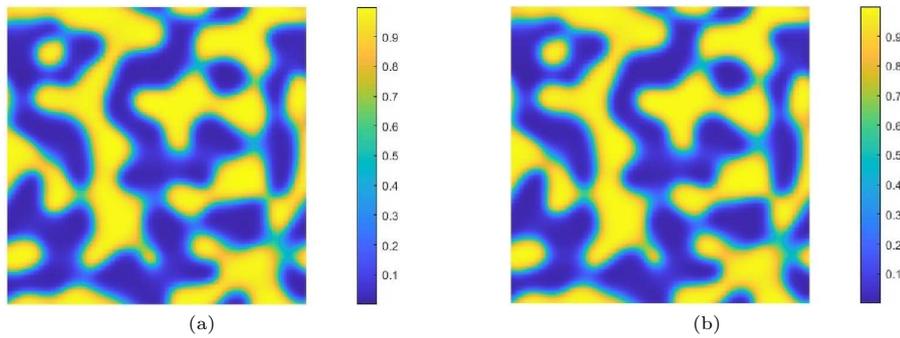Fig. 4.2. (a) The TGMFECNDR solution of ψ at $t = 50$. (b) The TGMFECN solution of ψ at $t = 50$.

Fig. 4.3. (a) The TGMFECNDR solution of ψ at $t = 100$. (b) The TGMFECN solution of ψ at $t = 100$.
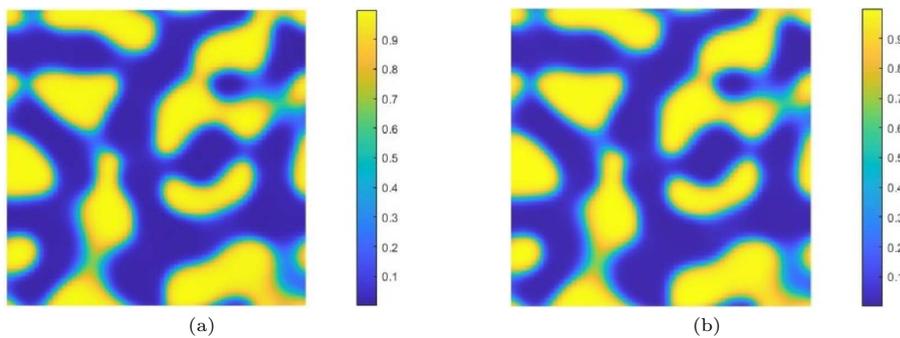

Fig. 4.4. (a) The TGMFECNDR solution of ψ at $t = 200$. (b) The TGMFECN solution of ψ at $t = 200$.

It is worth noting that, in practical application, it is unnecessary to compute the TGMFECN solutions, which may be replaced by the observations on the coarse grid division $\mathfrak{J}_H$ and fine grid division $\mathfrak{J}_h$ and to find directly the TGMFECNDR solutions ψ_v^k by the above four steps.

By comparing each pair of graphs in Figs. 4.1-4.4, it can be easy to know that the TGMFECNDR solutions are very closed to the TGMFECN solutions at $t = 0, 50, 100$, and 200.

By the above divisions $\mathfrak{J}_H$ and $\mathfrak{J}_h$, we can reckon that the TGMFECN method has $(10^2 + 10^4)$ unknowns at per time step, but the TGMFECNDR method has only 2×6 unknowns at per time step. Therefore, when the TGMFECNDR method is used to solve the INFOCH equation, it can greatly reduce the unknowns so as to greatly reduce the calculated workload, save CPU running time, reduce the accumulation of calculation errors, and improve the computation efficiency.

In order to further show that the TGMFECNDR method is superior to the TGMFECN method, we compared the errors and the CPU operating time of both TGMFECN and TGM-FECNDR solutions at $t = 50, 100$, and 200 calculated by Problems 2.4 and 3.1, respectively, as shown in Table 4.1. Because the analytical solution for the INFOCH equation is incalculable, the errors of the TGMFECN and TGMFECNDR solutions in Table 4.1 are approximatively reckoned by $\|\psi_h^{k+1} - \psi_h^k\|_0$ and $\|\psi_v^{k+1} - \psi_v^k\|_0$, respectively, which are the accumulation of computing errors on the entire time domain.

The data of Table 4.1 also signify that as the time node moves forward, the CPU operating time for calculating the TGMFECN solutions (including 10100 unknowns at per time step) increases quickly, but the CPU operating time for calculating the TGMFECNDR solutions (only including 12 unknowns at per time step) increases very tardily. For example, when $t = 200$, the CPU operating time for calculating the TGMFECN solution is about 40 times as that for calculating the TGMFECNDR solution. Hence, the TGMFECNDR method can highly save the calculating time. In addition, because the TGMFECN method contains too many unknowns, the errors of TGMFECN solutions amass gradually in the calculating process, while the TGMFECNDR method contains very few unknowns, the errors of the TGMFECNDR solutions amass slowly. Luckily, the numerical calculating errors are yet consistent with the theory errors and both achieve $\mathcal{O}(10^{-4})$. It is further shown that the TGMFECNDR method is far better than the TGMFECN method and is very valid for solving the INFOCH equation.

Table 4.1: The CPU operating time and errors of the TGMFECN and TGMFECNDR solutions.

t	k	TGMFECN solutions		TGMFECNDR solutions	
		Errors	CPU runtime	Errors	CPU runtime
50	5000	2.326×10^{-4}	956 s	2.326×10^{-4}	26 s
100	10000	4.652×10^{-4}	1916 s	3.453×10^{-4}	48 s
200	20000	9.311×10^{-4}	3828 s	4.616×10^{-4}	96 s

4.2. Three-dimensional numerical experiments

In the three-dimensional INFOCH equation, we take $\bar{\Omega} = [0, 1]^3 \subset \mathbb{R}^3, \epsilon = 0.1, G_0 = 1$, $\tilde{f}(\psi) = f_+(\psi) + f_-(\psi)$, $f(\psi) = \psi^3 - \psi$, and the initial function $\psi^0(\boldsymbol{x}) = 0.5 + 0.01\text{rand}(\boldsymbol{x})$. The fine grid division $\mathfrak{J}_h$ consists of the equal cubes with length of side $h = 1/100$. When $l = 1$, in order to meet the optimal order error estimates condition $h = \mathcal{O}(H^2)$, the coarse grid division $\mathfrak{J}_H$ consists of the equal cubes with length of side $H = 1/10$. When time step

$\Delta t = 1/100$ and $\sqrt{\lambda_{V(d_V+1)}} \le \sqrt{\lambda_{v(d_v+1)}} + \sqrt{\lambda_{V(d_V+1)}} \le 10^{-4}$, according to Theorems 2.2 and 3.1, the L^2 norm error estimates for the TGMFECN solutions ψ_h^k and the TGMFECNDR solutions ψ_v^k of the INFOCH equations can also theoretically achieve $\mathcal{O}(10^{-4})$.

Similarly, by the flow chart for finding the TGMFECNDR solutions in Section 4.1 and Problem 3.1, we can calculate the TGMFECNDR solutions ψ_v^k at $t = 0$, 50, 150, and 300, as shown in Figs. 4.5(a)-4.8(a).

In order to compare the TGMFECNDR method with the TGMFECN method, we also find the TGMFECN solutions ψ_h^k of the INFOCH equation at $t = 0$, 50, 150, and 300 by Problem 2.4, as shown in Figs. 4.5(b)-4.8(b).

(a)   (b)
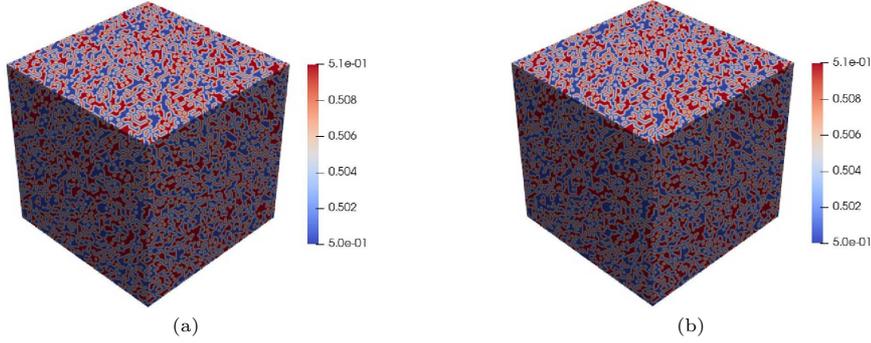
Fig. 4.5. (a) The TGMFECN solution ψ_v^k at $t = 0$. (b) The TGMFECN solution ψ_h^k at $t = 0$.

(a)   (b)
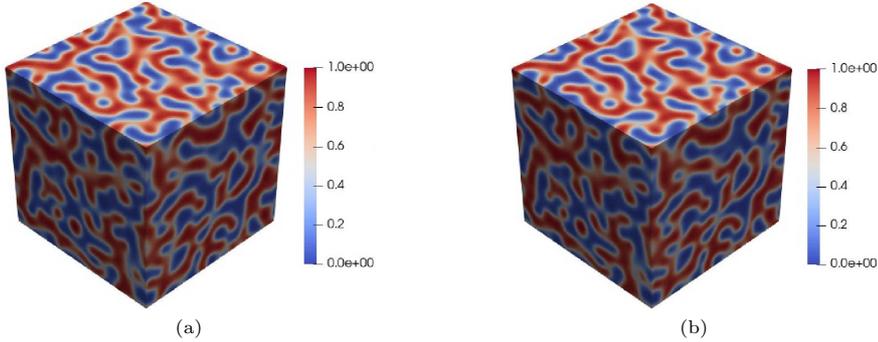
Fig. 4.6. (a) The TGMFECN solution ψ_v^k at $t = 50$. (b) The TGMFECN solution ψ_h^k at $t = 50$.

(a)   (b)
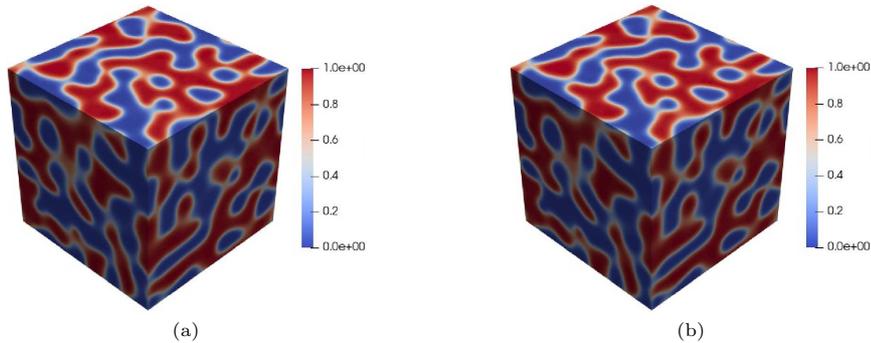
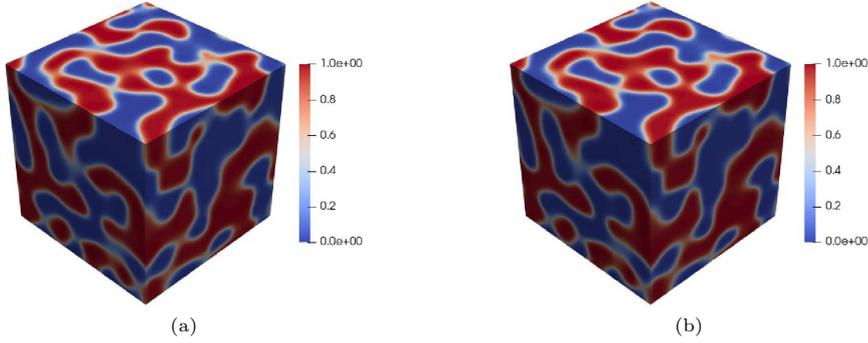Fig. 4.7. (a) The TGMFECN solution ψ_v^k at $t = 150$. (b) The TGMFECN solution ψ_h^k at $t = 150$.

Fig. 4.8. (a) The TGMFECN solution ψ_v^k at $t = 300$. (b) The TGMFECN solution ψ_h^k at $t = 300$.

By comparing the each pair of graphics in Figs. 4.5-4.8, it can be easy to see that the TGMFECNDR solutions are almost identical to the TGMFECN solutions at $t = 0, 50, 150$, and 300.

By the above divisions $\mathfrak{J}_H$ and $\mathfrak{J}_h$, it is easy to know that the TGMFECN method has $(10^3 + 10^6)$ unknowns at per time step, whereas the TGMFECNDR method has also only 2×6 unknowns at per time step. Therefore, when the TGMFECNDR method is used to find the numerical solutions for the three-dimensional INFOCH equation, it can indeed greatly reduce unknowns so as to greatly mitigate the calculated workload, save the CPU operating time, lessen the accumulation of computing errors, and enhance the computing efficiency.

In order to further exhibit the advantage of the TGMFECNDR method, we also compared the errors and the CPU operating time for calculating the TGMFECN and TGMFECNDR solutions by Problems 2.4 and 3.1 at $t = 50, 150$, and 300, respectively, as shown in Table 4.2.

The errors of the TGMFECN and the TGMFECNDR solutions in Table 4.2 are also approximatively reckoned by $\|\psi_h^{k+1} - \psi_h^k\|_0$ and $\|\psi_v^{k+1} - \psi_v^k\|_0$, respectively.

The data of Table 4.2 also signify that as the time node moves forward, the CPU operating time for calculating the TGMFECN solutions (including 1001000 unknowns at per time step) increases promptly, but the CPU runtime for finding the TGMFECNDR solutions (only including 12 unknowns at per time step) increases very slowly. For example, when $t = 300$, the CPU operating time for calculating the TGMFECN solution is about 50 times as that for calculating the TGMFECNDR solution. Therefore, the TGMFECNDR method can indeed highly save the computing time. In addition, because the TGMFECN method contains too many unknowns, the errors of TGMFECN solutions gradually amass in the calculating process, while the TGMFECNDR method contains very few unknowns, the errors of the TGMFECNDR solutions amass slowly. Luckily, the numerical calculation errors are yet consistent with the the-

Table 4.2: The CPU operating time and errors of the TGMFECN and TGMFECNDR solutions.

t	k	TGMFECN solutions		TGMFECNDR solutions	
		Errors	CPU runtime	Errors	CPU runtime
50	5000	3.252×10^{-4}	1276 s	2.326×10^{-4}	26 s
150	15000	6.495×10^{-4}	3849 s	3.323×10^{-4}	76 s
300	30000	9.881×10^{-4}	7658 s	4.436×10^{-4}	153 s

ory errors and both also reach $\mathcal{O}(10^{-4})$. Hence, the TGMFECNDR method is indeed much better than the TGMFECN method and is indeed very valid for solving the three-dimensional INFOCH equation.

5. Conclusions and Discussions

In this article, in order to solve the INFOCH equation, we have created a new TSDMCN scheme, a new TGMFECN method, and a new TGMFECNDR method. We have austerely analyzed the existence, stability, and errors of the TSDMCN, TGMFECN, and TGMFECNDR solutions, theoretically. We have also used two sets of numerical experiments to testify the correctness of the obtained theoretical results and revealed the superiority of the TGMFECNDR method. It is worth noting that the TSDMCN, TGMFECN, and TGMFECNDR methods for the INFOCH equation are firstly established in this paper, which are entirely distinguished from all the previous works. Undoubtedly, they are also distinguished from the existing FE and POD-based reduced-dimension methods with only time first-order accuracy, single layer grid, and conditional stability. Hence, they are original and bran-new.

The biggest advantage of the INFOCH equation is that the symmetry and positive definiteness of TSDMCN, TGMFECN, and TGMFECNDR methods are guaranteed, so they have unique series of solutions, achieving order error estimates.

Although we have merely developed the TGMFECNDR method for the INFOCH equation, the method presented in this paper can be generalized to more complex unsteady nonlinear PDEs, even to the practical engineering problems. Hence, the TGMFECNDR technique holds a very extensive applying room.

In this paper, we only focus on the numerical solutions of original unknown function ψ for the INFOCH equation. Therefore, the approximative solutions of the auxiliary function ϖ in the TGMFECN and TGMFECNDR matrix methods are eliminated and not solved. If ones care about the TGMFECNDR solutions of the auxiliary function ϖ, they can be added into the TGMFECN and TGMFECNDR matrix methods and their existence and unconditional stability as well as error estimates can also be proved, which should be worth to lucubrate.

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