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A LOW ORDER GLOBALLY DIVERGENCE-FREE WG FINITE ELEMENT METHOD FOR STEADY THERMALLY COUPLED INCOMPRESSIBLE MHD FLOW*

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Abstract

This paper develops a low order weak Galerkin (WG) finite element method for the steady thermally coupled incompressible magnetohydrodynamics flow. In the interior of elements, the WG scheme uses piecewise linear polynomials for the approximations of the velocity, the magnetic field and the temperature, and piecewise constants for the approximations of the pressure and the magnetic pseudo-pressure; and on the interfaces of elements, the scheme uses piecewise constants for the numerical traces of velocity and the temperature, and piecewise linear polynomials for the numerical traces of the magnetic fields, the pressure and the magnetic pseudo-pressure. This WG method is shown to yield globally divergence-free approximations of the velocity and magnetic fields. Existence and uniqueness results as well as optimal a priori error estimates for the discrete scheme are obtained. A convergent linearized iterative algorithm is presented. Numerical experiments are provided to verify the theoretical analysis.

Mathematics subject classification: 65N30, 65M60, 65M12.
Key words: Thermally coupled incompressible magnetohydrodynamics flow, Weak Galerkin method, Globally divergence-free, Error estimate.

1. Introduction

Magnetohydrodynamics (MHD) equations describe the basic physics laws of electrically conducting fluid flow interacting with magnetic fields, and are widely used in engineering areas; see, e.g. several monographs [8, 9, 26, 30, 33] and the references therein. In this paper we consider the steady thermally coupled incompressible MHD model, which is a coupled system of incompressible Navier-Stokes equations, Maxwell equations and a thermal equation.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a polygonal/polyhedral domain. The considered steady thermally coupled incompressible MHD model reads as follows: Find the velocity vector $\mathbf{u} = (u_1, u_2, \ldots, u_d)^\top$, the pressure p, the magnetic field $\mathbf{B} = (B_1, B_2, \ldots, B_d)^\top$, the magnetic pseudo-pressure r and the temperature T such that

$$-\frac{1}{H_a^2}\Delta\mathbf{u} + \frac{1}{N}(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \frac{1}{R_m}\nabla\times\mathbf{B}\times\mathbf{B} = \mathbf{f}_1 - \frac{G_r}{NR_e^2}\frac{\mathbf{g}}{g}T \quad \text{in } \Omega, \tag{1.1a}$$

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$$\nabla\cdot\mathbf{u}=0 \qquad \text{in } \Omega, \tag{1.1b}$$

$$\frac{1}{R_m}\nabla\times\nabla\times\mathbf{B}-\nabla\times(\mathbf{u}\times\mathbf{B})+\nabla r=\mathbf{f}_2 \qquad \text{in } \Omega, \tag{1.1c}$$

$$\nabla\cdot\mathbf{B}=0 \qquad \text{in } \Omega, \tag{1.1d}$$

$$-\frac{1}{P_rR_e}\Delta T+(\mathbf{u}\cdot\nabla)T=f_3 \qquad \text{in } \Omega, \tag{1.1e}$$

subject to the homogenous boundary conditions

$$\mathbf{u}|_{\partial\Omega}=0, \quad (\mathbf{B}\times\mathbf{n})|_{\partial\Omega}=0, \quad T|_{\partial\Omega}=0, \quad r|_{\partial\Omega}=0. \tag{1.2}$$

Here H_a is the Hartmann number, N the interaction parameter, R_e the Reynolds number, P_r the Prandtl number, R_m the magnetic Reynolds number, and G_r the Grashof number. $\mathbf{g}$ is the vector of gravitational acceleration with $g=|\mathbf{g}|$. $\mathbf{f}_1$ and $\mathbf{f}_2$ are the forcing functions, and f_3 denotes the heat source term. We refer to [2,23,24] for the study of the existence and uniqueness of weak solutions to related steady thermally coupled incompressible MHD models.

There are limited works in the literature on the finite element analysis of the steady thermally coupled incompressible MHD equations. Meir [24] proposed a Galerkin mixed finite element method and established optimal error estimates. Codina and Hernández [7] developed a stabilized finite element method. Yang and Zhang [39] analyzed three iteration algorithms, i.e. the Stokes, Newton and Oseen iterations, based on conforming mixed finite element discretization. We refer to [22, 28, 29] for several works on fully discrete mixed finite element methods for unsteady thermally coupled incompressible MHD model equations.

It is well-known that the two divergence constraints on the velocity and magnetic fields in the steady thermally coupled incompressible MHD model (1.1) are corresponding to the conservation of mass and magnetic flux, respectively, and that poor conservation of such physical properties in the algorithm design may lead to numerical instabilities [1, 3, 15, 16, 21, 27, 32]. For incompressible MHD equations, there have developed some divergence-free finite element methods, e.g. the mixed interior-penalty discontinuous Galerkin (DG) method with the exactly divergence-free velocity [10], the central DG method with the exactly divergence-free magnetic field [14, 18, 19], the mixed DG method with the exactly divergence-free velocity and magnetic field [13], the constrained transport finite element method with the exactly divergence-free velocity and magnetic field [20], and the weak Galerkin method with the exactly divergence-free velocity and magnetic field [41].

This paper is to develop a low order WG method with exactly divergence-free velocity and magnetic field for the steady thermally coupled incompressible MHD model (1.1). The WG method, pioneered by Wang and Ye [34, 35] for second-order elliptic problems, is of the same advantages as the DG method and has the local elimination property, i.e. the unknowns defined in the interior of elements can be locally eliminated by using the numerical traces defined on the interfaces of elements. We refer to [6, 11, 12, 25, 36–38, 40, 42, 43] for some applications of the WG method to the incompressible fluid flows and Maxwell equations.

Our WG discretization for (1.1) is of the following main features:

- It uses in the interior of elements piecewise linear polynomials for the approximations of the velocity, the magnetic field and the temperature, and piecewise constants for the approximations of the pressure and the magnetic pseudo-pressure, and uses on the interfaces of elements piecewise constants for the numerical traces of velocity and the temperature, and piecewise linear polynomials for the numerical traces of the magnetic fields, the pressure and the magnetic pseudo-pressure.

- It is "parameter-friendly" in the sense that no "sufficiently large" stabilization parameters are required.
- It yields globally and exactly divergence-free approximations of the velocity and magnetic fields, thus leading to pressure-robustness of the method.
- The obtained error estimates are optimal.

The rest of this paper is arranged as follows. Section 2 gives weak formulations of the model problem. Section 3 is devoted to the WG scheme and some preliminary results. In Section 4 we discuss the existence and uniqueness of the discrete solution. Section 5 derives a priori error estimates. Section 6 proposes an iteration algorithm for the nonlinear WG scheme. Finally, we provide some numerical results in Section 7.

2. Weak Problem

2.1. Notation

For any bounded domain $D \subset R^s$ $(s = d, d-1)$, nonnegative integer m and real number $1 \le q < \infty$, let $W^{m,q}(D)$ and $W_0^{m,q}(D)$ be the usual Sobolev spaces defined on D with norm $\|\cdot\|_{m,q,D}$ and semi-norm $|\cdot|_{m,q,D}$. In particular, $H^m(D) := W^{m,2}(D)$ and $H_0^m(D) := W_0^{m,2}(D)$, with $\|\cdot\|_{m,D} := \|\cdot\|_{m,2,D}$ and $|\cdot|_{m,D} := |\cdot|_{m,2,D}$. We use $(\cdot\,,\cdot)_{m,D}$ to denote the inner product of $H^m(D)$, with $(\cdot\,,\cdot)_D := (\cdot\,,\cdot)_{0,D}$. When $D = \Omega$, we set $\|\cdot\|_m := \|\cdot\|_{m,\Omega}$, $|\cdot|_m := |\cdot|_{m,\Omega}$, and $(\cdot\,,\cdot) := (\cdot\,,\cdot)_\Omega$. Especially, when $D \subset R^{d-1}$ we use $\langle\cdot\,,\cdot\rangle_D$ to replace $(\cdot\,,\cdot)_D$. For any integer $k \ge 0$, let $P_k(D)$ denote the set of all polynomials on D with degree no more than k. We also need the following spaces:

$$
\begin{aligned}
&L_0^2(\Omega) := \{v \in L^2(\Omega) : (v, 1) = 0\},\\
&\mathbf{H}(\mathrm{div}, \Omega) := \{\mathbf{v} \in [L^2(\Omega)]^d : \nabla\cdot\mathbf{v} \in L^2(\Omega)\},\\
&\mathbf{H}(\mathrm{curl}; \Omega) := \{\mathbf{v} \in [L^2(\Omega)]^d : \nabla\times\mathbf{v} \in [L^2(\Omega)]^{2d-3}\},\\
&\mathbf{H}_0(\mathrm{curl}; \Omega) := \{\mathbf{v} \in \mathbf{H}(\mathrm{curl}; \Omega) : \mathbf{n}\times\mathbf{v} = 0 \text{ on } \partial\Omega\},
\end{aligned}
$$

where the cross product $\times$ of two vectors is defined as following: for $\mathbf{v} = (v_1, \cdots, v_d)^\top$, $\mathbf{w} = (w_1, \cdots, w_d)^\top$,

$$
\mathbf{v}\times\mathbf{w} = \begin{cases} v_1w_2 - v_2w_1, & \text{if } \quad d = 2,\\ (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)^\top, & \text{if } \quad d = 3. \end{cases}
$$

Let $\mathcal{T}_h$ be a shape regular partition of Ω into closed simplexes, and let ε_h be the set of all edges (faces) of all the elements in Ω. For any $K \in \mathcal{T}_h, e \in \varepsilon_h$, we denote by h_K and h_e the diameters of K and e, respectively, and denote by ∂K the set of edges/faces of K. Let $\mathbf{n}_K$ and $\mathbf{n}_e$ denote the outward unit normal vectors along the boundary ∂K and e, respectively. Sometimes we may abbreviate $\mathbf{n}_K$ as $\mathbf{n}$.

We use $\nabla_h, \nabla_h\cdot$ and $\nabla_h\times$ to denote respectively the operators of piecewise-defined gradient, divergence and curl with respect to the decomposition $\mathcal{T}_h$. We also introduce the following mesh-dependent inner products and norms:

$$\langle u, v\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \langle u, v\rangle_{\partial K}, \quad \|v\|_{0,\partial\mathcal{T}_h} := \left(\sum_{K\in\mathcal{T}_h} \|v\|_{0,\partial K}^2\right)^{\frac{1}{2}}.$$

Throughout this paper, we use $\alpha \lesssim \beta$ to denote $\alpha \le C\beta$, where C is a positive constant independent of the mesh size h.

2.2. Weak form

For simplicity, we set

$$\mathbf{V} := \left[H_0^1(\Omega)\right]^d, \quad \mathbf{W} := \mathbf{H}_0(\mathrm{curl};\Omega).$$

For all $\mathbf{u}, \mathbf{v}, \Phi \in \mathbf{V}, \mathbf{B}, \mathbf{w} \in \mathbf{W}, T, z \in H_0^1(\Omega), q \in L_0^2(\Omega), \theta \in H_0^1(\Omega)$, we define the following bilinear and trilinear forms:

$$\begin{aligned}
&a_1(\mathbf{u},\mathbf{v}) := \frac{1}{H_a^2}(\nabla\mathbf{u},\nabla\mathbf{v}), && b_1(\mathbf{v},q) := (q,\nabla\cdot\mathbf{v}),\\
&a_2(\mathbf{B},\mathbf{w}) := \frac{1}{R_m^2}(\nabla\times\mathbf{B},\nabla\times\mathbf{w}), && b_2(\mathbf{w},\theta) := \frac{1}{R_m}(\nabla\theta,\mathbf{w}),\\
&a_3(T,z) := \frac{1}{P_r R_e}(\nabla T,\nabla z), && G_3(T,\mathbf{v}) := \left(\frac{G_r}{NR_e^2}\frac{\mathbf{g}}{g}T,\mathbf{v}\right),\\
&c_1(\Phi;\mathbf{u},\mathbf{v}) := \frac{1}{N}\left\{\frac{1}{2}\big(\nabla\cdot(\Phi\otimes\mathbf{u}),\mathbf{v}\big) - \frac{1}{2}\big(\nabla\cdot(\Phi\otimes\mathbf{v}),\mathbf{u}\big)\right\},\\
&c_2(\mathbf{v};\mathbf{B},\mathbf{w}) := \frac{1}{R_m}(\nabla\times\mathbf{w},\mathbf{v}\times\mathbf{B}),\\
&c_3(\mathbf{u};T,z) := \frac{1}{2}\big(\nabla\cdot(\mathbf{u}T),z\big) - \frac{1}{2}\big(\nabla\cdot(\mathbf{u}z),T\big).
\end{aligned}$$

It is easy to see that $c_1(\Phi;\mathbf{v},\mathbf{v}) = 0$ and $c_3(\mathbf{u};z,z) = 0$.

The weak form of the problem (1.1) reads: Find $\mathbf{u} \in \mathbf{V}, \mathbf{B} \in \mathbf{W}, T \in H_0^1(\Omega), p \in L_0^2(\Omega)$, $r \in H_0^1(\Omega)$ such that

$$\begin{aligned}
&a_1(\mathbf{u},\mathbf{v}) + a_2(\mathbf{B},\mathbf{w}) + b_1(\mathbf{u},q) - b_1(\mathbf{v},p) + b_2(\mathbf{w},r)\\
&\quad - b_2(\mathbf{B},\theta) + c_1(\mathbf{u};\mathbf{u},\mathbf{v}) + c_2(\mathbf{v};\mathbf{B},\mathbf{B}) - c_2(\mathbf{u};\mathbf{B},\mathbf{w})\\
&= (\mathbf{f}_1,\mathbf{v}) + \frac{1}{R_m}(\mathbf{f}_2,\mathbf{w}) - G_3(T,\mathbf{v}), \quad \forall\,(\mathbf{v},\mathbf{w},q,\theta) \in \mathbf{V}\times\mathbf{W}\times L_0^2(\Omega)\times H_0^1(\Omega), \qquad (2.1a)\\
&a_3(T,z) + c_3(\mathbf{u};T,z) = (f_3,z), \qquad \forall\, z \in H_0^1(\Omega). \qquad (2.1b)
\end{aligned}$$

3. Weak Galerkin Method

3.1. WG scheme

To establish the WG finite element scheme for the problem (1.1), we firstly introduce the discrete weak gradient operator $\nabla_{w,s}$, the discrete weak divergence operator $\nabla_{w,s}\cdot$ and the discrete weak curl operator $\nabla_{w,s}\times$ for $s = 0, 1$.

Definition 3.1 (Discrete Weak Gradient). *For any*

$$\mathrm{v} \in \mathrm{V}(K) := \big\{\mathrm{v} = \{\mathrm{v}_o, \mathrm{v}_b\} : \mathrm{v}_o \in L^2(K),\ \mathrm{v}_b \in H^{\frac{1}{2}}(\partial K)\big\},$$

and $K \in \mathcal{T}_h$*, the discrete weak gradient,* $\nabla_{w,s,K}\mathrm{v} \in [P_s(K)]^d$*, of* v *on* K *is defined by*

$$(\nabla_{w,s,K}\mathrm{v}, \boldsymbol{\phi})_K = -(\mathrm{v}_o, \nabla\cdot \boldsymbol{\phi})_K + \langle \mathrm{v}_b, \boldsymbol{\phi}\cdot \mathbf{n}_K\rangle_{\partial K}, \quad \forall\, \boldsymbol{\phi} \in [\mathcal{P}_s(K)]^d. \tag{3.1}$$

Then the global discrete weak gradient operator $\nabla_{w,s}$ *is defined by*

$$\nabla_{w,s}|_K := \nabla_{w,s,K}, \quad \forall\, K \in \mathcal{T}_h.$$

Moreover, for a vector $\mathbf{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_d)^\top$*, the discrete weak gradient* $\nabla_{w,s}\mathbf{v}$ *is defined by*

$$\nabla_{w,s}\mathbf{v} := (\nabla_{w,s}\mathrm{v}_1, \ldots, \nabla_{w,s}\mathrm{v}_d)^\top.$$

Definition 3.2 (Discrete Weak Divergence). *For any*

$$\mathbf{w} \in \mathbf{W}(K) := \big\{\mathbf{w} = \{\mathbf{w}_o, \mathbf{w}_b\} : \mathbf{w}_o \in [L^2(K)]^d,\ \mathbf{w}_b\cdot \mathbf{n}_K \in H^{-\frac{1}{2}}(\partial K)\big\},$$

and $K \in \mathcal{T}_h$*, the discrete weak divergence,* $\nabla_{w,s,K}\cdot \mathbf{w} \in \mathcal{P}_s(K)$*, of* $\mathbf{w}$ *on* K *is defined by*

$$(\nabla_{w,s,K}\cdot \mathbf{w}, \phi)_K = -(\mathbf{w}_o, \nabla\phi)_K + \langle \mathbf{w}_b\cdot \mathbf{n}_K, \phi\rangle_{\partial K}, \quad \forall\, \phi \in \mathcal{P}_s(K).$$

Then the global discrete weak divergence operator $\nabla_{w,\alpha}\cdot$ *is defined by*

$$\nabla_{w,s}\cdot|_K := \nabla_{w,s,K}\cdot, \quad \forall\, K \in \mathcal{T}_h.$$

Moreover, for a tensor $\hat{\mathbf{w}} = (\mathbf{w}_1, \ldots, \mathbf{w}_d)$*, the discrete weak divergence* $\nabla_{w,s}\cdot\hat{\mathbf{w}}$ *is defined by*

$$\nabla_{w,s}\cdot\hat{\mathbf{w}} := (\nabla_{w,s}\cdot\mathbf{w}_1, \ldots, \nabla_{w,s}\cdot\mathbf{w}_d)^\top.$$

Definition 3.3 (Discrete Weak Curl). *For any*

$$\boldsymbol{w} \in \boldsymbol{\mathcal{W}}(K) := \Big\{\boldsymbol{w} = \{\boldsymbol{w}_o, \boldsymbol{w}_b\} : \boldsymbol{w}_o \in [L^2(K)]^d, \boldsymbol{w}_b \times \mathbf{n}_K \in \big[H^{-\frac{1}{2}}(\partial K)\big]^{2d-3}\Big\},$$

and $K \in \mathcal{T}_h$*, the discrete weak curl* $\nabla_{w,s,K}\times \boldsymbol{w} \in [\mathcal{P}_\alpha(K)]^{2d-3}$ *on* K *is defined by*

$$(\nabla_{w,s,K}\times \boldsymbol{w}, \phi)_K = (\boldsymbol{w}_o, \nabla\times\phi)_K + \langle \boldsymbol{w}_b\times \mathbf{n}_K, \phi\rangle_{\partial K}, \quad \forall\, \phi \in [\mathcal{P}_s(K)]^{2d-3}. \tag{3.2}$$

Then the global discrete weak curl operator $\nabla_{w,s}\times$ *is defined by*

$$\nabla_{w,s}\times|_K := \nabla_{w,s,K}\times, \quad \forall\, K \in \mathcal{T}_h.$$

For any $K \in \mathcal{T}_h$ and $e \in \varepsilon_h$, let $Q_s^o : L^2(K) \to \mathcal{P}_s(K)$ and $Q_s^b : L^2(e) \to \mathcal{P}_s(e)$ be the standard L^2 projection operators. For vector spaces, we use $\mathbf{Q}_s^o$ to replace Q_s^o and $\mathbf{Q}_s^b$ to replace Q_s^b.

We introduce the following finite dimensional spaces:

$$\begin{aligned}
&\mathbf{V}_h = \big\{\mathbf{v}_h = \{\mathbf{v}_{ho}, \mathbf{v}_{hb}\} : \mathbf{v}_{ho}|_K \in [\mathcal{P}_1(K)]^d,\ \mathbf{v}_{hb}|_e \in [\mathcal{P}_0(e)]^d, \forall\, K \in \mathcal{T}_h,\ \forall\, e \in \varepsilon_h\big\},\\
&\mathbf{V}_h^0 = \big\{\mathbf{v}_h = \{\mathbf{v}_{ho}, \mathbf{v}_{hb}\} \in \mathbf{V}_h;\ \mathbf{v}_{hb}|_{\partial\Omega} = 0\big\},\\
&\mathbf{W}_h = \big\{\mathbf{w}_h = \{\mathbf{w}_{ho}, \mathbf{w}_{hb}\} : \mathbf{w}_{ho}|_K \in [\mathcal{P}_1(K)]^d, \mathbf{w}_{hb}|_e \in [\mathcal{P}_1(e)]^d,\ \forall\, K \in \mathcal{T}_h,\ \forall\, e \in \varepsilon_h\big\},\\
&\mathbf{W}_h^0 = \big\{\mathbf{w}_h = \{\mathbf{w}_{ho}, \mathbf{w}_{hb}\} \in \mathbf{V}_h;\ \mathbf{w}_{hb}\times \mathbf{n}|_{\partial\Omega} = 0\big\},\\
&Z_h = \big\{z_h = \{z_{ho}, z_{hb}\} : z_{ho}|_K \in [\mathcal{P}_1(K)],\ z_{hb}|_e \in [\mathcal{P}_0(e)],\ \forall\, K \in \mathcal{T}_h,\ \forall\, e \in \varepsilon_h\big\},\\
&Z_h^0 = \big\{z_h = \{z_{ho}, z_{hb}\} \in Z_h : z_{hb}|_{\partial\Omega=0}\big\},\\
&Q_h = \big\{q_h = \{q_{ho}, q_{hb}\} : q_{ho}|_K \in [\mathcal{P}_0(K)],\ q_{hb}|_e \in [\mathcal{P}_1(e)],\ \forall\, K \in \mathcal{T}_h,\ \forall\, e \in \varepsilon_h\big\},\\
&Q_h^0 = \big\{q_h = \{q_{ho}, q_{hb}\} \in Q_h : q_{ho} \in L_0^2(\Omega)\big\},\\
&R_h^0 = \big\{r_h = \{r_{ho}, r_{hb}\} \in Q_h;\ r_{hb}|_{\partial\Omega} = 0\big\}.
\end{aligned}$$

We also define the following bilinear forms and trilinear terms:

$$
\begin{aligned}
a_{1h}(\mathbf{u}_h, \mathbf{v}_h) &:= \frac{1}{H_a^2}(\nabla_{w,0}\mathbf{u}_h, \nabla_{w,0}\mathbf{v}_h) + s_{1h}(\mathbf{u}_h, \mathbf{v}_h),\\
s_{1h}(\mathbf{u}_h, \mathbf{v}_h) &:= \frac{1}{H_a^2}\big\langle \tau\big(\mathbf{Q}_0^b\mathbf{u}_{ho} - \mathbf{u}_{hb}\big), \mathbf{Q}_0^b\mathbf{v}_{ho} - \mathbf{v}_{hb}\big\rangle_{\partial\mathcal{T}_h},\\
a_{2h}(\mathbf{B}_h, \mathbf{w}_h) &:= \frac{1}{R_m^2}(\nabla_{w,0}\times\mathbf{B}_h, \nabla_{w,0}\times\mathbf{w}_h) + s_{2h}(\mathbf{B}_h, \mathbf{w}_h),\\
s_{2h}(\mathbf{B}_h, \mathbf{w}_h) &:= \frac{1}{R_m^2}\langle \tau(\mathbf{B}_{ho} - \mathbf{B}_{hb})\times\mathbf{n}, (\mathbf{w}_{ho} - \mathbf{w}_{hb})\times\mathbf{n}\rangle_{\partial\mathcal{T}_h},\\
a_{3h}(T_h, z_h) &:= \frac{1}{P_rR_e}(\nabla_{w,0}T_h, \nabla_{w,0}z_h) + s_{3h}(T_h, z_h),\\
s_{3h}(T_h, z_h) &:= \frac{1}{P_rR_e}\big\langle \tau\big(Q_0^bT_{ho} - T_{hb}\big), Q_0^bz_{ho} - z_{hb}\big\rangle_{\partial\mathcal{T}_h},\\
b_{1h}(\mathbf{v}_h, q_h) &:= (\nabla_{w,1}q_h, \mathbf{v}_{ho}),\\
b_{2h}(\mathbf{w}_h, \theta_h) &:= \frac{1}{R_m}(\nabla_{w,1}\theta_h, \mathbf{w}_{ho}),\\
G_{3h}(T_h, v_h) &:= \frac{G_r}{NR_e^2}\left(\frac{\mathbf{g}}{g}T_{ho}, \mathbf{v}_{h0}\right),\\
c_{1h}(\Phi_h; \mathbf{u}_h, \mathbf{v}_h) &:= \frac{1}{2N}(\nabla_{w,1}\cdot\{\mathbf{u}_{ho}\otimes\Phi_{ho}, \mathbf{u}_{hb}\otimes\Phi_{hb}\}, \mathbf{v}_{ho})\\
&\quad - \frac{1}{2N}(\nabla_{w,1}\cdot\{\mathbf{v}_{ho}\otimes\Phi_{ho}, \mathbf{v}_{hb}\otimes\Phi_{hb}\}, \mathbf{u}_{ho}),\\
c_{2h}(\mathbf{v}_h; \mathbf{B}_h, \mathbf{w}_h) &:= \frac{1}{R_m}(\nabla_{w,1}\times\mathbf{w}_h, \mathbf{v}_{ho}\times\mathbf{B}_{ho}),\\
c_{3h}(\mathbf{u}_h; T_h, z_h) &:= \frac{1}{2}\big(\nabla_{w,1}\cdot(\mathbf{u}_{ho}T_{ho}, \mathbf{u}_{hb}T_{hb}), z_{ho}\big) - \frac{1}{2}\big(\nabla_{w,1}\cdot(\mathbf{u}_{ho}z_{ho}, \mathbf{u}_{hb}z_{hb}), T_{ho}\big)
\end{aligned}
$$

for

$$
\begin{array}{lll}
\mathbf{u}_h = \{\mathbf{u}_{ho}, \mathbf{u}_{hb}\}, & \mathbf{v}_h = \{\mathbf{v}_{ho}, \mathbf{v}_{hb}\}, & \Phi_h = \{\Phi_{ho}, \Phi_{hb}\} \in \mathbf{V}_h^0,\\
\mathbf{B}_h = \{\mathbf{B}_{ho}, \mathbf{B}_{hb}\}, & \mathbf{w}_h = \{\mathbf{w}_{ho}, \mathbf{w}_{hb}\} \in \mathbf{W}_h^0, & \\
T_h = \{T_{ho}, T_{hb}\}, & Z_h = \{Z_{ho}, Z_{hb}\} \in Z_h^0, & \\
q_h = \{q_{ho}, q_{hb}\} \in Q_h^0, & \theta_h = \{\theta_{ho}, \theta_{hb}\} \in R_h^0, &
\end{array}
$$

where the stabilization parameter τ in $s_{1h}(\cdot;\cdot,\cdot), s_{2h}(\cdot;\cdot,\cdot)$ and $s_{3h}(\cdot;\cdot,\cdot)$ is given by

$$\tau|_{\partial K} = h_K^{-1}, \quad \forall K \in \mathcal{T}_h.$$

We easily see that

$$
\begin{aligned}
c_{1h}(\Phi_h; \mathbf{v}_h, \mathbf{v}_h) &= 0, \quad \forall \Phi_h, \mathbf{v}_h,\\
c_{3h}(\mathbf{u}_h; z_h, z_h) &= 0, \quad \forall \mathbf{u}_h, z_h.
\end{aligned}
$$

The WG finite element scheme for the model (1.1) reads as follows: Find $\mathbf{u}_h = \{\mathbf{u}_{ho}, \mathbf{u}_{hb}\} \in \mathbf{V}_h^0$, $\mathbf{B}_h = \{\mathbf{B}_{ho}, \mathbf{B}_{hb}\} \in \mathbf{W}_h^0$, $T_h = \{T_{ho}, T_{hb}\} \in Z_h^0$, $p_h = \{p_{ho}, p_{hb}\} \in Q_h^0$, $r_h = \{r_{ho}, r_{hb}\} \in R_h^0$ such that

$$a_{1h}(\mathbf{u}_h, \mathbf{v}_h) + a_{2h}(\mathbf{B}_h, \mathbf{w}_h) + b_{1h}(\mathbf{v}_h, p_h) - b_{1h}(\mathbf{u}_h, q_h) + b_{2h}(\mathbf{w}_h, r_h) - b_{2h}(\mathbf{B}_h, \theta_h)$$

$$
\begin{aligned}
&+ c_{1h}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) + c_{2h}(\mathbf{v}_h;\mathbf{B}_h,\mathbf{B}_h) - c_{2h}(\mathbf{u}_h;\mathbf{B}_h,\mathbf{w}_h) + G_{3h}(T_h,\mathbf{v}_h) \\
&= (\mathbf{f}_1,\mathbf{v}_{ho}) + \frac{1}{R_m}(\mathbf{f}_2,\mathbf{w}_{ho}), \quad \forall\,(\mathbf{v}_h,\mathbf{w}_h,q_h,\theta_h)\in \mathbf{V}_h^0\times\mathbf{W}_h^0\times Q_h^0\times R_h^0, \qquad (3.3a)\\
&a_{3h}(T_h,z_h) + c_{3h}(\mathbf{u}_h;T_h,z_h) = (f_3,z_{ho}), \quad \forall\, z_h\in Z_h^0. \qquad (3.3b)
\end{aligned}
$$

Notice that the above scheme can be rewritten as the following system: Find

$$
\begin{aligned}
&\mathbf{u}_h=\{\mathbf{u}_{ho},\mathbf{u}_{hb}\}\in\mathbf{V}_h^0, \quad \mathbf{B}_h=\{\mathbf{B}_{ho},\mathbf{B}_{hb}\}\in\mathbf{W}_h^0,\\
&T_h=\{T_{ho},T_{hb}\}\in Z_h^0, \quad p_h=\{p_{ho},p_{hb}\}\in Q_h^0, \quad r_h=\{r_{ho},r_{hb}\}\in R_h^0
\end{aligned}
$$

such that

$$
\begin{aligned}
&a_{1h}(\mathbf{u}_h,\mathbf{v}_h)+b_{1h}(\mathbf{v}_h,p_h)+c_{1h}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)+c_{2h}(\mathbf{v}_h;\mathbf{B}_h,\mathbf{B}_h)\\
&=(\mathbf{f}_1,\mathbf{v}_{ho})-G_{3h}(T_h,\mathbf{v}_h), && \forall\,\mathbf{v}_h\in\mathbf{V}_h^0, \qquad (3.4a)\\
&b_{1h}(\mathbf{u}_h,q_h)=0, && \forall\, q_h\in Q_h^0, \qquad (3.4b)\\
&a_{2h}(\mathbf{B}_h,\mathbf{w}_h)+b_{2h}(\mathbf{w}_h,r_h)-c_{2h}(\mathbf{u}_h;\mathbf{B}_h,\mathbf{w}_h)=\frac{1}{R_m}(\mathbf{f}_2,\mathbf{w}_{ho}), && \forall\,\mathbf{w}_h\in\mathbf{W}_h^0, \qquad (3.4c)\\
&b_{2h}(\mathbf{B}_h,\theta_h)=0, && \forall\,\theta_h\in R_h^0, \qquad (3.4d)\\
&a_{3h}(T_h,z_h)+c_{3h}(\mathbf{u}_h;T_h,z_h)=(f_3,z_{ho}), && \forall\, z_h\in Z_h^0. \qquad (3.4e)
\end{aligned}
$$

As shown in [41, Theorem 3.1], the Eqs. (3.4b) and (3.4d) lead to the globally divergence-free discrete solutions of velocity and magnetic field, respectively, i.e. there hold

$$
\begin{aligned}
&\mathbf{u}_h\in\mathbf{H}(\mathrm{div},\Omega), \quad \nabla\cdot\mathbf{u}_h=0, \qquad (3.5)\\
&\mathbf{B}_h\in\mathbf{H}(\mathrm{div},\Omega), \quad \nabla\cdot\mathbf{B}_h=0. \qquad (3.6)
\end{aligned}
$$

To discuss the existence and uniqueness of the discrete solution of the scheme (3.3) and derive error estimates, we will give some preliminary results in next subsection.

3.2. Preliminary results

In view of the definitions of weak gradient and curl operators, the Green's formula, the Cauchy-Schwarz inequality, the trace inequality and the inverse inequality, we can easily derive the following inequalities on $\mathbf{V}_h, \mathbf{W}_h$ and Z_h.

Lemma 3.1 ([6]). *For any $K\in\mathcal{T}_h$ and $\mathbf{v}_h=\{\mathbf{v}_{ho},\mathbf{v}_{hb}\}\in\mathbf{V}_h, \mathbf{w}_h=\{\mathbf{w}_{ho},\mathbf{w}_{hb}\}\in\mathbf{W}_h$, $z_h=\{z_{ho},z_{hb}\}\in Z_h$, and $s=0,1$, there hold*

$$
\begin{aligned}
&\|\nabla\mathbf{v}_{ho}\|_{0,K}\lesssim\|\nabla_{w,s}\mathbf{v}_h\|_{0,K}+h_K^{-\frac12}\left\|\mathbf{Q}_0^b\mathbf{v}_{ho}-\mathbf{v}_{hb}\right\|_{0,\partial K}, \qquad (3.7a)\\
&\|\nabla_{w,s}\mathbf{v}_h\|_{0,K}\lesssim\|\nabla\mathbf{v}_{ho}\|_{0,K}+h_K^{-\frac12}\left\|\mathbf{Q}_0^b\mathbf{v}_{ho}-\mathbf{v}_{hb}\right\|_{0,\partial K}, \qquad (3.7b)\\
&\|\nabla\times\mathbf{w}_{ho}\|_{0,K}\lesssim\|\nabla_{w,s}\times\mathbf{w}_h\|_{0,K}+h_K^{-\frac12}\|(\mathbf{w}_{ho}-\mathbf{w}_{hb})\times\mathbf{n}\|_{0,\partial K}, \qquad (3.7c)\\
&\|\nabla_{w,s}\times\mathbf{w}_h\|_{0,K}\lesssim\|\nabla\times\mathbf{w}_{ho}\|_{0,K}+h_K^{-\frac12}\|(\mathbf{w}_{ho}-\mathbf{w}_{hb})\times\mathbf{n}\|_{0,\partial K}, \qquad (3.7d)\\
&\|\nabla z_{ho}\|_{0,K}\lesssim\|\nabla_{w,s}z_h\|_{0,K}+h_K^{-\frac12}\left\|Q_0^b z_{ho}-z_{hb}\right\|_{0,\partial K}, \qquad (3.7e)\\
&\|\nabla_{w,s}z_h\|_{0,K}\lesssim\|\nabla z_{ho}\|_{0,K}+h_K^{-\frac12}\left\|Q_0^b z_{ho}-z_{hb}\right\|_{0,\partial K}. \qquad (3.7f)
\end{aligned}
$$

Introduce the following semi-norms respectively on $\mathbf{V}_h^0, \mathbf{W}_h^0, Z_h^0, Q_h^0$ and R_h^0:

$$
\begin{aligned}
|||\mathbf{v}_h|||_V &:= \left(\|\nabla_{w,0}\mathbf{v}_h\|_0^2 + \left\|\tau^{\frac{1}{2}}\left(\mathbf{Q}_0^b\mathbf{v}_{ho} - \mathbf{v}_{hb}\right)\right\|_{0,\partial\mathcal{T}_h}^2\right)^{\frac{1}{2}}, && \forall\, \mathbf{v}_h \in \mathbf{V}_h^0,\\
|||\mathbf{w}_h|||_W &:= \left(\|\nabla_{w,0}\times\mathbf{w}_h\|_0^2 + \left\|\tau^{\frac{1}{2}}(\mathbf{w}_{ho} - \mathbf{w}_{hb})\times\mathbf{n}\right\|_{0,\partial\mathcal{T}_h}^2\right)^{\frac{1}{2}}, && \forall\, \mathbf{w}_h \in \mathbf{W}_h^0,\\
|||z_h|||_Z &:= \left(\|\nabla_{w,0}z_h\|_0^2 + \left\|\tau^{\frac{1}{2}}\left(Q_0^b z_{ho} - z_{hb}\right)\right\|_{0,\partial\mathcal{T}_h}^2\right)^{\frac{1}{2}}, && \forall\, z_h \in Z_h^0,\\
|||q_h|||_Q &:= \left(\|q_{ho}\|_0^2 + \sum_{K\in\mathcal{T}_h} h_K^2\|\nabla_{w,1}q_h\|_{0,K}^2\right)^{\frac{1}{2}}, && \forall\, q_h \in Q_h^0,\\
|||\theta_h|||_R &:= \left(\|\theta_{ho} - \bar{\theta}_{ho}\|_0^2 \sum_{K\in\mathcal{T}_h} h_K^2\|\nabla_{w,1}\theta_h\|_{0,K}^2\right)^{\frac{1}{2}}, && \forall\, \theta_h \in R_h^0,
\end{aligned}
$$

where

$$\bar{\theta}_{ho} := \frac{1}{|\Omega|}\int_\Omega \theta_{ho} d\mathbf{x}$$

denotes the mean value of θ_{ho} and we recall that $\tau|_{\partial K} = h_K^{-1}$. It is easy to see that $|||\cdot|||_V$, $|||\cdot|||_Z$, $|||\cdot|||_Q$ and $|||\cdot|||_R$ are norms on $\mathbf{V}_h^0, Z_h^0, Q_h^0$ and R_h^0, respectively (cf. [6]). $|||\cdot|||_W$ is a norm on $\bar{\mathbf{W}}_h$ (cf. [41]).

Lemma 3.2. *There hold*

$$
\begin{aligned}
\|\nabla_h\mathbf{v}_{ho}\|_0 &\lesssim |||\mathbf{v}_h|||_V, && \forall\, \mathbf{v}_h \in \mathbf{V}_h^0, && (3.8a)\\
\|\nabla_h\times\mathbf{w}_{ho}\|_0 &\lesssim |||\mathbf{w}_h|||_W, && \forall\, \mathbf{w}_h \in \mathbf{W}_h^0, && (3.8b)\\
\|\nabla_h z_{ho}\|_0 &\lesssim |||z_h|||_Z, && \forall\, z_h \in Z_h^0. && (3.8c)
\end{aligned}
$$

In addition,

$$
\begin{aligned}
\|\mathbf{v}_{h0}\|_{0,q} &\lesssim |||\mathbf{v}_h|||_V, && \forall\, \mathbf{v}_h \in \mathbf{V}_h^0, && (3.9a)\\
\|z_{h0}\|_{0,q} &\lesssim |||z_h|||_Z, && \forall\, z_h \in Z_h^0 && (3.9b)
\end{aligned}
$$

for $1 \le q < \infty$ *when* $d = 2$, *and for* $1 \le q \le 6$ *when* $d = 3$.

Proof. The inequality (3.8) follows from Lemma 3.1 directly and (3.9) comes from [12, Lemma 3.5]. □

Lemma 3.3 ([41]). *There holds*

$$\|\mathbf{w}_{ho}\|_{0,3,\Omega} \lesssim |||\mathbf{w}_h|||_W, \quad \forall\, \mathbf{w}_h = \{\mathbf{w}_{ho}, \mathbf{w}_{hb}\} \in \bar{\mathbf{W}}_h. \tag{3.10}$$

In light of the trace theorem, the inverse inequality and scaling arguments, we can get the following lemma (cf. [31]).

Lemma 3.4. *For all* $K \in \mathcal{T}_h, \psi \in H^1(K)$, *and* $1 \le q \le \infty$, *there holds*

$$\|\psi\|_{0,q,\partial K} \lesssim h_K^{-\frac{1}{q}}\|\psi\|_{0,q,K} + h_K^{1-\frac{1}{q}}|\psi|_{1,q,K}.$$

In particular, for all $\psi \in \mathcal{P}_k(K)$,

$$\|\psi\|_{0,q,\partial K} \lesssim h_K^{-\frac{1}{q}}\|\psi\|_{0,q,K}.$$

Lemma 3.5 ([31]). *For any $K \in \mathcal{T}_h, e \in \varepsilon_h$, and $s = 0, 1$, there hold*

$$\begin{aligned}
&\left\|\mathrm{v} - Q_s^o \mathrm{v}\right\|_{0,K} + h_K \left|\mathrm{v} - Q_s^o \mathrm{v}\right|_{1,K} \lesssim h_K^j |\mathrm{v}|_{j,K}, && \forall\, \mathrm{v} \in H^j(K), \quad 1 \le j \le s+1,\\
&\left\|\mathrm{v} - Q_s^o \mathrm{v}\right\|_{0,\partial K} + \left\|\mathrm{v} - Q_s^b \mathrm{v}\right\|_{0,\partial K} \lesssim h_K^{j-\frac{1}{2}} |\mathrm{v}|_{j,K}, && \forall\, \mathrm{v} \in H^j(K), \quad 1 \le j \le s+1,\\
&\left\|Q_s^o \mathrm{v}\right\|_{0,K} \le \|\mathrm{v}\|_{0,K}, && \forall\, \mathrm{v} \in L^2(K),\\
&\left\|Q_s^b \mathrm{v}\right\|_{0,e} \le \|\mathrm{v}\|_{0,e}, && \forall\, \mathrm{v} \in L^2(e).
\end{aligned}$$

For any $K \in \mathcal{T}_h$, we introduce the local Raviart-Thomas ($\mathcal{RT}$) element space

$$\mathbb{RT}_1(K) = [\mathcal{P}_1(K)]^d + \mathbf{x}\mathcal{P}_1(K),$$

and the $\mathcal{RT}$ projection operator $\mathbf{P}_1^{\mathcal{RT}} : [H^1(K)]^d \to \mathbb{RT}_1(K)$ (cf. [5]) defined by

$$\left\langle \mathbf{P}_1^{\mathcal{RT}} \mathbf{v} \cdot \mathbf{n}_e, w \right\rangle_e = \langle \mathbf{v} \cdot \mathbf{n}_e, w \rangle_e, \quad \forall\, w \in \mathcal{P}_1(e), \quad e \in \partial K, \tag{3.11a}$$

$$\left(\mathbf{P}_1^{\mathcal{RT}} \mathbf{v}, \mathbf{w}\right)_K = (\mathbf{v}, \mathbf{w})_K, \quad \forall\, \mathbf{w} \in [\mathcal{P}_0(K)]^d. \tag{3.11b}$$

Lemmas 3.6-3.8 give some properties of the $\mathcal{RT}$ element space and the $\mathcal{RT}$ projection.

Lemma 3.6 ([5]). *For any $\mathbf{v}_{ho} \in \mathbb{RT}_1(K), \nabla \cdot \mathbf{v}_{ho}|_K = 0$ gives $\mathbf{v}_{ho} \in [\mathcal{P}_1(K)]^d$.*

Lemma 3.7 ([5]). *For any $K \in \mathcal{T}_h$ and $\mathbf{v} \in [H^1(K)]^d$, the following property holds:*

$$\left(\nabla \cdot \mathbf{P}_1^{RT} \mathbf{v}, \phi_h\right)_K = (\nabla \cdot \mathbf{v}, \phi_h)_K, \quad \forall\, \mathbf{v} \in [H^1(K)]^d, \quad \phi_h \in \mathcal{P}_1(K).$$

By using the triangle inequality, the inverse inequality, Lemmas 3.5 and 3.7 we can get more estimates for the $\mathcal{RT}$ projection (cf. [12]).

Lemma 3.8. *Let j be a nonnegative integer. For any $K \in \mathcal{T}_h$ and $\mathbf{v} \in [H^j(K)]^d$, the following estimates hold:*

$$\begin{aligned}
&\left|\mathbf{v} - \mathbf{P}_1^{\mathcal{RT}} \mathbf{v}\right|_{1,K} \lesssim h_K^{j-1} |\mathbf{v}|_{j,K}, && 1 \le j \le 2,\\
&\left|\mathbf{v} - \mathbf{P}_1^{\mathcal{RT}} \mathbf{v}\right|_{0,\partial K} \lesssim h_K^{j-\frac{1}{2}} |\mathbf{v}|_{j,K}, && 0 \le j \le 2,\\
&\left|\mathbf{v} - \mathbf{P}_1^{\mathcal{RT}} \mathbf{v}\right|_{0,3,K} \lesssim h_K^{j-\frac{d}{6}} |\mathbf{v}|_{j,K}, && 1 \le j \le 2,\\
&\left|\mathbf{v} - \mathbf{P}_1^{\mathcal{RT}} \mathbf{v}\right|_{0,3,\partial K} \lesssim h_K^{j-\frac{1}{3}-\frac{d}{6}} |\mathbf{v}|_{j,K}, && 1 \le j \le 2.
\end{aligned}$$

We also have the following commutativity properties for the $\mathcal{RT}$ projection, the L^2 projections and the discrete weak operators.

Lemma 3.9 ([6, 25]). *There hold*

$$\begin{aligned}
&\nabla_{w,0}\left\{\mathbf{P}_1^{\mathcal{RT}} \mathbf{v}, \mathbf{Q}_0^b \mathbf{v}\right\} = \mathbf{Q}_0^o(\nabla \mathbf{v}), && \forall\, \mathbf{v} \in [H^1(\Omega)]^d,\\
&\nabla_{w,0} \times \left\{\mathbf{P}_1^{\mathcal{RT}} \mathbf{w}, \mathbf{Q}_0^b \mathbf{w}\right\} = \mathbf{Q}_0^o(\nabla \times \mathbf{w}), && \forall\, \mathbf{w} \in H(\mathrm{curl}; \Omega),\\
&\nabla_{w,0}\left\{Q_1^o z, Q_0^b z\right\} = \mathbf{Q}_0^o(\nabla z), && \forall\, z \in H^1(\Omega),\\
&\nabla_{w,1}\left\{Q_0^o q, Q_1^b q\right\} = \mathbf{Q}_1^o(\nabla q), && \forall\, q \in H^1(\Omega).
\end{aligned}$$

4. Existence and Uniqueness of the Discrete Solution

4.1. Stability conditions

Lemma 4.1. *For any* $\mathbf{u}_h, \mathbf{v}_h, \Phi_h \in \mathbf{V}_h^0, \mathbf{B}_h, \mathbf{w}_h \in \mathbf{W}_h^0$, *and* $T_h, z_h \in Z_h^0$, *there hold*

$$a_{1h}(\mathbf{u}_h, \mathbf{v}_h) \leq \frac{1}{H_a^2} |||\mathbf{u}_h|||_V |||\mathbf{v}_h|||_V, \tag{4.1a}$$

$$a_{1h}(\mathbf{v}_h, \mathbf{v}_h) = \frac{1}{H_a^2} |||\mathbf{v}_h|||_V^2, \tag{4.1b}$$

$$a_{2h}(\mathbf{B}_h, \mathbf{w}_h) \leq \frac{1}{R_m^2} |||\mathbf{B}_h|||_W |||\mathbf{w}_h|||_W, \tag{4.1c}$$

$$a_{2h}(\mathbf{w}_h, \mathbf{w}_h) = \frac{1}{R_m^2} |||\mathbf{w}_h|||_W^2, \tag{4.1d}$$

$$a_{3h}(T_h, z_h) \leq \frac{1}{P_r R_e} |||T_h|||_Z |||z_h|||_Z, \tag{4.1e}$$

$$a_{3h}(z_h, z_h) = \frac{1}{P_r R_e} |||z_h|||_Z^2, \tag{4.1f}$$

$$G_{3h}(T_h, \mathbf{v}_h) \leq \frac{G_r}{N R_e^2} \frac{\mathbf{g}}{g} |||T_h|||_Z |||\mathbf{v}_h|||_V, \tag{4.1g}$$

$$c_{1h}(\Phi_h; \mathbf{v}_h, \mathbf{v}_h) = 0, \tag{4.1h}$$

$$c_{1h}(\Phi_h; \mathbf{u}_h, \mathbf{v}_h) \lesssim |||\Phi_h|||_V |||\mathbf{u}_h|||_V |||\mathbf{v}_h|||_V, \tag{4.1i}$$

$$c_{2h}(\mathbf{v}_h; \mathbf{B}_h, \mathbf{w}_h) \lesssim |||\mathbf{B}_h|||_W |||\mathbf{w}_h|||_W |||\mathbf{v}_h|||_V, \tag{4.1j}$$

$$c_{3h}(\mathbf{u}_h; z_h, z_h) = 0, \tag{4.1k}$$

$$c_{3h}(\mathbf{u}_h; T_h, z_h) \lesssim |||\mathbf{u}_h|||_V |||T_h|||_Z |||z_h|||_Z. \tag{4.1l}$$

Proof. The results (4.1a)-(4.1d) and (4.1h)-(4.1j) are from [41].

From the definitions of $a_{3h}(\cdot,\cdot)$ and $G_{3h}(\cdot,\cdot)$, Cauchy-Schwarz inequality and Lemma 3.1, we can easily get (4.1e)-(4.1g). The relation (4.1k) follows directly from the definition of $c_{3h}(\cdot;\cdot,\cdot)$. Finally, similar to (4.1i), the estimate (4.1l) follows from the Hölder's inequality, the inverse inequality, Lemmas 3.4 and 3.2. □

Lemma 4.2 ([41]). *There hold the following inf-sup inequalities:*

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h^0} \frac{b_{1h}(\mathbf{v}_h, q_h)}{|||\mathbf{v}_h|||_V} \gtrsim |||q_h|||_Q, \qquad \forall\, q_h \in Q_h^0, \tag{4.2}$$

$$\sup_{\mathbf{w}_h \in \mathbf{W}_h^0} \frac{b_{2h}(\mathbf{w}_h, \theta_h)}{|||\mathbf{w}_h|||_W} \gtrsim |||\theta_h|||_R, \quad \forall\, \theta_h \in R_h^0. \tag{4.3}$$

4.2. Existence and uniqueness results

We introduce the following two spaces:

$$\begin{aligned}
\bar{\mathbf{V}}_h &:= \{\mathbf{v}_h \in \mathbf{V}_h^0 : b_{1h}(\mathbf{v}_h, q_h) = 0,\ \forall q_h \in Q_h^0\} \\
&= \{\mathbf{v}_h \in \mathbf{V}_h^0 : \mathbf{v}_{ho} \in \mathbf{H}(\mathrm{div}, \Omega), \nabla \cdot \mathbf{v}_{ho} = 0\}, \\
\bar{\mathbf{W}}_h &:= \{\mathbf{w}_h \in \mathbf{W}_h^0 : b_{2h}(\mathbf{w}_h, \theta_h) = 0,\ \forall \theta_h \in R_h^0\} \\
&= \{\mathbf{w}_h \in \mathbf{W}_h^0 : \mathbf{w}_{ho} \in \mathbf{H}(\mathrm{div}, \Omega), \nabla \cdot \mathbf{w}_{ho} = 0\}.
\end{aligned}$$

Thus, the solution $(\mathbf{u}_h, \mathbf{B}_h, T_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Z_h^0$ of the scheme (3.3) also solves the following discretization problem: find $(\mathbf{u}_h, \mathbf{B}_h, T_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h \times Z_h^0$ such that

$$\begin{aligned} & a_{1h}(\mathbf{u}_h, \mathbf{v}_h) + a_{2h}(\mathbf{B}_h, \mathbf{w}_h) + c_{1h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \\ & \quad + c_{2h}(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) - c_{2h}(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h) + G_{3h}(T_h, \mathbf{v}_h) \\ = & (\mathbf{f}_1, \mathbf{v}_{ho}) + \frac{1}{R_m}(\mathbf{f}_2, \mathbf{w}_{ho}), \qquad \forall (\mathbf{v}_h, \mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h, \end{aligned} \tag{4.4a}$$

$$a_{3h}(T_h, z_h) + c_{3h}(\mathbf{u}_h; T_h, z_h) = (f_3, z_{ho}), \quad \forall z_h \in Z_h^0. \tag{4.4b}$$

For the discretization problems (3.3) and (4.4), we easily have the following equivalence result.

Lemma 4.3. *The problems* (3.3) *and* (4.4) *are equivalent in the sense that (I) and (II) hold:*

(I) If $(\mathbf{u}_h, \mathbf{B}_h, T_h, p_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Z_h^0 \times Q_h^0 \times R_h^0$ *is the solution to the problem* (3.3), *then* $(\mathbf{u}_h, \mathbf{B}_h, T_h)$ *is also the solution to the problem* (4.4).

(II) If $(\mathbf{u}_h, \mathbf{B}_h, T_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h \times Z_h^0$ *is the solution to the problem* (4.4), *then* $(\mathbf{u}_h, \mathbf{B}_h, T_h, p_h, r_h)$ *is also the solution to the problem* (3.3), *where* $(p_h, r_h) \in Q_h^0 \times R_h^0$ *is given by*

$$\begin{aligned} b_{1h}(\mathbf{v}_h, p_h) = & (\mathbf{f}_1, \mathbf{v}_{ho}) - a_{1h}(\mathbf{u}_h, \mathbf{v}_h) - c_{1h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \\ & - c_{2h}(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) - G_{3h}(T_h, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^0, \end{aligned} \tag{4.5a}$$

$$b_{2h}(\mathbf{w}_h, r_h) = \frac{1}{R_m}(\mathbf{f}_2, \mathbf{w}_{ho}) - a_{2h}(\mathbf{B}_h, \mathbf{w}_h) + c_{2h}(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h), \quad \forall \mathbf{w}_h \in \mathbf{W}_h^0. \tag{4.5b}$$

Proof. We only need to show (II). In fact, since the problem (3.3) is equivalent to the system (3.4), it suffices to show $(\mathbf{u}_h, \mathbf{B}_h, T_h, p_h, r_h)$ satisfies (3.4a)-(3.4e).

Notice that (4.4b) is as same as (3.4e). From the definitions of $\bar{\mathbf{V}}_h$ and $\bar{\mathbf{W}}_h$ we know that (3.4b) and (3.4d) hold. Finally, (4.5a) and (4.5b) imply (3.4a) and (3.4c), respectively. This completes the proof. □

Denote

$$M_{1h} := \sup_{\mathbf{0} \neq \Phi_h, \mathbf{u}_h, \mathbf{v}_h \in \bar{\mathbf{V}}_h} \frac{c_{2h}(\Phi_h; \mathbf{u}_h, \mathbf{v}_h)}{|||\Phi_h|||_V |||\mathbf{u}_h|||_V |||\mathbf{v}_h|||_V}, \tag{4.6}$$

$$M_{2h} := \sup_{\substack{\mathbf{0} \neq \mathbf{B}_h, \mathbf{w}_h \in \bar{\mathbf{W}}_h, \\ \mathbf{0} \neq \mathbf{v}_h \in \bar{\mathbf{V}}_h}} \frac{c_{2h}(\mathbf{v}_h; \mathbf{B}_h, \mathbf{w}_h)}{|||\mathbf{B}_h|||_W |||\mathbf{v}_h|||_V |||\mathbf{w}_h|||_W}, \tag{4.7}$$

$$M_{3h} := \sup_{\substack{\mathbf{0} \neq T_h, z_h \in Z_h^0, \\ \mathbf{0} \neq \mathbf{u}_h \in \bar{\mathbf{V}}_h}} \frac{c_{3h}(\mathbf{u}_h; T_h, z_h)}{|||\mathbf{u}_h|||_V |||T_h|||_Z |||z_h|||_Z}. \tag{4.8}$$

From Lemma 4.1, it is easy to know that M_{1h}, M_{2h}, and M_{3h} are bounded from above by a positive constant independent of the mesh size h.

Lemma 4.4. *The WG scheme* (4.4) *admits at least one solution* $(\mathbf{u}_h, \mathbf{B}_h, T_h) \in \bar{\mathbf{V}} \times \bar{\mathbf{W}} \times Z_h^0$. *In addition, there hold*

$$|||\mathbf{u}_h|||_V + |||\mathbf{B}_h|||_W \leq 2\zeta^2(\|\mathbf{f_1}\|_{1h} + \|\mathbf{f_2}\|_{2h} + \|f_3\|_{3h}), \tag{4.9}$$

$$|||T_h|||_Z \leq P_r R_e \|f_3\|_{3h}, \tag{4.10}$$

where

$$\zeta := \max\left\{H_a, \frac{H_aP_rR_eG_r\mathbf{g}}{NR_e^2g}\right\}, \qquad \|\mathbf{f}_1\|_{1h} := \sup_{\mathbf{0}\neq\mathbf{v}_h\in\bar{\mathbf{V}}_h}\frac{(\mathbf{f}_1,\mathbf{v}_{ho})}{\|\!|\mathbf{v}_h|\!\|_V},$$
$$\|\mathbf{f}_2\|_{2h} := \sup_{\mathbf{0}\neq\mathbf{w}_h\in\bar{\mathbf{W}}_h}\frac{1/R_m(\mathbf{f}_2,\mathbf{w}_{ho})}{\|\!|\mathbf{w}_h|\!\|_W}, \quad \|f_3\|_{3h} := \sup_{0\neq z_h\in Z_h^0}\frac{(\mathbf{f}_3,z_{ho})}{\|\!|z_h|\!\|_Z}.$$

Proof. First, by Lemma 4.1 it is easy to see that, for a given $\mathbf{u}_h \in \bar{\mathbf{V}}_h$, the bilinear form $a_{3h}(\cdot,\cdot) + c_{3h}(\mathbf{u}_h;\cdot,\cdot)$ is continuous and coercive on $Z_h^0 \times Z_h^0$. Then the Lax-Milgram theorem implies that the problem (4.4b) has a unique $T_h = T_h(\mathbf{u}_h) \in Z_h^0$ with the boundedness estimate (4.10).

We easily see that the scheme (4.4) is equivalent to the following problem: Find $(\mathbf{u}_h,\mathbf{B}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$ such that

$$\begin{aligned}&a_{1h}(\mathbf{u}_h,\mathbf{v}_h) + a_{2h}(\mathbf{B}_h,\mathbf{w}_h) + c_{1h}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) + c_{2h}(\mathbf{v}_h;\mathbf{B}_h,\mathbf{B}_h) - c_{2h}(\mathbf{u}_h;\mathbf{B}_h,\mathbf{w}_h)\\ &= (\mathbf{f}_1,\mathbf{v}_h) + \frac{1}{R_m}(\mathbf{f}_2,\mathbf{w}_h) - G_{3h}\big(T_h(\mathbf{u}_h),\mathbf{v}_h\big), \quad \forall(\mathbf{v}_h,\mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h. \end{aligned} \tag{4.11}$$

Taking $\mathbf{v}_h = \mathbf{u}_h$ and $\mathbf{w}_h = \mathbf{B}_h$ in (4.11), by Lemma 4.1 we obtain

$$\begin{aligned}&\frac{1}{H_a^2}\|\!|\mathbf{u}_h|\!\|_V^2 + \frac{1}{R_m^2}\|\!|\mathbf{B}_h|\!\|_W^2\\ &\leq -G_{3h}\big(T_h(\mathbf{u}_h),\mathbf{u}_h\big) + (\mathbf{f}_1,\mathbf{u}_h) + \frac{1}{R_m}(\mathbf{f}_2,\mathbf{B}_h)\\ &\leq \left(\frac{P_rR_eG_r\mathbf{g}}{NR_e^2g}\|f_3\|_{3h} + \|\mathbf{f}_1\|_{1h}\right)\|\!|\mathbf{u}_h|\!\|_V + \frac{1}{R_m}\|\mathbf{f}_2\|_{2h}\|\!|\mathbf{B}_h|\!\|_W,\end{aligned}$$

which further yields

$$\begin{aligned}&\frac{1}{2}\min\left\{\frac{1}{H_a^2},\frac{1}{R_m^2}\right\}\big(\|\!|\mathbf{u}_h|\!\|_V + \|\!|\mathbf{B}_h|\!\|_W\big)^2\\ &\leq \min\left\{\frac{1}{H_a^2},\frac{1}{R_m^2}\right\}\big(\|\!|\mathbf{u}_h|\!\|_V^2 + \|\!|\mathbf{B}_h|\!\|_W^2\big)\\ &\leq \frac{1}{H_a^2}\|\!|\mathbf{u}_h|\!\|_V^2 + \frac{1}{R_m^2}\|\!|\mathbf{B}_h|\!\|_W^2\\ &\leq H_a^2\left(\frac{P_rR_eG_r\mathbf{g}}{NR_e^2g}\|f_3\|_{3h} + \|\mathbf{f}_1\|_{1h}\right)^2 + \|\mathbf{f}_2\|_{2h}^2\\ &\leq \left(\frac{H_aP_rR_eG_r\mathbf{g}}{NR_e^2g}\|f_3\|_{3h} + H_a\|\mathbf{f}_1\|_{1h} + \|\mathbf{f}_2\|_{2h}\right)^2.\end{aligned}$$

This indicates the boundedness result (4.9).

To show the existence of solution to the problem (4.11), we define a mapping $\mathbb{A} : \bar{\mathbf{V}}_h\times\bar{\mathbf{W}}_h \to \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$ with $\mathbb{A}(\mathbf{u}_h,\mathbf{B}_h) = (\mathbf{x}_u,\mathbf{x}_B)$, where $(\mathbf{x}_u,\mathbf{x}_B) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$ is given by

$$\begin{aligned}&a_{1h}(\mathbf{x}_u,\mathbf{v}_h) + a_{2h}(\mathbf{x}_B,\mathbf{w}_h)\\ &= (\mathbf{f}_1,\mathbf{v}_h) + \frac{1}{R_m}(\mathbf{f}_2,\mathbf{w}_h) - G_{3h}\big(T_h(\mathbf{u}_h),\mathbf{v}_h\big) - c_{1h}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)\\ &\quad - c_{2h}(\mathbf{v}_h;\mathbf{B}_h,\mathbf{B}_h) + c_{2h}(\mathbf{u}_h;\mathbf{B}_h,\mathbf{w}_h), \quad \forall(\mathbf{v}_h,\mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h.\end{aligned} \tag{4.12}$$

Clearly, $(\mathbf{u}_h, \mathbf{B}_h)$ is a solution to (4.11) if it is a fixed point of $\mathbb{A}$, i.e.

$$\mathbb{A}(\mathbf{u}_h, \mathbf{B}_h) = (\mathbf{u}_h, \mathbf{B}_h). \tag{4.13}$$

In order to show the system (4.13) has a solution, from the Schaefer fixed point theorem [17, Theorem 2.11] it suffices to prove the following two assertions:

(i) $\mathbb{A}$ is a continuous and compact mapping.

(ii) The set

$$\mathbf{\Theta}_h := \{(\mathbf{v}_h, \mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h; (\mathbf{v}_h, \mathbf{w}_h) = \lambda\mathbb{A}(\mathbf{v}_h, \mathbf{w}_h) \text{ for some } 0 < \lambda \leq 1\}$$

is bounded.

To show (i), let $\mathbf{u}_{h1}, \mathbf{u}_{h2} \in \bar{\mathbf{V}}_h$ and $\mathbf{B}_{h1}, \mathbf{B}_{h2} \in \bar{\mathbf{W}}_h$ be such that $\mathbb{A}(\mathbf{u}_{h1}, \mathbf{B}_{h1}) = (\mathbf{x}_{1u}, \mathbf{x}_{1B})$ and $\mathbb{A}(\mathbf{u}_{h2}, \mathbf{B}_{h2}) = (\mathbf{x}_{2u}, \mathbf{x}_{2B})$, then we have

$$\begin{aligned} &a_{1h}(\mathbf{x}_{1u}, \mathbf{v}_h) + a_{2h}(\mathbf{x}_{1B}, \mathbf{w}_h) + c_{1h}(\mathbf{u}_{h1}; \mathbf{u}_{h1}, \mathbf{v}_h) \\ &\quad + c_{2h}(\mathbf{v}_h; \mathbf{B}_{h1}, \mathbf{B}_{h1}) - c_{2h}(\mathbf{u}_{h1}; \mathbf{B}_{h1}, \mathbf{w}_h) \\ &= (\mathbf{f}_1, \mathbf{v}_h) + \frac{1}{R_m}(\mathbf{f}_2, \mathbf{w}_h) - G_{3h}\big(T_h(\mathbf{u}_{h1}), \mathbf{v}_h\big), \end{aligned} \tag{4.14}$$

$$\begin{aligned} &a_{1h}(\mathbf{x}_{2u}, \mathbf{v}_h) + a_{2h}(\mathbf{x}_{2B}, \mathbf{w}_h) + c_{1h}(\mathbf{u}_{h2}; \mathbf{u}_{h2}, \mathbf{v}_h) \\ &\quad + c_{2h}(\mathbf{v}_h; \mathbf{B}_{h2}, \mathbf{B}_{h2}) - c_{2h}(\mathbf{u}_{h2}; \mathbf{B}_{h2}, \mathbf{w}_h) \\ &= (\mathbf{f}_1, \mathbf{v}_h) + \frac{1}{R_m}(\mathbf{f}_2, \mathbf{w}_h) - G_{3h}\big(T_h(\mathbf{u}_{h2}), \mathbf{v}_h\big) \end{aligned} \tag{4.15}$$

for all $(\mathbf{v}_h, \mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$. Subtracting (4.15) from (4.14), and taking $\mathbf{v}_h = \mathbf{x}_{1u} - \mathbf{x}_{2u}$, $\mathbf{w} = \mathbf{x}_{1B} - \mathbf{x}_{2B}$, we get

$$\begin{aligned} &a_{1h}(\mathbf{x}_{1u} - \mathbf{x}_{2u}, \mathbf{x}_{1u} - \mathbf{x}_{2u}) + a_{2h}(\mathbf{x}_{1B} - \mathbf{x}_{2B}, \mathbf{x}_{1B} - \mathbf{x}_{2B}) \\ &= -c_{1h}(\mathbf{u}_{h1} - \mathbf{u}_{h2}; \mathbf{u}_{h1}, \mathbf{x}_{1u} - \mathbf{x}_{2u}) - c_{1h}(\mathbf{u}_{h2}; \mathbf{u}_{h1} - \mathbf{u}_{h2}, \mathbf{x}_{1u} - \mathbf{x}_{2u}) \\ &\quad - c_{2h}(\mathbf{x}_{1u} - \mathbf{x}_{2u}; \mathbf{B}_{h1} - \mathbf{B}_{h2}, \mathbf{B}_{h1}) - c_{2h}(\mathbf{x}_{1u} - \mathbf{x}_{2u}; \mathbf{B}_{h2}, \mathbf{B}_{h1} - \mathbf{B}_{h2}) \\ &\quad + c_{2h}(\mathbf{u}_{h1}; \mathbf{B}_{h1} - \mathbf{B}_{h2}, \mathbf{x}_{1B} - \mathbf{x}_{2B}) + c_{2h}(\mathbf{u}_{h1} - \mathbf{u}_{h2}; \mathbf{B}_{h2}, \mathbf{x}_{1B} - \mathbf{x}_{2B}) \\ &\quad - G_{3h}\big(T_h(\mathbf{u}_{h1}) - T_h(\mathbf{u}_{h2}), \mathbf{x}_{1u} - \mathbf{x}_{2u}\big). \end{aligned} \tag{4.16}$$

Substitute $T_{h1} = T_h(\mathbf{u}_{h1})$ and $T_{h2} = T_h(\mathbf{u}_{h2})$ into (4.4b), respectively, and then subtract the first resulting equation from the second one, we can obtain

$$\begin{aligned} &a_{3h}\big(T_h(\mathbf{u}_{h1}) - T_h(\mathbf{u}_{h2}), z_h\big) \\ &= -c_{3h}\big(\mathbf{u}_{h1} - \mathbf{u}_{h2}; T_h(\mathbf{u}_{h1}), z_h\big) \\ &\quad - c_{3h}\big(\mathbf{u}_{h2}; T_h(\mathbf{u}_{h1}) - T_h(\mathbf{u}_{h2}), z_h\big), \quad \forall\, z \in Z_h^0. \end{aligned} \tag{4.17}$$

Taking $z_h = T_h(\mathbf{u}_{h1}) - T_h(\mathbf{u}_{h2})$ in (4.17) and using Lemma 4.1 and (4.10), we get

$$\begin{aligned} &\frac{1}{P_r R_e} |||T_h(\mathbf{u}_{h1}) - T_h(\mathbf{u}_{h2})|||_Z \\ &\lesssim |||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V |||T_h(\mathbf{u}_{h1})|||_Z \\ &\leq M_{3h} P_r R_e \|f_3\|_{3h} |||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V, \end{aligned} \tag{4.18}$$

which, together with Lemma 4.1 and (4.16), implies

$$
\begin{aligned}
&\frac{1}{H_a^2}|||\mathbf{x}_{1u}-\mathbf{x}_{2u}|||_V^2+\frac{1}{R_m^2}|||\mathbf{x}_{1B}-\mathbf{x}_{2B}|||_W^2\\
\le\ & M_{1h}\big(|||\mathbf{u}_{h1}|||_V+|||\mathbf{u}_{h2}|||_V\big)|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V|||\mathbf{x}_{1u}-\mathbf{x}_{2u}|||_V\\
&+M_{2h}\big(|||\mathbf{B}_{h1}|||_W+|||\mathbf{B}_{h2}|||_W\big)|||\mathbf{x}_{1u}-\mathbf{x}_{2u}|||_V|||\mathbf{B}_{1h}-\mathbf{B}_{h2}|||_W\\
&+M_{2h}|||\mathbf{x}_{1B}-\mathbf{x}_{2B}|||_W\big(|||\mathbf{u}_{h1}|||_W|||\mathbf{B}_{h1}-\mathbf{B}_{h2}|||_W+|||\mathbf{B}_{h2}|||_W|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V\big)\\
&+M_{3h}P_r^2R_e^2\frac{G_r\mathbf{g}}{NR_e^2g}\|f_3\|_{3h}|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V|||\mathbf{x}_{1u}-\mathbf{x}_{2u}|||_V.
\end{aligned}
$$

This estimate plus (4.9) yields

$$
\begin{aligned}
&|||\mathbf{x}_{1u}-\mathbf{x}_{2u}|||_V+|||\mathbf{x}_{1B}-\mathbf{x}_{2B}|||_W\\
\le\ & 2\zeta\Big[H_aM_{1h}\big(|||\mathbf{u}_{h1}|||_V+|||\mathbf{u}_{h2}|||_V\big)|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V\\
&\quad+H_aM_{2h}\big(|||\mathbf{B}_{h1}|||_W+|||\mathbf{B}_{h2}|||_W\big)|||\mathbf{B}_{1h}-\mathbf{B}_{h2}|||_W\\
&\quad+R_mM_{2h}\big(|||\mathbf{u}_{h1}|||_W|||\mathbf{B}_{h1}-\mathbf{B}_{h2}|||_W+|||\mathbf{B}_{h2}|||_W|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V\big)\Big]\\
&+2\zeta M_{3h}H_aP_r^2R_e^2\frac{G_r\mathbf{g}}{NR_e^2g}\|f_3\|_{3h}|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V\\
\le\ & 4\zeta^3\big(\|\mathbf{f_1}\|_{1h}+\|\mathbf{f_2}\|_{2h}+\|f_3\|_{3h}\big)\Big[2H_aM_{1h}|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V+2H_aM_{2h}|||\mathbf{B}_{1h}-\mathbf{B}_{h2}|||_W\\
&\qquad+R_mM_{2h}|||\mathbf{B}_{h1}-\mathbf{B}_{h2}|||_W+R_mM_{2h}|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V\Big]\\
&+2\zeta M_{3h}H_aP_r^2R_e^2\frac{G_r\mathbf{g}}{NR_e^2g}\|f_3\|_{3h}|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V\\
\le\ & \Bigg(4\zeta^3(\|\mathbf{f_1}\|_{1h}+\|\mathbf{f_2}\|_{2h}+\|f_3\|_{3h})(2H_aM_{1h}+R_mM_{2h})\\
&\quad+2\zeta M_{3h}H_aP_r^2R_e^2\frac{G_r\mathbf{g}}{NR_e^2g}\|f_3\|_{3h}\Bigg)|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V\\
&+4\zeta^3(\|\mathbf{f_1}\|_{1h}+\|\mathbf{f_2}\|_{2h}+\|f_3\|_{3h})(2H_aM_{2h}+R_mM_{2h})|||\mathbf{B}_{1h}-\mathbf{B}_{h2}|||_W\\
\le\ & \Bigg(4\zeta^3(\|\mathbf{f_1}\|_{1h}+\|\mathbf{f_2}\|_{2h}+\|f_3\|_{3h})(2H_aM_{1h}+R_mM_{2h})+2\zeta M_{3h}H_aP_r^2R_e^2\frac{G_r\mathbf{g}}{NR_e^2g}\|f_3\|_{3h}\Bigg)\\
&\times(|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V+|||\mathbf{B}_{1h}-\mathbf{B}_{h2}|||_W)\\
\le\ & \max\{M_{1h},M_{2h},M_{3h}\}\big(12\zeta^4(\|\mathbf{f_1}\|_{1h}+\|\mathbf{f_2}\|_{2h}+\|f_3\|_{3h})+2\zeta^2P_rR_e\|f_3\|_{3h}\big)\\
&\times\big(|||\mathbf{u}_{h1}-\mathbf{u}_{h2}|||_V+|||\mathbf{B}_{1h}-\mathbf{B}_{h2}|||_W\big).
\end{aligned}
$$

This implies that $\mathbb{A}$ is equicontinuous and uniformly bounded, since

$$\mathbb{A}(\mathbf{u}_{h1},\mathbf{B}_{h1})-\mathbb{A}(\mathbf{u}_{h2},\mathbf{B}_{h2})=(\mathbf{x}_{1u}-\mathbf{x}_{2u},\mathbf{x}_{1B}-\mathbf{x}_{2B}).$$

Hence, $\mathbb{A}$ is compact by the Arzelá-Ascoli theorem [4], and (i) holds.

The thing left is to prove (ii). For any $(\mathbf{v}_u,\mathbf{v}_B)\in\mathbf{\Theta}_h$, using (4.12) we have

$$
\begin{aligned}
&\lambda^{-1}\big(a_{1h}(\tilde{\mathbf{x}}_u,\mathbf{v}_h)+a_{2h}(\tilde{\mathbf{x}}_B,\mathbf{w}_h)\big)+c_{1h}(\tilde{\mathbf{x}}_u;\tilde{\mathbf{x}}_u,\mathbf{v}_h)+c_{2h}(\mathbf{v}_h;\tilde{\mathbf{x}}_B,\tilde{\mathbf{x}}_B)-c_{2h}(\tilde{\mathbf{x}}_u;\tilde{\mathbf{x}}_B,\mathbf{w}_h)\\
=\ &(\mathbf{f}_1,\mathbf{v}_h)+\frac{1}{R_m}(\mathbf{f}_2,\mathbf{w}_h)-G_{3h}\big(T_h(\tilde{\mathbf{x}}_u),\mathbf{v}_h\big),\quad\forall(\mathbf{v}_h,\mathbf{w}_h)\in\bar{\mathbf{V}}_h\times\bar{\mathbf{W}}_h.
\end{aligned}
$$

Similar to (4.9), there holds

$$|||\tilde{\mathbf{x}}_u|||_V + |||\tilde{\mathbf{x}}_B|||_W \le 2\zeta\lambda(\|\mathbf{f_1}\|_{1h} + \|\mathbf{f_2}\|_{2h} + \|f_3\|_{3h}).$$

As a result, (ii) holds. This completes the proof. □

Lemma 4.5. *Under the smallness condition*

$$\max\{M_{1h}, M_{2h}, M_{3h}\}\big(12\zeta^4\|\mathbf{f_1}\|_{1h} + 12\zeta^4\|\mathbf{f_2}\|_{2h} + \big(12\zeta^4 H_a + 2\zeta^2 P_r R_e\big)\|f_3\|_{3h}\big) < 1, \quad (4.19)$$

the problem (4.4) *admits a unique solution* $(\mathbf{u}_h, \mathbf{B}_h, T_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h \times Z_h^0$.

Proof. Let $(\mathbf{u}_{h1}, \mathbf{B}_{h1})$, $(\mathbf{u}_{h2}, \mathbf{B}_{h2}) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$ be two solutions to problem (4.11), then for any $(\mathbf{v}_h, \mathbf{w}_h) \in \bar{\mathbf{V}}_h \times \bar{\mathbf{W}}_h$ we have

$$\begin{aligned}
&a_{1h}(\mathbf{u}_{h1}, \mathbf{v}_h) + a_{2h}(\mathbf{B}_{h1}, \mathbf{w}_h) + c_{1h}(\mathbf{u}_{h1}; \mathbf{u}_{h1}, \mathbf{v}_h)\\
&\quad + c_{2h}(\mathbf{v}_h; \mathbf{B}_{h1}, \mathbf{B}_{h1}) - c_{2h}(\mathbf{u}_{h1}; \mathbf{B}_{h1}, \mathbf{w}_h)\\
&= (\mathbf{f}_1, \mathbf{v}_h) + \frac{1}{R_m}(\mathbf{f}_2, \mathbf{w}_h) - G_{3h}\big(T_h(\mathbf{u}_{h1}), \mathbf{v}_h\big),\\
&a_{1h}(\mathbf{u}_{h2}, \mathbf{v}_h) + a_{2h}(\mathbf{B}_{h2}, \mathbf{w}_h) + c_{1h}(\mathbf{u}_{h2}; \mathbf{u}_{h2}, \mathbf{v}_h)\\
&\quad + c_{2h}(\mathbf{v}_h; \mathbf{B}_{h2}, \mathbf{B}_{h2}) - c_{2h}(\mathbf{u}_{h2}; \mathbf{B}_{h2}, \mathbf{w}_h)\\
&= (\mathbf{f}_1, \mathbf{v}_h) + \frac{1}{R_m}(\mathbf{f}_2, \mathbf{w}_h) - G_{3h}\big(T_h(\mathbf{u}_{h2}), \mathbf{v}_h\big).
\end{aligned}$$

Subtracting the above first equation from the second one and choosing $\mathbf{v}_h = \mathbf{u}_{1h} - \mathbf{u}_{2h}$, $\mathbf{w}_h = \mathbf{B}_{1h} - \mathbf{B}_{2h}$, we get

$$\begin{aligned}
&a_{1h}(\mathbf{u}_{h1} - \mathbf{u}_{h2}, \mathbf{u}_{h1} - \mathbf{u}_{h2}) + a_{2h}(\mathbf{B}_{h1} - \mathbf{B}_{h2}, \mathbf{B}_{h1} - \mathbf{B}_{h2})\\
&= -c_{1h}(\mathbf{u}_{h1}; \mathbf{u}_{h1} - \mathbf{u}_{h2}, \mathbf{u}_{h1} - \mathbf{u}_{h2}) - c_{1h}(\mathbf{u}_{h1} - \mathbf{u}_{h2}; \mathbf{u}_{h2}, \mathbf{u}_{h1} - \mathbf{u}_{h2})\\
&\quad - c_{2h}(\mathbf{u}_{h1} - \mathbf{u}_{h2}; \mathbf{B}_{h1} - \mathbf{B}_{h2}; \mathbf{B}_{h1}) - c_{2h}(\mathbf{u}_{h1} - \mathbf{u}_{h2}; \mathbf{B}_{h2}, \mathbf{B}_{h1} - \mathbf{B}_{h2})\\
&\quad + c_{2h}(\mathbf{u}_{h1}; \mathbf{B}_{h1} - \mathbf{B}_{h2}, \mathbf{B}_{h1} - \mathbf{B}_{h2}) + c_{2h}(\mathbf{u}_{h1} - \mathbf{u}_{h2}; \mathbf{B}_{h2}, \mathbf{B}_{h1} - \mathbf{B}_{h2})\\
&\quad - G_{3h}\big(\big(T_h(\mathbf{u}_{h1}) - \big(T_h(\mathbf{u}_{h2}), \mathbf{v}_h\big),
\end{aligned}$$

which, together with Lemma 4.1 and (4.18), leads to

$$\begin{aligned}
&\frac{1}{H_a^2}|||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V^2 + \frac{1}{R_m^2}|||\mathbf{B}_{h1} - \mathbf{B}_{h2}|||_W^2\\
&\le M_{1h}|||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V^2|||\mathbf{u}_{h2}|||_V + M_{2h}\big(|||\mathbf{B}_{h1}|||_W + |||\mathbf{B}_{h2}|||_W\big)|||\mathbf{B}_{h1} - \mathbf{B}_{h2}|||_W|||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V\\
&\quad + M_{2h}|||\mathbf{B}_{h1} - \mathbf{B}_{h2}|||_W^2|||\mathbf{u}_{h1}|||_V + M_{2h}|||\mathbf{B}_{h2}|||_W|||\mathbf{B}_{h1} - \mathbf{B}_{h2}|||_W|||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V\\
&\quad + M_{3h}P_r^2R_e^2\frac{G_r\mathbf{g}}{NR_eg}\|f_3\|_{3h}|||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V^2.
\end{aligned}$$

This estimate plus (4.9) yields

$$\begin{aligned}
&|||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V + |||\mathbf{B}_{h1} - \mathbf{B}_{h2}|||_W\\
&\le 2\zeta\Big(H_aM_{1h}|||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_1|||\mathbf{u}_{h2}|||_V + H_aM_{2h}\big(|||\mathbf{B}_{h1}|||_W + |||\mathbf{B}_{h2}|||_W\big)|||\mathbf{B}_{h1} - \mathbf{B}_{h2}|||_W
\end{aligned}$$

$$
\begin{aligned}
&\quad + R_m M_{2h} |||\mathbf{B}_{h1} - \mathbf{B}_{h2}|||_W |||\mathbf{u}_{h1}|||_V + R_m M_{2h} |||\mathbf{B}_{h2}|||_W |||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V \\
&\quad + H_a M_{3h} P_r^2 R_e^2 \frac{G_r \mathbf{g}}{N R_e g} \|f_3\|_{3h} |||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V \Big) \\
&\le 4\zeta^3 (\|\mathbf{f_1}\|_{1h} + \|\mathbf{f_2}\|_{2h} + \|f_3\|_{3h}) \big(H_a M_{1h} |||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V + 2 H_a M_{2h} |||\mathbf{B}_{h1} - \mathbf{B}_{h2}|||_W \\
&\qquad\qquad + R_m M_{2h} |||\mathbf{B}_{h1} - \mathbf{B}_{h2}|||_W + R_m M_{2h} |||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V \big) \\
&\quad + 2\zeta H_a M_{3h} P_r^2 R_e^2 \frac{G_r \mathbf{g}}{N R_e g} \|f_3\|_{3h} |||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V \\
&\le \Bigg(4\zeta^3 (\|\mathbf{f_1}\|_{1h} + \|\mathbf{f_2}\|_{2h} + \|f_3\|_{3h}) (H_a M_{1h} + R_m M_{2h}) \\
&\quad + 2\zeta M_{3h} H_a P_r^2 R_e^2 \frac{G_r \mathbf{g}}{N R_e^2 g} \|f_3\|_{3h} \Bigg) |||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V \\
&\quad + 4\zeta^3 (\|\mathbf{f_1}\|_{1h} + \|\mathbf{f_2}\|_{2h} + \|f_3\|_{3h}) (2 H_a M_{2h} + R_m M_{2h}) |||\mathbf{B}_{1h} - \mathbf{B}_{h2}|||_W \\
&\le \max\{M_{1h}, M_{2h}, M_{3h}\} \Bigg(4\zeta^3 (\|\mathbf{f_1}\|_{1h} + \|\mathbf{f_2}\|_{2h} + \|f_3\|_{3h}) (2H_a + R_m) \\
&\qquad + 2\zeta H_a P_r^2 R_e^2 \frac{G_r \mathbf{g}}{N R_e^2 g} \|f_3\|_{3h} \Bigg) \times \big(|||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V + |||\mathbf{B}_{1h} - \mathbf{B}_{h2}|||_W \big) \\
&\le \max\{M_{1h}, M_{2h}, M_{3h}\} \big(12\zeta^4 (\|\mathbf{f_1}\|_{1h} + \|\mathbf{f_2}\|_{2h} + \|f_3\|_{3h}) + 2\zeta^2 P_r R_e \|f_3\|_{3h} \big) \\
&\quad \times \big(|||\mathbf{u}_{h1} - \mathbf{u}_{h2}|||_V + |||\mathbf{B}_{1h} - \mathbf{B}_{h2}|||_W \big),
\end{aligned}
$$

which, together with the assumption (4.19), implies

$$
\mathbf{u}_{1h} = \mathbf{u}_{2h}, \quad \mathbf{B}_{1h} = \mathbf{B}_{2h}.
$$

This completes the proof. □

Finally, we obtain the following existence and uniqueness results for the WG scheme (3.3).

Theorem 4.1. *The scheme* (3.3) *admits at least one solution*

$$
(\mathbf{u}_h, \mathbf{B}_h, T_h, p_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Z_h^0 \times Q_h^0 \times R_h^0
$$

and, under the condition (4.19)*, admits a unique solution.*

Proof. The existence and uniqueness of the discrete solutions $\mathbf{u}_h, \mathbf{B}_h$ and T_h follow from Lemmas 4.3-4.5, and the existence and uniqueness of the discrete solution, p_h, to (4.5a) and the discrete solution, r_h, to (4.5b) follow from the two discrete inf-sup inequalities in Lemma 4.2. The proof is complete. □

5. Error Estimates

This section is devoted to establishing error estimates for the WG scheme (3.3). To this end, we assume that the weak solution, $(\mathbf{u}, \mathbf{B}, T, p, r)$, to the problem (1.1) satisfies the following regularity conditions:

$$
\begin{aligned}
&\mathbf{u} \in \mathbf{V} \cap [H^2(\Omega)]^d, \quad \mathbf{B} \in \mathbf{W} \cap [H^2(\Omega)]^d, \\
&T \in H_0^1 \cap [H^2(\Omega)]^d, \quad p \in L_0^2(\Omega) \cap H^1(\Omega), \quad r \in H_0^1(\Omega) \cap H^1(\Omega).
\end{aligned} \tag{5.1}
$$

We set

$$
\begin{aligned}
\Pi_1\mathbf{u}|_K &:= \big\{\mathbf{P}_1^{\mathcal{RT}}(\mathbf{u}|_K), \mathbf{Q}_0^b(\mathbf{u}|_K)\big\},\\
\Pi_2\mathbf{B}|_K &:= \big\{\mathbf{P}_1^{\mathcal{RT}}(\mathbf{B}|_K), \mathbf{Q}_1^b(\mathbf{B}|_K)\big\},\\
\Pi_3 T|_K &:= \big\{Q_1^o(T|_K), Q_0^b(T|_K)\big\},\\
\Pi_4 p|_K &:= \big\{Q_0^o(p|_K), Q_1^b(p|_K)\big\},\\
\Pi_5 r|_K &:= \big\{Q_0^o(r|_K), Q_1^b(r|_K)\big\}
\end{aligned}
$$

for any $K \in \mathcal{T}_h$.

Lemma 5.1. *For any* $(\mathbf{v}_h, \mathbf{w}_h, z_h, q_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Z_h^0 \times Q_h^0 \times R_h^0$*, there hold*

$$
\begin{aligned}
&a_{1h}(\Pi_1\mathbf{u}, \mathbf{v}_h) + a_{2h}(\Pi_2\mathbf{B}, \mathbf{w}_h) + b_{1h}(\mathbf{v}_h, \Pi_4 p)\\
&\quad - b_{1h}(\Pi_1\mathbf{u}, q_h) + b_{2h}(\mathbf{w}_h, \Pi_5 r) - b_{2h}(\Pi_2\mathbf{B}, \theta_h)\\
&\quad + c_{1h}(\Pi_1\mathbf{u}; \Pi_1\mathbf{u}, \mathbf{v}_h) + c_{2h}(\mathbf{v}_h; \Pi_2\mathbf{B}, \Pi_2\mathbf{B})\\
&\quad - c_{2h}(\Pi_1\mathbf{u}; \Pi_2\mathbf{B}, \mathbf{w}_h) + G_{3h}(\Pi_3 T_h, \mathbf{v}_h)\\
&= (\mathbf{f}_1, \mathbf{v}_h) + \frac{1}{R_m}(\mathbf{f}_2, \mathbf{w}_h) + E_u(\mathbf{u}, \mathbf{v}_h) + E_B(\mathbf{B}, \mathbf{w}_h)\\
&\quad + E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h) + E_{\tilde{B}1}(\mathbf{B}, \mathbf{v}_h) + E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \mathbf{v}_h), \qquad (5.2a)\\
&a_{3h}(\Pi_3 T, z_h) + c_{3h}(\Pi_1\mathbf{u}; \Pi_3 T, z_h)\\
&= (f_3, z_h) + E_T(T, z_h) + E_{\tilde{T}}(\mathbf{u}; T, z_h), \qquad (5.2b)
\end{aligned}
$$

where

$$
\begin{aligned}
E_u(\mathbf{u}, \mathbf{v}_h) &:= \frac{1}{H_a^2}\big\langle(\nabla\mathbf{u} - \mathbf{Q}_0^o\nabla\mathbf{u})\cdot\mathbf{n}, \mathbf{v}_{hb} - \mathbf{v}_{ho}\big\rangle_{\partial\mathcal{T}_h} + \frac{1}{H_a^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{u} - \mathbf{u}\big), \mathbf{Q}_0^b\mathbf{v}_{ho} - \mathbf{v}_{hb}\big\rangle_{\partial\mathcal{T}_h},\\
E_B(\mathbf{B}, \mathbf{w}_h) &:= -\frac{1}{R_m^2}\big\langle\nabla\times\mathbf{B} - \mathbf{Q}_0^o\nabla\times\mathbf{B}, (\mathbf{w}_{ho} - \mathbf{w}_{hb})\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h}\\
&\quad + \frac{1}{R_m^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{B} - \mathbf{Q}_1^b\mathbf{B}\big)\times\mathbf{n}, (\mathbf{w}_{ho} - \mathbf{w}_{hb})\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h},\\
E_T(T, z_h) &:= \frac{1}{P_r R_e}\big\langle\big(\nabla T - Q_0^o\nabla T\big)\cdot\mathbf{n}, z_{ho} - z_{hb}\big\rangle_{\partial\mathcal{T}_h} + \frac{1}{P_r R_e}\big\langle h_K^{-1}\big(Q_1^o T - T\big), Q_0^b z_{ho} - z_{hb}\big\rangle_{\partial\mathcal{T}_h},\\
E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h) &:= \frac{1}{2N}\big(\mathbf{u}\otimes\mathbf{u} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\otimes\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}, \nabla_h\mathbf{v}_{ho}\big) - \frac{1}{2N}\big\langle\big(\mathbf{u}\otimes\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\otimes\mathbf{Q}_0^b\mathbf{u}\big)\mathbf{n}, \mathbf{v}_{ho}\big\rangle_{\partial\mathcal{T}_h}\\
&\quad - \frac{1}{2N}\big(\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\cdot\nabla_h\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}, \mathbf{v}_{ho}\big) - \frac{1}{2N}\big\langle\mathbf{v}_{hb}\otimes\mathbf{Q}_0^b\mathbf{u}\mathbf{n}, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big\rangle_{\partial\mathcal{T}_h},\\
E_{\tilde{B}1}(\mathbf{B}, \mathbf{v}_h) &:= -\frac{1}{R_m}\big(\nabla_h\times\big(\mathbf{B} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{B}\big), \mathbf{v}_{ho}\times\mathbf{B}\big) + \frac{1}{R_m}\big(\nabla_h\times\mathbf{P}_1^{\mathcal{RT}}\mathbf{B}, \mathbf{v}_{ho}\times\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{B} - \mathbf{B}\big)\big)\\
&\quad - \frac{1}{R_m}\big\langle\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{B} - \mathbf{Q}_0^b\mathbf{B}\big)\times\mathbf{n}, \mathbf{v}_{ho}\times\mathbf{P}_1^{\mathcal{RT}}\mathbf{B}\big\rangle_{\partial\mathcal{T}_h},\\
E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \mathbf{w}_h) &:= -\frac{1}{R_m}\big(\nabla_h\times\mathbf{w}_{ho}, \big(\mathbf{u}\times\mathbf{B} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\times\mathbf{P}_1^{\mathcal{RT}}\mathbf{B}\big)\big) - \frac{1}{R_m}\langle\mathbf{w}_{ho}\times\mathbf{n}, \mathbf{u}\times\mathbf{B}\rangle_{\partial\mathcal{T}_h}\\
&\quad - \frac{1}{R_m}\big\langle(\mathbf{w}_{ho} - \mathbf{w}_{hb})\times\mathbf{n}, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\times\mathbf{Q}_1^o\mathbf{B}\big\rangle_{\partial\mathcal{T}_h},\\
E_{\tilde{T}}(\mathbf{u}; T, z_h) &:= \frac{1}{2}\big(\mathbf{u}T - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}Q_1^0 T, \nabla_h z_{ho}\big) - \frac{1}{2}\big\langle\big(\mathbf{u}T - \mathbf{Q}_0^b\mathbf{u}Q_0^b T\big)\cdot\mathbf{n}, z_{ho}\big\rangle_{\partial\mathcal{T}_h}\\
&\quad - \frac{1}{2}\big(\mathbf{u}\cdot\nabla T - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\cdot\nabla_h Q_1^o T, z_{ho}\big) - \frac{1}{2}\big\langle\big(\mathbf{Q}_0^b\mathbf{u}z_{hb}\big)\cdot\mathbf{n}, Q_1^o T\big\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

In addition, there hold

$$\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}|_K \in [\mathcal{P}_1(K)]^d, \quad \mathbf{P}_1^{\mathcal{RT}}\mathbf{B}|_K \in [\mathcal{P}_1(K)]^d, \quad \forall K \in \mathcal{T}_h. \tag{5.3}$$

Proof. We first show (5.3). For all $K \in \mathcal{T}_h$, by Lemma 3.7 we have

$$\begin{aligned} \big(\nabla\cdot\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}, \varphi_h\big)_K &= (\nabla\cdot\mathbf{u}, \varphi_h)_K = 0, \quad \forall \varphi_h \in \mathcal{P}_1(K),\\ \big(\nabla\cdot\mathbf{P}_1^{\mathcal{RT}}\mathbf{B}, \theta_h\big)_K &= (\nabla\cdot\mathbf{B}, \theta_h)_K = 0, \quad \forall \theta_h \in \mathcal{P}_1(K), \end{aligned}$$

which mean that

$$\nabla\cdot\mathbf{P}_1^{\mathcal{RT}}\mathbf{u} = 0, \quad \nabla\cdot\mathbf{P}_1^{\mathcal{RT}}\mathbf{B} = 0.$$

Then the result (5.3) follows from Lemma 3.6.

From the definitions of the bilinear form $a_{1h}(\cdot\,,\cdot)$ and the weak gradient, the first commutativity property in Lemma 3.9, the properties of the projection $\mathbf{Q}_0^o$, the Green's formula, the relation $\langle\nabla\mathbf{u}\,\mathbf{n}, \mathbf{v}_{hb}\rangle_{\partial\mathcal{T}_h} = 0$, the properties of the projection $\mathbf{Q}_0^b$ and the definition of $E_u(\mathbf{u}, \mathbf{v}_h)$, we immediately get, for any $\mathbf{v}_h \in \mathbf{V}_h^0$,

$$\begin{aligned} a_{1h}(\Pi_1\mathbf{u}, \mathbf{v}_h) &= \frac{1}{H_a^2}(\nabla_{w,0}\Pi_1\mathbf{u}, \nabla_{w,0}\mathbf{v}_h) + \frac{1}{H_a^2}\big\langle h_K^{-1}\mathbf{Q}_0^b\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}-\mathbf{u}\big), \mathbf{Q}_0^b\mathbf{v}_{ho}-\mathbf{v}_{hb}\big\rangle_{\partial\mathcal{T}_h}\\ &= \frac{1}{H_a^2}\big(\mathbf{Q}_0^o\nabla\mathbf{u}, \nabla_{w,0}\mathbf{v}_h\big) + \frac{1}{H_a^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}-\mathbf{u}\big), \mathbf{Q}_0^b\mathbf{v}_{ho}-\mathbf{v}_{hb}\big\rangle_{\partial\mathcal{T}_h}\\ &= -\frac{1}{H_a^2}\big(\nabla_h\cdot\mathbf{Q}_0^o\nabla\mathbf{u}, \mathbf{v}_{ho}\big) + \frac{1}{H_a^2}\big\langle\mathbf{Q}_0^o\nabla\mathbf{u}\,\mathbf{n}, \mathbf{v}_{hb}\big\rangle_{\partial\mathcal{T}_h}\\ &\quad + \frac{1}{H_a^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}-\mathbf{u}\big), \mathbf{Q}_0^b\mathbf{v}_{ho}-\mathbf{v}_{hb}\big\rangle_{\partial\mathcal{T}_h}\\ &= \frac{1}{H_a^2}\big(\mathbf{Q}_0^o\nabla\mathbf{u}, \nabla_h\mathbf{v}_{ho}\big) + \frac{1}{H_a^2}\big\langle\mathbf{Q}_0^o\nabla\mathbf{u}\,\mathbf{n}, \mathbf{v}_{hb}-\mathbf{v}_{ho}\big\rangle_{\partial\mathcal{T}_h}\\ &\quad + \frac{1}{H_a^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}-\mathbf{u}\big), \mathbf{Q}_0^b\mathbf{v}_{ho}-\mathbf{v}_{hb}\big\rangle_{\partial\mathcal{T}_h}\\ &= -\frac{1}{H_a^2}(\Delta\mathbf{u}, \mathbf{v}_{ho}) + \frac{1}{H_a^2}\big\langle\big(\nabla\mathbf{u}-\mathbf{Q}_0^o\nabla\mathbf{u}\big)\,\mathbf{n}, \mathbf{v}_{hb}-\mathbf{v}_{ho}\big\rangle_{\partial\mathcal{T}_h}\\ &\quad + \frac{1}{H_a^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}-\mathbf{u}\big), \mathbf{Q}_0^b\mathbf{v}_{ho}-\mathbf{v}_{hb}\big\rangle_{\partial\mathcal{T}_h}\\ &= -\frac{1}{H_a^2}(\Delta\mathbf{u}, \mathbf{v}_{ho}) + E_u(\mathbf{u}, \mathbf{v}_h). \end{aligned} \tag{5.4}$$

Similarly, in light of the definitions of the bilinear form $a_{2h}(\cdot\,,\cdot)$ and the weak curl, the second commutativity property in Lemma 3.9, the properties of the projection $\mathbf{Q}_0^o$, the Green's formula, the relation $\langle\nabla\times\mathbf{B}, \mathbf{w}_{hb}\times\mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0$, and the definition of $E_B(\mathbf{B}, \mathbf{w}_h)$, we obtain, for any $\mathbf{w}_h \in \mathbf{W}_h^0$,

$$\begin{aligned} a_{2h}(\Pi_2\mathbf{B}, \mathbf{w}_h) &= \frac{1}{R_m^2}(\nabla_{w,0}\times\Pi_2\mathbf{B}, \nabla_{w,0}\times\mathbf{w}_h)\\ &\quad + \frac{1}{R_m^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{B}-\mathbf{Q}_0^b\mathbf{B}\big)\times\mathbf{n}, (\mathbf{w}_{ho}-\mathbf{w}_{hb})\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h}\\ &= \frac{1}{R_m^2}\big(\mathbf{Q}_0^o(\nabla\times\mathbf{B}), \nabla_{w,0}\times\mathbf{w}_h\big)\\ &\quad + \frac{1}{R_m^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{B}-\mathbf{Q}_0^b\mathbf{B}\big)\times\mathbf{n}, (\mathbf{w}_{ho}-\mathbf{w}_{hb})\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{R_m^2}\big(\nabla_h \times \big(\mathbf{Q}_0^o(\nabla\times\mathbf{B})\big), \mathbf{w}_{ho}\big) + \frac{1}{R_m^2}\big\langle \mathbf{Q}_0^o(\nabla\times\mathbf{B}), \mathbf{w}_{hb}\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h} \\
&+ \frac{1}{R_m^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{B} - \mathbf{Q}_0^b\mathbf{B}\big)\times\mathbf{n}, (\mathbf{w}_{ho}-\mathbf{w}_{hb})\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h} \\
&= \frac{1}{R_m^2}\big(\mathbf{Q}_0^o(\nabla\times\mathbf{B}), \nabla_h\times\mathbf{w}_{ho}\big) - \frac{1}{R_m^2}\big\langle \mathbf{Q}_0^o(\nabla\times\mathbf{B}), (\mathbf{w}_{ho}-\mathbf{w}_{hb})\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h} \\
&\quad + \frac{1}{R_m^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{B} - \mathbf{Q}_0^b\mathbf{B}\big)\times\mathbf{n}, (\mathbf{w}_{ho}-\mathbf{w}_{hb})\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h} \\
&= \frac{1}{R_m^2}(\nabla\times\nabla\times\mathbf{B}, \mathbf{w}_{ho}) - \frac{1}{R_m^2}\big\langle \nabla\times\mathbf{B} - \mathbf{Q}_0^o(\nabla\times\mathbf{B}), (\mathbf{w}_{ho}-\mathbf{w}_{hb})\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h} \\
&\quad + \frac{1}{R_m^2}\big\langle h_K^{-1}\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{B} - \mathbf{Q}_0^b\mathbf{B}\big)\times\mathbf{n}, (\mathbf{w}_{ho}-\mathbf{w}_{hb})\times\mathbf{n}\big\rangle_{\partial\mathcal{T}_h} \\
&= \frac{1}{R_m^2}(\nabla\times\nabla\times\mathbf{B}, \mathbf{w}_{ho}) + E_B(\mathbf{B}, \mathbf{w}_h).
\end{aligned}
$$

In view of the definitions of $b_{1h}(\cdot,\cdot)$ and the weak gradient, the fourth commutativity property in Lemma 3.9, the projection property, and the relations (3.11a), (5.3) and $\langle \mathbf{u}\cdot\mathbf{n}, q_{hb}\rangle_{\partial\mathcal{T}_h} = 0$, we get

$$
\begin{aligned}
& b_{1h}(\mathbf{v}_h, \Pi_3 p) - b_h(\Pi_1\mathbf{u}, q_h) \\
&= \big(\nabla_{w,1}\{Q_0^o p, Q_1^b p\}, \mathbf{v}_{ho}\big) - \big(\nabla_{w,1} q_h, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big) \\
&= \big(Q_1^o \nabla p, \mathbf{v}_{ho}\big) + \big(\nabla\cdot\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}, q_{ho}\big) - \big\langle \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\cdot\mathbf{n}, q_{hb}\big\rangle_{\partial\mathcal{T}_h} \\
&= (\nabla p, \mathbf{v}_{ho}) - \langle \mathbf{u}\cdot\mathbf{n}, q_{hb}\rangle_{\partial\mathcal{T}_h} \\
&= (\nabla p, \mathbf{v}_{ho}), \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h^0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& b_{2h}(\mathbf{w}_h, \Pi_4 r) - \tilde{b}_{2h}(\Pi_2\mathbf{B}, \theta_h) \\
&= \frac{1}{R_m}\big(\nabla_{w,1}\{Q_0^o r, Q_1^b r\}, \mathbf{w}_{ho}\big) - \frac{1}{R_m}\big(\nabla_{w,1}\theta_h, \mathbf{P}_1^{\mathcal{RT}}\mathbf{B}\big) \\
&= \frac{1}{R_m}\big(Q_1^o\nabla r, \mathbf{w}_{ho}\big) + \frac{1}{R_m}\big(\nabla\cdot\mathbf{P}_1^{\mathcal{RT}}\mathbf{B}, \theta_{ho}\big) - \frac{1}{R_m}\big\langle \mathbf{P}_1^{\mathcal{RT}}\mathbf{B}\cdot\mathbf{n}, \theta_{hb}\big\rangle_{\partial\mathcal{T}_h} \\
&= \frac{1}{R_m}(\nabla r, \mathbf{w}_{ho}) - \frac{1}{R_m}\langle \mathbf{B}\cdot\mathbf{n}, \theta_{hb}\rangle_{\partial\mathcal{T}_h} \\
&= \frac{1}{R_m}(\nabla r, \mathbf{w}_{ho}), \quad \forall\, \mathbf{w}_h \in \mathbf{W}_h^0.
\end{aligned}
$$

By the Green's formula and the definitions of $c_{1h}(\cdot;\cdot,\cdot)$, the weak divergence and $E_{\tilde{u}}(\cdot,\cdot)$ we get

$$
\begin{aligned}
& c_{1h}(\Pi_1\mathbf{u}; \Pi_1\mathbf{u}, \mathbf{v}_h) \\
&= \frac{1}{2N}\big(\nabla_{w,1}\cdot\big\{\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\otimes\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}, \mathbf{Q}_0^b\mathbf{u}\otimes\mathbf{Q}_0^b\mathbf{u}\big\}, \mathbf{v}_{ho}\big) \\
&\quad - \frac{1}{2N}\big(\nabla_{w,1}\cdot\big\{\mathbf{v}_{ho}\otimes\mathbf{P}_1^{\mathcal{RT}}\Phi, \mathbf{v}_{hb}\otimes\mathbf{Q}_0^b\mathbf{u}\big\}, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big) \\
&= \frac{1}{2N}\big(\nabla\cdot(\mathbf{u}\otimes\mathbf{u}), \mathbf{v}_{ho}\big) + \frac{1}{2N}\big(\mathbf{u}\otimes\mathbf{u} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\otimes\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}, \nabla_h\mathbf{v}_{ho}\big) \\
&\quad - \frac{1}{2N}\big\langle\big(\mathbf{u}\otimes\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\otimes\mathbf{Q}_0^b\mathbf{u}\big)\mathbf{n}, \mathbf{v}_{ho}\big\rangle_{\partial\mathcal{T}_h}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2N}\big(\nabla\cdot(\mathbf{u}\otimes\mathbf{u}),\mathbf{v}_{ho}\big)+\frac{1}{2N}\big(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\cdot\nabla_h\mathbf{P}_1^{\mathcal{RT}}\mathbf{u},\mathbf{v}_{ho}\big)\\
&+\frac{1}{2N}\big\langle\big(\mathbf{v}_{hb}\otimes\mathbf{Q}_0^b\mathbf{u}\,\mathbf{n},\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big\rangle_{\partial\mathcal{T}_h}\\
=&\frac{1}{N}\big(\nabla\cdot(\mathbf{u}\otimes\mathbf{u}),\mathbf{v}_{ho}\big)+E_{\tilde{u}}(\mathbf{u},\mathbf{v}_h).
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
c_{2h}(\mathbf{v}_h;\Pi_2\mathbf{B},\Pi_2\mathbf{B})&=-\frac{1}{R_m}(\nabla\times\mathbf{B}\times\mathbf{B},\mathbf{v}_{ho})+E_{\tilde{B}1}(\mathbf{B}_h,\mathbf{v}_h),\\
c_{2h}(\Pi_1\mathbf{u};\Pi_2\mathbf{B},\mathbf{w}_h)&=\frac{1}{R_m}\big(\nabla\times(\mathbf{u}\times\mathbf{B}),\mathbf{w}_{ho}\big)+E_{\tilde{B}2}(\mathbf{u}_h;\mathbf{B}_h,\mathbf{w}_h).
\end{aligned}
$$

By the definition of the projection Q_1^o, we have

$$
G_{3h}(\Pi_3T_h,\mathbf{v}_h)=\frac{G_r}{NR_e^2}\left(\frac{\mathbf{g}}{g}Q_1^oT_{ho},\mathbf{v}_{ho}\right)=\frac{G_r}{NR_e^2}\left(\frac{\mathbf{g}}{g}T,\mathbf{v}_{ho}\right).
$$

Combining the above relations and (1.1), we finally arrive at the desired conclusion (5.2a).

Similarly, we can get the relation (5.2b). This completes the proof. □

Lemma 5.2. *For $\mathbf{v}_h\in\mathbf{V}_h^0$, $\mathbf{w}_h\in\mathbf{W}_h^0$, and $z_h\in Z_h^0$, there hold*

$$
\begin{aligned}
|E_u(\mathbf{u},\mathbf{v}_h)|&\lesssim h|\mathbf{u}|_2|\!|\!|\mathbf{v}_h|\!|\!|_V, &(5.5a)\\
|E_B(\mathbf{B},\mathbf{w}_h)|&\lesssim h|\mathbf{B}|_2|\!|\!|\mathbf{w}_h|\!|\!|_W, &(5.5b)\\
|E_T(T,z_h)|&\lesssim h|T|_2|\!|\!|z_h|\!|\!|_Z, &(5.5c)\\
|E_{\tilde{u}}(\mathbf{u},\mathbf{v})|&\lesssim h\|\mathbf{u}\|_2^2|\!|\!|\mathbf{v}_h|\!|\!|_V, &(5.5d)\\
|E_{\tilde{B}1}(\mathbf{B},\mathbf{v}_h)|&\lesssim h\|\mathbf{B}\|_2^2|\!|\!|\mathbf{v}_h|\!|\!|_W, &(5.5e)\\
|E_{\tilde{B}2}(\mathbf{u};\mathbf{B},\mathbf{w}_h)|&\lesssim h\|\mathbf{u}\|_2\|\mathbf{B}\|_2|\!|\!|\mathbf{w}_h|\!|\!|_W, &(5.5f)\\
|E_{\tilde{T}}(\mathbf{u};T,z_h)|&\lesssim h\|\mathbf{u}\|_2\|T\|_2|\!|\!|z_h|\!|\!|_Z. &(5.5g)
\end{aligned}
$$

Proof. We only show (5.5d), since the other results can be derived similarly.

Let us estimate the four terms of $E_{\tilde{u}}(\mathbf{u},\mathbf{v}_h)$ one by one. Using the Cauchy-Schwarz inequality, the Hölder's inequality, the Sobolev embedding theorem, and Lemmas 3.7, 3.8 and 3.2, we have

$$
\begin{aligned}
&\big|\big(\mathbf{u}\otimes\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\otimes\mathbf{P}_1^{\mathcal{RT}}\mathbf{u},\nabla_h\mathbf{v}_{ho}\big)\big|\\
\le&\big|\big((\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u})\otimes\mathbf{u},\nabla_h\mathbf{v}_{ho}\big)\big|+\big|\big(\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\otimes(\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}),\nabla_h\mathbf{v}_{ho}\big)\big|\\
\le&|\mathbf{u}|_{0,\infty,\Omega}\sum_{K\in\mathcal{T}_h}\big|\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big|_{0,K}\|\nabla_h\mathbf{v}_{ho}\|_{0,K}\\
&+\sum_{K\in\mathcal{T}_h}\big|\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big|_{0,3,K}\big|\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big|_{0,6,K}\|\nabla_h\mathbf{v}_{ho}\|_{0,K}\\
\le&|\mathbf{u}|_{0,\infty,\Omega}\sum_{K\in\mathcal{T}_h}\big|\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big|_{0,K}\|\nabla_h\mathbf{v}_{ho}\|_{0,K}\\
&+\sum_{K\in\mathcal{T}_h}\big|\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big|_{0,3,K}\big(\big|\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big|_{0,6,K}+|\mathbf{u}|_{0,6,K}\big)\|\nabla_h\mathbf{v}_{ho}\|_{0,K}\\
\le&|\mathbf{u}|_{0,\infty,\Omega}\sum_{K\in\mathcal{T}_h}\big|\mathbf{u}-\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\big|_{0,K}\|\nabla_h\mathbf{v}_{ho}\|_{0,K}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\left|\mathbf{u} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\right|_{0,6,\Omega} + |\mathbf{u}|_{0,6,\Omega}\right) \sum_{K\in\mathcal{T}_h} \left|\mathbf{u} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\right|_{0,3,K} \|\nabla_h \mathbf{v}_{ho}\|_{0,K} \\
\lesssim{}& h^2 |\mathbf{u}|_{0,\infty,\Omega} |\mathbf{u}|_2 \|\!|\mathbf{v}_h|\!\|_V + \|\mathbf{u}\|_1 \sum_{K\in\mathcal{T}_h} \left|\mathbf{u} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\right|_{0,3,K} \|\nabla_h \mathbf{v}_{ho}\|_{0,K} \\
\lesssim{}& h^2 |\mathbf{u}|_{0,\infty} |\mathbf{u}|_2 \|\!|\mathbf{v}_h|\!\|_V + h^{2-\frac{d}{6}} \|\mathbf{u}\|_1 |\mathbf{u}|_2 \|\nabla_h \mathbf{v}_{ho}\|_0 \lesssim h \|\mathbf{u}\|_2^2 \|\!|\mathbf{v}_h|\!\|_V.
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
&\left|\left\langle \left(\mathbf{u}\otimes\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\otimes\mathbf{Q}_0^b\mathbf{u}\right)\mathbf{n}, \mathbf{v}_{ho}\right\rangle_{\partial\mathcal{T}_h}\right| \\
={}& \left|\left\langle \left(\mathbf{u}\otimes\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\otimes\mathbf{Q}_0^b\mathbf{u}\right)\mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right| \\
\le{}& \left|\left\langle \left(\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right)\otimes\left(\mathbf{u} - \mathbf{Q}_1^o\mathbf{u}\right)\mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right| \\
&+ \left|\left\langle \left(\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right)\otimes\mathbf{Q}_1^o\mathbf{u}\mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right| \\
&+ \left|\left\langle \left(\mathbf{Q}_1^o\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right)\otimes\left(\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right)\mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right| \\
&+ \left|\left\langle \mathbf{Q}_1^o\mathbf{u}\otimes\left(\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right)\mathbf{n}, \mathbf{v}_{ho} - \mathbf{v}_{hb}\right\rangle_{\partial\mathcal{T}_h}\right| \\
\le{}& \sum_{K\in\mathcal{T}_h} \left|\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right|_{0,\partial K} \left|\mathbf{u} - \mathbf{Q}_1^o\mathbf{u}\right|_{0,\partial K} |\mathbf{v}_{ho} - \mathbf{v}_{hb}|_{0,\infty,\partial K} \\
&+ \sum_{K\in\mathcal{T}_h} \left|\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right|_{0,\partial K} \left|\mathbf{Q}_1^o\mathbf{u}\right|_{0,\infty,\partial K} |\mathbf{v}_{ho} - \mathbf{v}_{hb}|_{0,\partial K} \\
&+ \sum_{K\in\mathcal{T}_h} \left|\mathbf{Q}_1^o\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right|_{0,\partial K} \left|\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right|_{0,\partial K} |\mathbf{v}_{ho} - \mathbf{v}_{hb}|_{0,\infty,\partial K} \\
&+ \sum_{K\in\mathcal{T}_h} \left|\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right|_{0,\partial K} \left|\mathbf{Q}_1^o\mathbf{u}\right|_{0,\infty,\partial K} |\mathbf{v}_{ho} - \mathbf{v}_{hb}|_{0,\partial K} \lesssim h\|\mathbf{u}\|_2^2 \|\!|\mathbf{v}_h|\!\|_V,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\left(\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\cdot\nabla_h\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}, \mathbf{v}_{ho}\right)\right| \\
\le{}& \left|\left(\left(\mathbf{u} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\right)\cdot\nabla\mathbf{u}, \mathbf{v}_{ho}\right)\right| + \left|\left(\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\cdot\left(\nabla\mathbf{u} - \nabla_h\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\right), \mathbf{v}_{ho}\right)\right| \\
\le{}& \sum_{K\in\mathcal{T}_h} \left|\mathbf{u} - \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\right|_{0,3,K} |\nabla\mathbf{u}|_{0,K} \|\mathbf{v}_{ho}\|_{0,6,K} \\
&+ \sum_{K\in\mathcal{T}_h} \left|\nabla\mathbf{u} - \nabla_h\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\right|_{0,K} \left|\mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\right|_{0,6,K} \|\mathbf{v}_{ho}\|_{0,3,K} \lesssim h\|\mathbf{u}\|_2^2 \|\!|\mathbf{v}_h|\!\|_V,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\left\langle \mathbf{v}_{hb}\otimes\mathbf{Q}_0^b\mathbf{u}\,\mathbf{n}, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u}\right\rangle_{\partial\mathcal{T}_h}\right| \\
={}& \left|\left\langle \mathbf{v}_{hb}\otimes\mathbf{Q}_0^b\mathbf{u}\,\mathbf{n}, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right\rangle_{\partial\mathcal{T}_h}\right| \\
\le{}& \left|\left\langle (\mathbf{v}_{ho} - \mathbf{v}_{hb})\otimes\left(\mathbf{Q}_1^b\mathbf{u} - \mathbf{Q}_1^o\mathbf{u}\right)\mathbf{n}, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right\rangle_{\partial\mathcal{T}_h}\right| \\
&+ \left|\left\langle \mathbf{v}_{ho}\otimes\left(\mathbf{Q}_0^b\mathbf{u} - \mathbf{Q}_1^o\mathbf{u}\right)\mathbf{n}, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right\rangle_{\partial\mathcal{T}_h}\right| \\
&+ \left|\left\langle (\mathbf{v}_{ho} - \mathbf{v}_{hb})\otimes\mathbf{Q}_1^o\mathbf{u}\,\mathbf{n}, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right\rangle_{\partial\mathcal{T}_h}\right| \\
&+ \left|\left\langle \mathbf{v}_{ho}\otimes\mathbf{Q}_1^o\mathbf{u}\,\mathbf{n}, \mathbf{P}_1^{\mathcal{RT}}\mathbf{u} - \mathbf{Q}_0^b\mathbf{u}\right\rangle_{\partial\mathcal{T}_h}\right| \lesssim h\|\mathbf{u}\|_2^2 \|\!|\mathbf{v}_h|\!\|_V.
\end{aligned}
$$

As a result, the desired estimate (5.5d) follows. □

Theorem 5.1. *Let $(\mathbf{u}_h, \mathbf{B}_h, T_h, p_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Z_h^0 \times Q_h^0 \times R_h^0$ be the solutions to the WG scheme* (3.3)*. Under the regularity assumption* (5.1) *and the smallness condition* (4.5)*, there hold the following estimates:*

$$|||\Pi_1 \mathbf{u} - \mathbf{u}_h|||_V + |||\Pi_2 \mathbf{B} - \mathbf{B}_h|||_W + |||\Pi_3 T - T_h|||_Z \lesssim h C_1(\mathbf{u}, \mathbf{B}, T), \tag{5.6a}$$

$$|||\Pi_4 p - p_h|||_Q + |||\Pi_5 r - r_h|||_R \lesssim h C_1(\mathbf{u}, \mathbf{B}, T) + h^2 C_2(\mathbf{u}, \mathbf{B}), \tag{5.6b}$$

where

$$\begin{aligned} &C_1(\mathbf{u}, \mathbf{B}, T) := (\|\mathbf{u}\|_2 + \|\mathbf{B}\|_2 + \|T\|_2)(1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2 + \|T\|_2), \\ &C_2(\mathbf{u}, \mathbf{B}) := (\|\mathbf{u}\|_2 + \|\mathbf{B}\|_2)^2 (1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2)^2. \end{aligned}$$

Proof. From (3.3) and Lemma 5.1 we get the error equations as follows:

$$\begin{aligned} &a_{1h}(\Pi_1 \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + a_{2h}(\Pi_2 \mathbf{B} - \mathbf{B}_h, \mathbf{w}_h) + b_{1h}(\mathbf{v}_h, \Pi_4 p - p_h) - b_{1h}(\Pi_1 \mathbf{u} - \mathbf{u}_h, q_h) \\ &\quad + b_{2h}(\mathbf{w}_h, \Pi_5 r - r_h) - b_{2h}(\Pi_2 \mathbf{B} - \mathbf{B}_h, \theta_h) + c_{1h}(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u}, \mathbf{v}_h) - c_{1h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \\ &\quad + c_{2h}(\mathbf{v}_h; \Pi_2 \mathbf{B}, \Pi_2 \mathbf{B}) - c_{2h}(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) - c_{2h}(\Pi_1 \mathbf{u}; \Pi_2 \mathbf{B}, \mathbf{w}_h) \\ &\quad + c_{2h}(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h) + G_{3h}(\Pi_3 T - T_h, \mathbf{v}_h) \\ &= E_u(\mathbf{u}, \mathbf{v}_h) + E_B(\mathbf{B}, \mathbf{w}_h) + E_{\tilde{u}}(\mathbf{u}, \mathbf{v}_h) + E_{\tilde{B}1}(\mathbf{B}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) \\ &\quad + E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h), \quad \forall (\mathbf{v}_h, \mathbf{w}_h, q_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times Q_h^0, \end{aligned} \tag{5.7a}$$

$$\begin{aligned} &a_{3h}(\Pi_3 T - T_h, z_h) + c_{3h}(\Pi_1 \mathbf{u}; \Pi_3 T, z_h) - c_{3h}(\mathbf{u}_h; T_h, z_h) \\ &= E_T(T, z_h) + E_{\tilde{T}}(\mathbf{u}; T, z_h), \qquad \forall z_h \in Z_h^0. \end{aligned} \tag{5.7b}$$

Taking

$$(\mathbf{v}_h, \mathbf{w}_h, z_h, q_h, \theta_h) = (\Pi_1 \mathbf{u} - \mathbf{u}_h, \Pi_2 \mathbf{B} - \mathbf{B}_h, \Pi_3 T - T_h, \Pi_4 p - p_h, \Pi_5 r - r_h)$$

in (5.7) and using the relation

$$\begin{aligned} &c_{1h}(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u} - \mathbf{u}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) = 0, \\ &c_{3h}(\Pi_3 T; \Pi_3 T - T_h, \Pi_3 T - T_h) = 0, \end{aligned}$$

we obtain

$$\begin{aligned} &a_{1h}(\Pi_1 \mathbf{u} - \mathbf{u}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) + a_{2h}(\Pi_2 \mathbf{B} - \mathbf{B}_h, \Pi_2 \mathbf{B} - \mathbf{B}_h) \\ &= E_u(\mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) + E_B(\mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) + E_{\tilde{u}}(\mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) \\ &\quad + E_{\tilde{B}1}(\mathbf{B}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) + E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) - c_{1h}(\Pi_1 \mathbf{u}; \Pi_1 \mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) \\ &\quad + c_{1h}(\mathbf{u}_h; \mathbf{u}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) - c_{2h}(\Pi_1 \mathbf{u} - \mathbf{u}_h; \Pi_2 \mathbf{B}, \Pi_2 \mathbf{B}) + c_{2h}(\Pi_1 \mathbf{u} - \mathbf{u}_h; \mathbf{B}_h, \mathbf{B}_h) \\ &\quad + c_{2h}(\Pi_1 \mathbf{u}; \Pi_2 \mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) - c_{2h}(\mathbf{u}_h; \mathbf{B}_h, \Pi_2 \mathbf{B} - \mathbf{B}_h) - G_{3h}(\Pi_3 T - T_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) \\ &= E_u(\mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) + E_B(\mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) + E_{\tilde{u}}(\mathbf{u}; \mathbf{u}, \Pi_1 \mathbf{u} - \mathbf{u}_h) + E_{\tilde{B}1}(\mathbf{B}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) \\ &\quad + E_{\tilde{B}2}(\mathbf{u}; \mathbf{B}, \Pi_2 \mathbf{B} - \mathbf{B}_h) - c_{1h}(\Pi_1 \mathbf{u} - \mathbf{u}_h; \mathbf{u}_h, \Pi_1 \mathbf{u} - \mathbf{u}_h) - c_{2h}(\mathbf{u}_h; \Pi_2 \mathbf{B} - \mathbf{B}_h, \Pi_2 \mathbf{B} - \mathbf{B}_h) \\ &\quad - c_{2h}(\Pi_1 \mathbf{u} - \mathbf{u}_h; \Pi_2 \mathbf{B} - \mathbf{B}_h, \mathbf{B}_h) - G_{3h}(\Pi_3 T - T_h, \Pi_1 \mathbf{u} - \mathbf{u}_h), \end{aligned} \tag{5.8a}$$

$$\begin{aligned} &a_{3h}(\Pi_3 T - T_h, \Pi_3 T - T_h) \\ &= E_T(T, \Pi_3 T - T_h) + E_{\tilde{T}}(\mathbf{u}; T, \Pi_3 T - T_h) \\ &\quad - c_{3h}(\Pi_1 \mathbf{u}; \Pi_3 T, \Pi_3 T - T_h) + c_{3h}(\mathbf{u}_h; T_h, \Pi_3 T - T_h) \\ &= E_T(T, \Pi_3 T - T_h) + E_{\tilde{T}}(\mathbf{u}; T, \Pi_3 T - T_h) - c_{3h}(\Pi_1 \mathbf{u} - \mathbf{u}_h; T_h, \Pi_3 T - T_h). \end{aligned} \tag{5.8b}$$

In view of Lemmas 4.1, 5.2, and the definitions of M_{ih} $(i=1,2,3)$ in (4.6)-(4.8), we further have

$$
\begin{aligned}
&|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + |||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W \\
&\le 2\zeta Ch\left(\|\mathbf{u}\|_2+\|\mathbf{B}\|_2+\|\mathbf{B}\|_2^2+\|\mathbf{u}\|_2^2+\|\mathbf{u}\|_2\|\mathbf{B}\|_2\right)\\
&\quad + 2\zeta\Bigg(H_aM_{1h}|||\mathbf{u}_h|||_V|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + R_mM_{2h}|||\mathbf{u}_h|||_V|||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W\\
&\qquad\quad + \frac{1}{2}H_aM_{2h}|||\mathbf{B}_h|||_W|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + \frac{1}{2}R_mM_{2h}|||\mathbf{B}_h|||_W|||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W\Bigg)\\
&\quad + 2\zeta H_aM_{3h}\frac{G_r\mathbf{g}}{NR_eg}|||\Pi_3T-T_h|||_Z,
\end{aligned} \tag{5.9a}
$$

$$
\begin{aligned}
&|||\Pi_3T-T_h|||_Z\\
&\le Ch(|T|_2+\|\mathbf{u}\|_2\|T\|_2+\|T\|_2\|\mathbf{u}\|_2) + M_{3h}|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_1|||T_h|||_Z\\
&\le Ch(|T|_2+\|\mathbf{u}\|_2\|T\|_2+\|T\|_2\|\mathbf{u}\|_2) + M_{3h}P_r^2R_e^2\|f_3\|_{3h}|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V.
\end{aligned} \tag{5.9b}
$$

Taking (5.9b) in (5.9a) we get

$$
\begin{aligned}
&|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + |||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W \\
&\le 2\zeta Ch\left(\|\mathbf{u}\|_2+\|\mathbf{B}\|_2+\|\mathbf{B}\|_2^2+\|\mathbf{u}\|_2^2+\|\mathbf{u}\|_2\|\mathbf{B}\|_2+\|\mathbf{B}\|_2\|\mathbf{u}\|_2\right)\\
&\quad + 2\zeta\Bigg(H_aM_{1h}|||\mathbf{u}_h|||_V|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + R_mM_{2h}|||\mathbf{u}_h|||_V|||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W\\
&\qquad\quad + \frac{1}{2}H_aM_{2h}|||\mathbf{B}_h|||_W|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + \frac{1}{2}R_mM_{2h}|||\mathbf{B}_h|||_W|||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W\Bigg)\\
&\quad + Ch(|T|_2+\|\mathbf{u}\|_2\|T\|_2+\|T\|_2\|\mathbf{u}\|_2) + 2\zeta H_aM_{3h}P_r^2R_e^2\frac{G_r\mathbf{g}}{NR_eg}\|f_3\|_{3h}|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V\\
&\le 2\zeta Ch\left(\|\mathbf{u}\|_2+\|\mathbf{B}\|_2+\|\mathbf{B}\|_2^2+\|\mathbf{u}\|_2^2+\|\mathbf{u}\|_2\|\mathbf{B}\|_2\right)\\
&\quad + 4\zeta^3(\|\mathbf{f_1}\|_{1h}+\|\mathbf{f_2}\|_{2h}+\|f_3\|_{3h})\Bigg(H_aM_{1h}|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + R_mM_{2h}|||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W\\
&\qquad\qquad + \frac{1}{2}H_aM_{2h}|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + \frac{1}{2}R_mM_{2h}|||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W\Bigg)\\
&\quad + Ch(|T|_2+\|\mathbf{u}\|_2\|T\|_2+\|T\|_2\|\mathbf{u}\|_2) + 2\zeta H_aM_{3h}P_r^2R_e^2\frac{G_r\mathbf{g}}{NR_eg}\|f_3\|_{3h}|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V\\
&\le \max\{M_{1h},M_{2h},M_{3h}\}\left(12\zeta^4(\|\mathbf{f_1}\|_{1h}+\|\mathbf{f_2}\|_{2h}+\|f_3\|_{3h}) + 2\zeta^2P_rR_e\|f_3\|_{3h}\right)\\
&\quad\times(|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + |||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W)\\
&\quad + 2\zeta Ch\left(\|\mathbf{u}\|_2+\|\mathbf{B}\|_2+\|\mathbf{B}\|_2^2+\|\mathbf{u}\|_2^2+\|\mathbf{u}\|_2\|\mathbf{B}\|_2+\|\mathbf{B}\|_2\|\mathbf{u}\|_2\right)\\
&\quad + Ch(|T|_2+\|\mathbf{u}\|_2\|T\|_2+\|T\|_2\|\mathbf{u}\|_2),
\end{aligned}
$$

which plus the smallness condition (4.19) yields

$$|||\Pi_1\mathbf{u}-\mathbf{u}_h|||_V + |||\Pi_2\mathbf{B}-\mathbf{B}_h|||_W \lesssim hC_1(\mathbf{u},\mathbf{B},T).$$

Hence, combining this estimate with (5.9b) leads to the desired estimate (5.6a).

Next let us estimate the pressure error. Taking $(\mathbf{w}_h, q_h, r_h) = (0,0,0)$ in the Eq. (5.2a), we have

$$a_{1h}(\Pi_1\mathbf{u},\mathbf{v}_h) + b_{1h}(\mathbf{v}_h,\Pi_3 p)$$

$$
\begin{aligned}
&+ c_{1h}(\Pi_1\mathbf{u};\Pi_1\mathbf{u},\mathbf{v}_h) + c_{2h}(\mathbf{v}_h;\Pi_2\mathbf{B},\Pi_2\mathbf{B}) + G_{3h}(\Pi_3 T,\mathbf{v}_h)\\
&= (\mathbf{f_1},\mathbf{v}_{ho}) + E_u(\mathbf{u},\mathbf{v}_h) + E_{\tilde{u}}(\mathbf{u},\mathbf{v}_h),
\end{aligned}
$$

which, together with (3.4a), gives

$$
\begin{aligned}
& b_{1h}(\mathbf{v}_h,\Pi_3 p - p_h)\\
&= E_u(\mathbf{u},\mathbf{v}_h) + E_{\tilde{u}}(\mathbf{u},\mathbf{v}_h) - G_{3h}(\Pi_3 T - T_h,\mathbf{v}_h) - a_{1h}(\Pi_1\mathbf{u} - \mathbf{u}_h,\mathbf{v}_h)\\
&\quad - c_{1h}(\Pi_1\mathbf{u};\Pi_1\mathbf{u},\mathbf{v}_h) + c_{1h}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) - c_{2h}(\mathbf{v}_h;\Pi_2\mathbf{B},\Pi_2\mathbf{B}) + c_{2h}(\mathbf{v}_h;\mathbf{B}_h,\mathbf{B}_h)\\
&= E_u(\mathbf{u},\mathbf{v}_h) + E_{\tilde{u}}(\mathbf{u},\mathbf{v}_h) - G_{3h}(\Pi_3 T - T_h,\mathbf{v}_h) - a_{1h}(\Pi_1\mathbf{u} - \mathbf{u}_h,\mathbf{v}_h)\\
&\quad - c_{1h}(\Pi_1\mathbf{u} - \mathbf{u}_h;\Pi_1\mathbf{u} - \mathbf{u}_h,\mathbf{v}_h) - c_{1h}(\mathbf{u}_h;\Pi_1\mathbf{u} - \mathbf{u}_h,\mathbf{v}_h) - c_{1h}(\Pi_1\mathbf{u} - \mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)\\
&\quad - c_{2h}(\mathbf{v}_h;\Pi_2\mathbf{B} - \mathbf{B}_h,\Pi_2\mathbf{B} - \mathbf{B}_h) - c_{2h}(\mathbf{v}_h;\mathbf{B}_h,\Pi_2\mathbf{B} - \mathbf{B}_h) - c_{2h}(\mathbf{v}_h;\Pi_2\mathbf{B} - \mathbf{B}_h,\mathbf{B}_h).
\end{aligned}
$$

Thus, using the inf-sup condition (4.2), Lemmas 4.1 and 5.2, and the estimate (5.6a), we get

$$
\begin{aligned}
\|\!|\Pi_3 p - p_h|\!\|_Q &\lesssim \sup_{0\neq\mathbf{v}_h\in\mathbf{V}_h^0} \frac{b_{1h}(\mathbf{v}_h,\Pi_3 p - p_h)}{\|\!|\mathbf{v}_h|\!\|_V}\\
&\lesssim h(\|\mathbf{u}\|_2 + \|\mathbf{B}\|_2 + \|T\|_2)(1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2 + \|T\|_2)\\
&\quad + h^2(\|\mathbf{u}\|_2 + \|\mathbf{B}\|_2)^2(1 + \|\mathbf{u}\|_2 + \|\mathbf{B}\|_2)^2\\
&\lesssim hC_1(\mathbf{u},\mathbf{B},T) + h^2C_2(\mathbf{u},\mathbf{B}).
\end{aligned}
$$

Similarly, by using the inf-sup condition (4.3), Lemmas 4.1 and 5.2, and (5.6a), we can obtain

$$
\|\!|\Pi_4 r - r_h|\!\|_R \lesssim hC_1(\mathbf{u},\mathbf{B},T) + h^2C_2(\mathbf{u},\mathbf{B}).
$$

Combining the above two inequalities leads to the desired result (5.6b). □

In light of Theorem 5.1, Lemmas 3.1, 3.5, 3.7 and 3.8, and the triangle inequality, we can finally obtain the following main error estimates.

Theorem 5.2. *Under the same conditions of Theorem* 5.1*, there hold*

$$
\begin{aligned}
&\|\nabla\mathbf{u} - \nabla_h\mathbf{u}_{ho}\|_0 + \|\nabla\mathbf{u} - \nabla_{w,0}\mathbf{u}_h\|_0 \lesssim hC_1(\mathbf{u},\mathbf{B},T), && \text{(5.10a)}\\
&\|\nabla\times\mathbf{B} - \nabla_h\times\mathbf{B}_{ho}\|_0 + \|\nabla\times\mathbf{B} - \nabla_{w,0}\times\mathbf{B}_h\|_0 \lesssim hC_1(\mathbf{u},\mathbf{B},T), && \text{(5.10b)}\\
&\|\nabla T - \nabla_h T_{ho}\|_0 + \|\nabla T - \nabla_{w,0}T_h\|_0 \lesssim hC_1(\mathbf{u},\mathbf{B},T), && \text{(5.10c)}\\
&\|p - p_{ho}\|_0 + h\|\nabla p - \nabla_{w,1}p\|_0 \lesssim hC_1(\mathbf{u},\mathbf{B},T) + h\|p\|_1 + h^2C_2(\mathbf{u},\mathbf{B}), && \text{(5.10d)}\\
&\|r - r_{ho} - (\bar{r} - \bar{r}_{ho})\|_0 + h\|\nabla r - \nabla_{w,1}r\|_0 \lesssim hC_1(\mathbf{u},\mathbf{B},T) + h\|r\|_1 + h^2C_2(\mathbf{u},\mathbf{B}), && \text{(5.10e)}
\end{aligned}
$$

where $\bar{r}$ and $\bar{r}_{ho}$ denote the mean values of r and r_{ho} on Ω, respectively.

Remark 5.1. From the estimates (5.10a) and (5.10b) we see that the upper bounds of the errors of the velocity and the magnetic field are independent of the approximations of the pressure and the magnetic pseudo-pressure. This means that our WG scheme is pressure-robust.

6. Oseen Iteration Scheme

Notice that the WG scheme (3.3) is a nonlinear system. We shall adopt the following Oseen iterative algorithm: given $\mathbf{u}_h^0$ and $\mathbf{B}_h^0$, find $(\mathbf{u}_h^n, \mathbf{B}_h^n, T_h, p_h^n, r_h^n)$ with $n = 1, 2, \ldots$ such that

$$\begin{aligned}
&a_{1h}\big(\mathbf{u}_h^n, \mathbf{v}_h\big) + b_{1h}\big(\mathbf{v}_h, p_h^n\big) - b_{1h}\big(\mathbf{u}_h^n, q_h\big) + c_{1h}\big(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h\big) + c_{2h}\big(\mathbf{v}_h; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n\big) \\
&\quad + a_{2h}\big(\mathbf{B}_h^n, \mathbf{w}_h\big) + b_{2h}\big(\mathbf{w}_h, r_h^n\big) - b_{2h}\big(\mathbf{B}_h^n, \theta_h\big) - c_{2h}\big(\mathbf{u}_h^{n-1}; \mathbf{B}_h^{n-1}, \mathbf{w}_h\big) \\
&= (\mathbf{f}_1, \mathbf{v}_{ho}) - G_{3h}\big(T_h^n, \mathbf{v}_h\big) + \frac{1}{R_m}(\mathbf{f}_2, \mathbf{w}_{ho}), \qquad (6.1a) \\
&a_{3h}\big(T_h^n, z_h\big) + c_{3h}\big(\mathbf{u}_h^{n-1}; T_h^n, z_h\big) = (f_3, z_{ho}) \qquad (6.1b)
\end{aligned}$$

for any $(\mathbf{v}_h, \mathbf{w}_h, q_h, \theta_h, z_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Q_h^0 \times R_h^0 \times Z_h^0$.

We have the following convergence theorem.

Theorem 6.1. *Let* $(\mathbf{u}_h, \mathbf{B}_h, T_h, p_h, r_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Z_h^0 \times Q_h^0 \times R_h^0$ *be the solution of the WG scheme* (3.3). *Under the smallness condition* (4.19) *the Oseen iteration scheme* (6.1) *is convergent in the following sense:*

$$\begin{aligned}
&\lim_{n\to\infty} \big|\!\big|\!\big|\mathbf{u}_h^n - \mathbf{u}_h\big|\!\big|\!\big|_V = 0, \quad \lim_{n\to\infty} \big|\!\big|\!\big|\mathbf{B}_h^n - \mathbf{B}_h\big|\!\big|\!\big|_W = 0, \\
&\lim_{n\to\infty} \big|\!\big|\!\big|T_h^n - T_h\big|\!\big|\!\big|_Z = 0, \quad \lim_{n\to\infty} \big|\!\big|\!\big|p_h^n - p_h\big|\!\big|\!\big|_Q = 0, \qquad \lim_{n\to\infty} \big|\!\big|\!\big|r_h^n - r_h\big|\!\big|\!\big|_R = 0.
\end{aligned}$$

Proof. Denote

$$e_u^n := \mathbf{u}_h^n - \mathbf{u}_h, \quad e_B^n := \mathbf{B}_h^n - \mathbf{B}_h, \quad e_T^n := T_h^n - T_h, \quad e_p^n := p_h^n - p_h, \quad e_r^n := r_h^n - r_h.$$

Subtracting (6.1) from (3.3), we have for all $(\mathbf{v}_h, \mathbf{w}_h, z_h, q_h, \theta_h, z_h) \in \mathbf{V}_h^0 \times \mathbf{W}_h^0 \times Z_h^0 \times Q_h^0 \times Q_h^0$,

$$\begin{aligned}
&a_{1h}\big(e_u^n, \mathbf{v}_h\big) + a_{2h}\big(e_B^n, \mathbf{w}_h\big) \\
&= -b_{1h}\big(\mathbf{v}_h, e_p^n\big) + b_{1h}\big(e_u^n, q_h\big) - b_{2h}\big(\mathbf{w}_h, e_r^n\big) + b_{2h}\big(e_B^n, \theta_h\big) + c_{1h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \\
&\quad - c_{1h}\big(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h\big) + c_{2h}(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) - c_{2h}\big(\mathbf{v}_h; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n\big) \\
&\quad - c_{2h}(\mathbf{u}_h; \mathbf{B}_h, \mathbf{w}_h) + c_{2h}\big(\mathbf{u}_h^{n-1}; \mathbf{B}_h^{n-1}, \mathbf{w}_h\big) - G_{3h}\big(e_T^n, \mathbf{v}_h\big), \qquad (6.2a) \\
&a_{3h}\big(e_T^n, z_h\big) = c_{3h}(\mathbf{u}_h; T_h, z_h) - c_{3h}\big(\mathbf{u}_h^{n-1}; T_h^n, z_h\big). \qquad (6.2b)
\end{aligned}$$

Taking $\mathbf{v}_h = e_u^n, \mathbf{w}_h = e_B^n, z_h = e_T^n, q_h = e_p^n, \theta_h = e_r^n$ in (6.2) and using Lemma 4.1 we get

$$\begin{aligned}
&\frac{1}{H_a^2}\big|\!\big|\!\big|e_u^n\big|\!\big|\!\big|_V^2 + \frac{1}{R_m^2}\big|\!\big|\!\big|e_B^n\big|\!\big|\!\big|_W^2 \\
&= c_{1h}\big(\mathbf{u}_h; \mathbf{u}_h, e_u^n\big) - c_{1h}\big(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, e_u^n\big) + c_{2h}\big(e_u^n; \mathbf{B}_h, \mathbf{B}_h\big) - c_{2h}\big(e_u^n; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n\big) \\
&\quad - c_{2h}\big(\mathbf{u}_h; \mathbf{B}_h, e_B^n\big) + c_{2h}\big(\mathbf{u}_h^{n-1}; \mathbf{B}_h^{n-1}, e_B^n\big) - G_{3h}\big(e_T^n, e_u^n\big) \\
&= -c_{1h}\big(e_u^{n-1}; \mathbf{u}_h^n, e_u^n\big) - c_{2h}\big(e_u^n; e_B^{n-1}, \mathbf{B}_h\big) + c_{2h}\big(\mathbf{u}_h; e_B^{n-1}, e_B^n\big) - G_{3h}\big(e_T^n, e_u^n\big) \\
&\le M_{1h}|\!|\!|\mathbf{u}_h|\!|\!|_V \big|\!\big|\!\big|e_u^{n-1}\big|\!\big|\!\big|_V \big|\!\big|\!\big|e_u^n\big|\!\big|\!\big|_V + M_{2h}\big|\!\big|\!\big|e_B^{n-1}\big|\!\big|\!\big|_W \big|\!\big|\!\big|e_u^n\big|\!\big|\!\big|_V |\!|\!|\mathbf{B}_h|\!|\!|_W \\
&\quad + M_{2h}\big|\!\big|\!\big|e_B^{n-1}\big|\!\big|\!\big|_W |\!|\!|\mathbf{u}_h|\!|\!|_1 |\!|\!|e_B^n|\!|\!|_W + M_{3h}\frac{G_r \mathbf{g}}{N R_e^2 g}\big|\!\big|\!\big|e_T^n\big|\!\big|\!\big|_Z \big|\!\big|\!\big|e_u^{n-1}\big|\!\big|\!\big|_V, \qquad (6.3a) \\
&\frac{1}{P_r R_e}\big|\!\big|\!\big|e_T^n\big|\!\big|\!\big|_V^2 = c_{3h}\big(\mathbf{u}_h; T_h, e_T^n\big) - c_{3h}\big(\mathbf{u}_h^{n-1}; T_h^n, e_T^n\big) \\
&\qquad\qquad = -c_{3h}\big(e_u^{n-1}; T_h, e_T^n\big) \le M_{3h}\big|\!\big|\!\big|e_u^{n-1}\big|\!\big|\!\big|_V |\!|\!|T_h|\!|\!|_Z \big|\!\big|\!\big|e_T^n\big|\!\big|\!\big|_Z. \qquad (6.3b)
\end{aligned}$$

The inequality (6.3b) further gives

$$\left|\!\left|\!\left|e_T^n\right|\!\right|\!\right|_Z \leq M_{3h} P_r R_e \left|\!\left|\!\left|e_u^{n-1}\right|\!\right|\!\right|_V |\!|\!| T_h |\!|\!|_Z, \tag{6.4}$$

which, together with (6.3a), (4.9) and (4.10), implies

$$\begin{aligned}
&\left|\!\left|\!\left|e_u^n\right|\!\right|\!\right|_V + \left|\!\left|\!\left|e_B^n\right|\!\right|\!\right|_W \\
\leq\ & M_{1h} |\!|\!|\mathbf{u}_h|\!|\!|_V \left|\!\left|\!\left|e_u^{n-1}\right|\!\right|\!\right|_V \left|\!\left|\!\left|e_u^n\right|\!\right|\!\right|_V + M_{2h} \left|\!\left|\!\left|e_B^{n-1}\right|\!\right|\!\right|_W \left|\!\left|\!\left|e_u^n\right|\!\right|\!\right|_V |\!|\!|\mathbf{B}_h|\!|\!|_W \\
&+ M_{2h} \left|\!\left|\!\left|e_B^{n-1}\right|\!\right|\!\right|_W |\!|\!|\mathbf{u}_h|\!|\!|_V \left|\!\left|\!\left|e_B^n\right|\!\right|\!\right|_W + M_{3h} \frac{G_r \mathbf{g}}{N R_e^2 g} \left|\!\left|\!\left|e_T^n\right|\!\right|\!\right|_Z \left|\!\left|\!\left|e_u^{n-1}\right|\!\right|\!\right|_V \\
&+ M_{3h} \left|\!\left|\!\left|e_u^{n-1}\right|\!\right|\!\right|_V |\!|\!|T_h|\!|\!|_Z \left|\!\left|\!\left|e_T^n\right|\!\right|\!\right|_Z \\
\leq\ & \max\{M_{1h}, M_{2h}, M_{3h}\} \left(12\zeta^4 \|\mathbf{f_1}\|_{1h} + 12\zeta^4 \|\mathbf{f_2}\|_{2h} + \left(12\zeta^4 H_a + 2\zeta^2 P_r R_e\right) \|f_3\|_{3h}\right) \\
&\times \left(\left|\!\left|\!\left|e_u^{n-1}\right|\!\right|\!\right|_V + \left|\!\left|\!\left|e_B^{n-1}\right|\!\right|\!\right|_W\right) \\
\leq\ & \cdots \\
\leq\ & \left(\max\{M_{1h}, M_{2h}, M_{3h}\} \left(12\zeta^4 \|\mathbf{f_1}\|_{1h} + 12\zeta^4 \|\mathbf{f_2}\|_{2h} + \left(12\zeta^4 H_a + 2\zeta^2 P_r R_e\right) \|f_3\|_{3h}\right)\right)^n \\
&\times \left(\left|\!\left|\!\left|e_u^0\right|\!\right|\!\right|_V + \left|\!\left|\!\left|e_B^0\right|\!\right|\!\right|_W\right).
\end{aligned}$$

The above estimate plus (4.19) yields

$$\lim_{n\to\infty} \left|\!\left|\!\left|\mathbf{u}_h^n - \mathbf{u}_h\right|\!\right|\!\right|_V = \lim_{n\to\infty} \left|\!\left|\!\left|e_u^n\right|\!\right|\!\right|_V = 0, \tag{6.5a}$$

$$\lim_{n\to\infty} \left|\!\left|\!\left|\mathbf{B}_h^n - \mathbf{B}_h\right|\!\right|\!\right|_W = \lim_{n\to\infty} \left|\!\left|\!\left|e_B^n\right|\!\right|\!\right|_W = 0, \tag{6.5b}$$

which plus (6.4) indicates

$$\lim_{n\to\infty} \left|\!\left|\!\left|T_h^n - T_h\right|\!\right|\!\right|_Z = \lim_{n\to\infty} \left|\!\left|\!\left|e_T^n\right|\!\right|\!\right|_Z = 0. \tag{6.6}$$

From (6.2) we see that for any $\mathbf{v}_h \in \mathbf{V}_h^0$,

$$\begin{aligned}
b_{1h}\left(\mathbf{v}_h, e_p^n\right) &= -a_{1h}\left(e_u^n, \mathbf{v}_h\right) + c_{1h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - c_{1h}\left(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h\right) \\
&\quad + c_{2h}(\mathbf{v}_h; \mathbf{B}_h, \mathbf{B}_h) - c_{2h}\left(\mathbf{v}_h; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n\right) - G_{3h}\left(e_T^n, \mathbf{v}_h\right) \\
&= -a_{1h}\left(e_u^n, \mathbf{v}_h\right) - c_{1h}\left(e_u^{n-1}; \mathbf{e}_u^n, \mathbf{v}_h\right) - c_{1h}\left(e_u^{n-1}; \mathbf{u}_h, \mathbf{v}_h\right) \\
&\quad - c_{2h}\left(\mathbf{v}_h; e_B^{n-1}, e_B^n\right) - c_{2h}\left(\mathbf{v}_h; e_B^{n-1}, \mathbf{B}_h\right) - G_{3h}\left(e_T^n, \mathbf{v}_h\right).
\end{aligned}$$

Using the inf-sup condition (4.2) and Lemma 4.1, we have

$$\begin{aligned}
\left|\!\left|\!\left|e_p^n\right|\!\right|\!\right|_Q &\leq \sup_{\mathbf{v}_h \in \mathbf{V}_h^0} \frac{b_{1h}\left(\mathbf{v}_h, e_p^n\right)}{|\!|\!|\mathbf{v}_h|\!|\!|_V} \\
&\leq \left|\!\left|\!\left|e_u^n\right|\!\right|\!\right|_V + M_{1h}\left(\left|\!\left|\!\left|e_u^{n-1}\right|\!\right|\!\right|_1 |\!|\!|\mathbf{u}_h|\!|\!|_V + \left|\!\left|\!\left|e_u^{n-1}\right|\!\right|\!\right|_V \left|\!\left|\!\left|e_u^n\right|\!\right|\!\right|_V\right) \\
&\quad + M_{2h}\left(\left|\!\left|\!\left|e_B^{n-1}\right|\!\right|\!\right|_W |\!|\!|\mathbf{B}_h|\!|\!|_W + \left|\!\left|\!\left|e_B^{n-1}\right|\!\right|\!\right|_W \left|\!\left|\!\left|e_B^n\right|\!\right|\!\right|_W\right) + \frac{G_r \mathbf{g}}{N R_e^2 g} \left|\!\left|\!\left|e_T^n\right|\!\right|\!\right|_Z,
\end{aligned}$$

which, together with (6.5) and (6.6), yields

$$\lim_{n\to\infty} \left|\!\left|\!\left|p_h^n - p_h\right|\!\right|\!\right|_Q = \lim_{n\to\infty} \left|\!\left|\!\left|e_p^n\right|\!\right|\!\right|_Q = 0.$$

Finally, using the inf-sup condition (4.3) and Lemma 4.1 and (6.5), we obtain

$$\lim_{n\to\infty} \left|\!\left|\!\left|r_h^n - r_h\right|\!\right|\!\right|_R = 0.$$

This completes the proof. □

Remark 6.1. Notice that the above Oseen iterative scheme can be rewritten as the following decoupled system: given $\mathbf{u}_h^0$ and $\mathbf{B}_h^0$, for $n = 1, 2, \ldots,$

Step 1. Find T_h^n such that

$$a_{3h}\big(T_h^n, z_h\big) + c_{3h}\big(\mathbf{u}_h^{n-1}; T_h^n, z_h\big) = (f_3, z_{ho}), \quad \forall\, z_h \in Z_h^0.$$

Step 2. Find $(\mathbf{B}_h^n, r_h^n)$ such that

$$\begin{aligned} & a_{2h}\big(\mathbf{B}_h^n, \mathbf{w}_h\big) + b_{2h}\big(\mathbf{w}_h, r_h^n\big) - b_{2h}\big(\mathbf{B}_h^n, \theta_h\big) - c_{2h}\big(\mathbf{u}_h^{n-1}; \mathbf{B}_h^{n-1}, \mathbf{w}_h\big) \\ = & \frac{1}{R_m}(\mathbf{f}_2, \mathbf{w}_{ho}), \quad \forall\, (\mathbf{w}_h, \theta_h) \in \mathbf{W}_h^0 \times R_h^0. \end{aligned}$$

Step 3. Find $(\mathbf{u}_h^n, p_h^n)$ such that

$$\begin{aligned} & a_{1h}\big(\mathbf{u}_h^n, \mathbf{v}_h\big) + b_{1h}\big(\mathbf{v}_h, p_h^n\big) - b_{1h}\big(\mathbf{u}_h^n, q_h\big) + c_{1h}\big(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h\big) + c_{2h}\big(\mathbf{v}_h; \mathbf{B}_h^{n-1}, \mathbf{B}_h^n\big) \\ = & (\mathbf{f}_1, \mathbf{v}_{ho}) - G_{3h}\big(T_h^n, \mathbf{v}_h\big), \quad \forall\, (\mathbf{v}_h, q_h) \in \mathbf{V}_h^0 \times Q_h^0. \end{aligned}$$

7. Numerical Examples

In this section, we give a 2D numerical example and a 3D example to verify the performance of the WG scheme (3.3) for the steady incompressible MHD flow (1.1). We apply the Oseen iterative scheme with the initial guess $(\mathbf{u}_{ho}^0, \mathbf{B}_{ho}^0) = (0, 0)$ and the stop criterion

$$\big\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\big\|_0 < 1e - 8$$

in all numerical experiments.

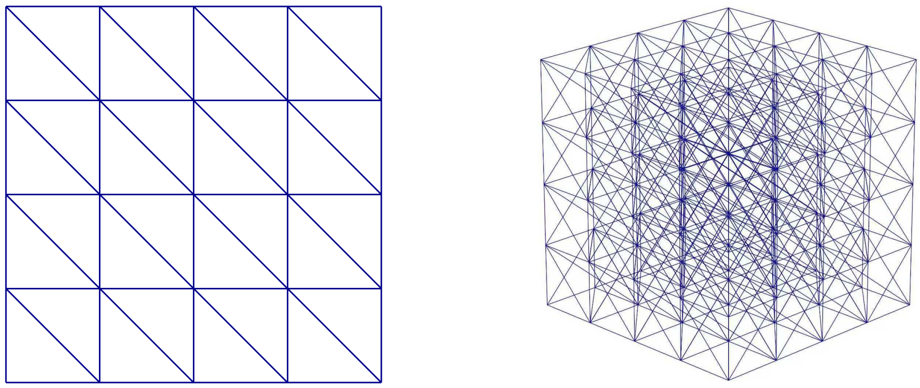
Example 7.1 (A 2D case). In the model (1.1) we set

$$\Omega = [0, 1]^2, \quad H_a = N = R_m = 1.$$

The exact solution $(\mathbf{u}, \mathbf{B}, T, p, r)$ is of the form

$$\begin{cases} u_1 = -x^2(x-1)^2 y(y-1)(2y-1), \\ u_2 = y^2(y-1)^2 x(x-1)(2x-1), \\ B_1 = -x^2(x-1)^2 y(y-1)(2y-1), \\ B_2 = y^2(y-1)^2 x(x-1)(2x-1), \\ p = x(x-1)\left(x - \dfrac{1}{2}\right) y(y-1)\left(y - \dfrac{1}{2}\right), \\ r = x(x-1)\left(x - \dfrac{1}{2}\right) y(y-1)\left(y - \dfrac{1}{2}\right), \\ T = x(x-1)y(y-1). \end{cases}$$

We compute the scheme (3.3) on $M \times M$ uniform regular triangular meshes (cf. Fig. 7.1) with $M = 4, 8, 16, 32, 64$. Numerical results are listed in Table 7.1.



(a) (b)

Fig. 7.1. The meshes: (a) 4×4 mesh for $\Omega = [0,1]^2$; (b) $4 \times 4 \times 4$ mesh for $\Omega = [0,1]^3$.

Table 7.1: Convergence history for Example 7.1: $M \times M$ meshes.

M	$\frac{\|\mathbf{u}-\mathbf{u}_{ho}\|_0}{\|\mathbf{u}\|_0}$		$\frac{\|\nabla\mathbf{u}-\nabla_h\mathbf{u}_{ho}\|_0}{\|\nabla\mathbf{u}\|_0}$		$\|\nabla\cdot\mathbf{u}_{ho}\|_{0,\infty,\Omega}$
	Error	Order	Error	Order	
4	5.9583e-01	–	5.1511e-01	–	2.4533e-18
8	1.5876e-01	1.90	2.7300e-01	0.91	1.0426e-17
16	4.1521e-02	1.93	1.3851e-01	0.97	1.9319e-17
32	1.0635e-02	1.96	6.9419e-02	0.99	9.1078e-17
64	2.6901e-03	1.98	3.4722e-02	1.00	1.1319e-15

M	$\frac{\|\mathbf{B}-\mathbf{B}_{ho}\|_0}{\|\mathbf{B}\|_0}$		$\frac{\|\nabla\times\mathbf{B}-\nabla_h\times\mathbf{B}_{ho}\|_0}{\|\nabla\times\mathbf{B}\|_0}$		$\|\nabla\cdot\mathbf{B}_{ho}\|_{0,\infty,\Omega}$
	Error	Order	Error	Order	
4	1.1606e+00	–	5.6441e-01	–	4.9065e-18
8	2.9482e-01	1.97	2.7675e-01	1.02	8.5864e-18
16	7.4064e-02	1.99	1.3198e-01	1.06	1.1040e-17
32	1.8525e-02	1.99	6.4624e-02	1.03	2.0914e-16
64	4.6310e-03	2.00	3.2115e-02	1.00	8.8931e-18

M	$\frac{\|T-T_{ho}\|_0}{\|T\|_0}$		$\frac{\|\nabla T-\nabla_h T_{ho}\|_0}{\|\nabla T\|_0}$		$\frac{\|p-p_{ho}\|_0}{\|p\|_0}$		$\frac{\|r-r_{ho}\|_0}{\|r\|_0}$	
	Error	Order	Error	Order	Error	Order	Error	Order
4	2.2913e-01	–	3.1028e-01	–	3.9131e-00	–	4.7146e-00	–
8	5.7836e-02	1.98	1.5776e-01	0.97	2.1280e-00	0.87	2.8776e-00	0.71
16	1.4496e-02	1.99	7.9217e-02	0.99	8.9070e-01	1.25	1.4532e-01	0.98
32	3.6265e-03	1.99	3.9651e-02	0.99	3.4464e-01	1.37	7.2538e-01	1.00
64	9.0678e-04	2.00	1.9831e-02	1.00	1.4036e-01	1.30	3.6249e-02	1.01

Example 7.2 (A 3D case). In the model (1.1) we set

$$\Omega = [0,1]^3, \quad H_a = N = R_m = 1.$$

The exact solution $(\mathbf{u}, \mathbf{B}, T, p, r)$ is of the form

$$\begin{cases} u_1 = -\frac{1}{20}\pi\big(\sin(\pi x)\big)^2 \sin(\pi y)\cos(\pi y)\sin(\pi z)\cos(\pi z), \\ u_2 = \frac{1}{10}\pi \sin(\pi x)\cos(\pi x)\big(\sin(\pi y)\big)^2 \sin(\pi z)\cos(\pi z), \\ u_3 = -\frac{1}{20}\pi \sin(\pi x)\cos(\pi x)\sin(\pi y)\cos(\pi y)\big(\sin(\pi z)\big)^2, \\ B_1 = -\frac{1}{20}\pi\big(\sin(\pi x)\big)^2 \sin(\pi y)\cos(\pi y)\sin(\pi z)\cos(\pi z), \\ B_2 = \frac{1}{10}\pi \sin(\pi x)\cos(\pi x)\big(\sin(\pi y)\big)^2 \sin(\pi z)\cos(\pi z), \\ B_3 = -\frac{1}{20}\pi \sin(\pi x)\cos(\pi x)\sin(\pi y)\cos(\pi y)\big(\sin(\pi z)\big)^2, \\ P = \frac{1}{10}\cos(\pi x)\cos(\pi y)\cos(\pi z), \\ r = \frac{1}{10}\sin(\pi x)\sin(\pi y)\sin(\pi z), \\ T = u_1 + u_2 + u_3. \end{cases}$$

We compute the scheme (3.3) on $M \times M \times M$ uniform regular tetrahedral meshes (cf. Fig. 7.1) with $M = 4, 8, 16, 32$. Numerical results are listed in Table 7.2.

Table 7.2: Convergence history for Example 7.2: $M \times M \times M$ meshes.

M	$\frac{\|\mathbf{u}-\mathbf{u}_{ho}\|_0}{\|\mathbf{u}\|_0}$ Error	Order	$\frac{\|\nabla\mathbf{u}-\nabla_h\mathbf{u}_{ho}\|_0}{\|\nabla\mathbf{u}\|_0}$ Error	Order	$\|\nabla\cdot\mathbf{u}_{ho}\|_{0,\infty,\Omega}$
4	1.2609e+00	–	6.1715e-01	–	1.7049e-13
8	3.2835e-01	1.94	2.9765e-01	1.05	5.3038e-15
16	7.9966e-02	2.03	1.5702e-01	0.92	4.4653e-16
32	2.0071e-02	2.00	7.3542e-02	1.09	3.0028e-15

M	$\frac{\|\mathbf{B}-\mathbf{B}_{ho}\|_0}{\|\mathbf{B}\|_0}$ Error	Order	$\frac{\|\nabla\times\mathbf{B}-\nabla_h\times\mathbf{B}_{ho}\|_0}{\|\nabla\times\mathbf{B}\|_0}$ Error	Order	$\|\nabla\cdot\mathbf{B}_{ho}\|_{0,\infty,\Omega}$
4	1.1658e+00	–	2.4345e+00	–	4.3347e-15
8	2.8945e-01	2.00	1.1778e+00	1.04	4.8867e-15
16	6.8306e-02	2.08	5.9735e-01	0.97	5.1961e-16
32	1.6447e-02	2.05	3.0755e-01	0.95	2.5105e-16

M	$\frac{\|T-T_{ho}\|_0}{\|T\|_0}$ Error	Order	$\frac{\|\nabla T-\nabla_h T_{ho}\|_0}{\|\nabla T\|_0}$ Error	Order	$\frac{\|p-p_{ho}\|_0}{\|p\|_0}$ Error	Order	$\frac{\|r-r_{ho}\|_0}{\|r\|_0}$ Error	Order
4	1.9942e-01	–	2.6572e-01	–	4.3113e+00	–	2.8566e+00	–
8	4.6543e-02	2.09	1.3456e-01	0.98	2.1655e+00	0.93	1.4746e+00	0.95
16	1.2088e-02	1.94	6.5097e-02	1.03	1.0606e+00	0.94	7.3616e-01	1.00
32	3.1578e-03	1.94	3.3348e-02	0.96	5.6356e-01	0.91	3.7211e-01	0.98

Tables 7.1 to 7.2 show the histories of convergence for the velocity $\mathbf{u}_{ho}$, the magnetic field $\mathbf{B}_{ho}$, the pressure p_{ho}, and the magnetic pseudo-pressure r_{ho}. Results of $\|\nabla\cdot\mathbf{u}_{ho}\|_{0,\infty,\Omega}$ and $\|\nabla\cdot\mathbf{B}_{ho}\|_{0,\infty,\Omega}$ are also listed to verify the divergence-free property. From the numerical results of the two examples, we have the following observations:

- The convergence rates of $\|\nabla\mathbf{u}-\nabla_h\mathbf{u}_{ho}\|_0$, $\|\nabla\times\mathbf{B}-\nabla_h\times\mathbf{B}_{ho}\|_0$, $\|\nabla T-\nabla_h T_{ho}\|_0$, $\|p-p_{ho}\|_0$ and $\|r-r_{ho}\|_0$ for the WG scheme are of first order, which are consistent with the established theoretical results in Theorem 5.2.

- The convergence rates of $\|\mathbf{u}-\mathbf{u}_{ho}\|_0$, $\|\mathbf{B}-\mathbf{B}_{ho}\|_0$, and $\|T-T_{ho}\|_0$ are of second order.

- Both the discrete velocity and the discrete magnetic field are globally divergence-free.

8. Conclusions

In this paper, we have developed a low order weak Galerkin finite element method for the steady thermally coupled incompressible magnetohydrodynamics flow. The well-posedness of the discrete scheme has been established. The method yields globally divergence-free approximations of velocity and magnetic field, and is of optimal first order convergence for the velocity, the magnetic field, the pressure, the magnetic pseudo-pressure, and temperature approximations. The proposed Oseen iteration algorithm is unconditionally convergent. Numerical experiments have verified the theoretical results.

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