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A STRUCTURE-PRESERVING NONCONFORMING FEM OF NONLINEAR KIRCHHOFF-TYPE EQUATION WITH DAMPING*

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Abstract

Superconvergent behavior for nonlinear Kirchhoff-type with damping is researched by a structure-preserving nonconforming finite element method (FEM). A new implicit energy dissipation scheme is developed and the numerical solution is bounded in energy norm. The existence of the numerical solution is obtained with the help of the Brouwer fixed-point theorem and then the uniqueness is gained. Superconvergence characteristics is revealed by the properties of the nonconforming FE and a special splitting technique. Numerical tests confirm the correctness of the theoretical research results.

Mathematics subject classification: 65N15, 65N30.
Key words: A structure-preserving nonconforming finite element method, A splitting technique, Kirchhoff-type equation with damping, Superconvergent behavior.

1. Introduction

The nonlinear Kirchhoff-type equation is a nonlinear wave equation that describes the motion of a thin elastic rod or a thin plate under certain assumptions. The equation can be used to study the propagation of waves in the structures. The nonlinear Kirchhoff-type equation is explored

$$\begin{cases}\psi_{tt}=\Delta\psi_t+\phi\big(\|\nabla\psi\|_0^2\big)\Delta\psi-\alpha|\psi|^2\psi, & (X,t)\in\Gamma\times(0,T],\\ \psi(X,t)=0, & (X,t)\in\ \partial\Gamma\times(0,T],\\ \psi(X,0)=\psi_0(X),\quad \psi_t(X,0)=\psi_1(X), & X\in\ \bar{\Gamma}.\end{cases}\tag{1.1}$$

Assume $\Gamma\subset\mathbb{R}^2$ is a rectangle, whose boundary is $\partial\Gamma$. Let $0<T<\infty, X=(x,y)$ and $\phi(s)=1+s, s\in R^+=(0,+\infty)$, where $\phi(s)\geq\phi_0>0$ and ϕ_0 is some positive constant. Moreover, $\psi_0(X)$ and $\psi_1(X)$ are two known functions.

The nonlinear Kirchhoff-type equation is a challenging one to be solved analytically, and the existence and uniqueness of (1.1) were showed in [5, 16, 37–39]. As a result, numerical methods were often employed to study its solutions. For instance, in [19], the authors employed a sophisticated algorithm that integrates the Galerkin method, the application of a modified Crank-Nicolson difference scheme and a Picard type iteration process. Errors of the three parts of the algorithm were estimated and the total error estimate was analyzed. Additionally, in [9],

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the author aimed to derive quasi-optimal error estimates for the FEM through proposing a novel approach to discuss the nonlocal term of the nonlinear Kirchhoff equations by substituting the conventional finite element system with an alternative equivalent system that possesses a sparse Jacobian matrix. Such method solved the difficulties caused by use of the Newton-Raphson method. In [6], by deriving and analyzing a priori and a posteriori estimates, and through the Helmholtz decomposition and Brouwer fixed point theorem, the authors provided a robust framework for the numerical solution of Kirchhoff equations using conforming and expanded mixed FEMs. This article delved into the use of the Helmholtz decomposition for L^2-vector-valued functions, a powerful tool in vector calculus that allowed for the separation of a vector field into its irrotational and solenoidal components.

On the one hand, the study on the nonconforming elements, such as the Wilson element in [1,17,21,28,29], Q_1^{rot} element in [20], the constrained nonconforming Q_1^{rot} element in [10,23], the P^1-nonconforming quadrilateral element in [18] and the nonconforming element EQ_1^{rot} in [14,22,24,25], is also a hot topic. In comparison to the conforming FEs, the nonconforming FEs possess several notable advantages. Firstly, nonconforming FEs allow for a relaxation of the continuity requirements of the function in the FE spaces. Secondly, for some Crouzeix-Raviart (C-R) type nonconforming elements in 2D, their unknowns of the elements are only associated with the element itself and its edges, and each degree of freedom belongs to at most two elements. Therefore, it facilitates the exchange of information across each subdomain, and provides spectral radius estimates for the iterative domain decomposition operator [12]. Thirdly, some C-R type nonconforming elements often possess certain very useful mathematical properties, such as commutative relations with respect to the gradient, divergence, and curl operators [3]. These properties are crucial for satisfying the stability conditions in mixed finite element methods, which are essential for accurate and reliable numerical solutions in a variety of physical problems. Owing to these favorable characteristics, nonconforming elements have been widely adopted and utilized in solving various types of problems across different fields, as evidenced by numerous studies and applications documented in the literatures. Their versatility and efficiency have made them a cornerstone in the realm of computational mechanics and numerical analysis. In [25], the nonconforming element EQ_1^{rot} was studied for (1.1) and optimal result was obtained. However, how to get the superconvergent results is not an easy work.

On the other hand, the backward differential formula (BDF) is a numerical method used for approximating the derivative of a function [2,4,7,8,13,15,35]. It is a finite difference method that calculates the derivative at a given point based on the function values at that point and a point value before. The BDF can be derived using Taylor series expansion. It is widely used for numerical calculations involving derivatives. Although, lots of studies devoted to BDF-FEM for nonlinear time-dependent PDEs in [30–33] recently, it is difficult to utilize BDF method to analyze (1.1).

In this paper, a second order nonconforming FE scheme is constructed combined with the BDF method and the superconvergent result for (1.1) is derived. Some of the unique features of our paper lay in two points.

The first one is that, the scheme is new in which we define

$$D_{\tau\tau}\sigma^{n+1} = \frac{2\sigma^{n+1} - 5\sigma^n + 4\sigma^{n-1} - \sigma^{n-2}}{\tau^2}, \quad n \geq 2$$

to simulate σ_{tt}^n and combined with the BDF method, the second order result for temporal direction is derived. The approach used here for time discretization is of three-step type, allows us to deduce the numerical solution of the last point comparing with the results in [36]. To preserve the energy dissipation,

$$\frac{1}{4}\big(|\sigma^{n+1}|^2+|2\sigma^{n+1}-\sigma^n|^2+|\sigma^n|^2+|2\sigma^n-\sigma^{n-1}|^2\big)\sigma^{n+1}$$

is used to simulation $|\sigma^{n+1}|^2\sigma^{n+1}$. Meanwhile, the boundedness of the numerical solution is derived by this way.

The second one is that, comparing with the results in [25], superconvergent result is discussed with the help of a splitting method. We split

$$\begin{aligned}&\int_K \nabla_h u^{n+1}\nabla_h\big(u^{n+1}-I_h u^{n+1}\big)dX\\ =&\int_K \nabla_h\big(u^{n+1}-I_h u^{n+1}\big)\nabla_h\big(u^{n+1}-I_h u^{n+1}\big)dX\\ &+\int_K \nabla_h\big(I_h u^{n+1}-U_h^{n+1}\big)\nabla_h\big(u^{n+1}-I_h u^{n+1}\big)dX\\ &+\int_K \nabla_h U_h^{n+1}\nabla_h\big(u^{n+1}-I_h u^{n+1}\big)dX \end{aligned} \tag{1.2}$$

with the property of nonconforming element EQ_1^{rot}, that is

$$\int_K \nabla_h U_h^{n+1}\nabla_h\big(u^{n+1}-I_h u^{n+1}\big)dX=0.$$

As a result, the superconvergent characteristics for the spatial direction is shown.

2. A New Energy Dissipation Nonconforming FE Scheme

Without loss of generality, we assume that Γ is a rectangle in X-plane, whose edges parallel to the coordinate axes. Moreover, assume that K_h is a regular rectangular subdivision. For all $K\in K_h$, we assume $l_i=\overline{a_i a_{i+1}}, i=1\sim 4$ (mod 4). The FE space V_h of nonconforming element EQ_1^{rot} is defined as follows:

$$V_h=\left\{v_h: v_h|_K\in \operatorname{span}\{1,x,y,x^2,y^2\},\ \int_F [v_h]ds=0,\ F\subset\partial K,\ \forall K\in K_h\right\}.$$

If F is an internal edge, $[v_h]$ means the jump of v_h across F and if F is a boundary edge, v_h is itself. Assume $I_h: H^1(\Omega)\to V_h$ is the interpolation operator on V_h, where $I_K=I_h|_K$ and

$$\int_{l_i}(I_K\psi-\psi)ds=0,\quad i=1\sim 4,\quad \int_K (I_K\psi-\psi)dX=0. \tag{2.1}$$

The following results can be referred to [14, 22].

Lemma 2.1. *If $\psi\in H^1(\Omega)$ and $\varphi\in (H^2(\Omega))^2$, for all $v_h\in V_h$, there hold*

$$\big(\nabla_h(\psi-I_h\psi),\nabla_h v_h\big)_h=0, \tag{2.2}$$

$$\langle\varphi, v_h\rangle_h=\mathcal{O}(h^2)|\varphi|_2\|v_h\|_h. \tag{2.3}$$

Start now, ∇_h means the gradient operator defined piecewisely, $(\star,\star)_h=\sum_K\int_K \star\cdot\star\, dX$, $\langle\star,\star\rangle_h=\sum_K\int_{\partial K}\star\cdot\star\cdot\vec{n}ds$, and $\|\cdot\|_h=(\sum_K|\cdot|_{1,K}^2)^{1/2}$ is a norm on V_h.

We assume that $\{t_n : t_n = n\tau; 0 \le n \le N\}$ is a division on the timeline of $[0,T]$. Denote $\tau = T/N$ and $\sigma^n = \sigma(X, t_n)$. Suppose$\{\sigma^n\}_{n=0}^N$ is a sequence of functions, we note

$$D_{\tau\tau}\sigma^{n+1} = \frac{2\sigma^{n+1} - 5\sigma^n + 4\sigma^{n-1} - \sigma^{n-2}}{\tau^2}, \qquad 2 \le n \le N,$$
$$D_\tau \sigma^{n+1} = \frac{3\sigma^{n+1} - 4\sigma^n + \sigma^{n-1}}{2\tau}, \quad \bar{\partial}_t \sigma^{n+1} = \frac{\sigma^{n+1} - \sigma^n}{\tau}, \quad 1 \le n \le N.$$

Then it could be found in [8,13] that

$$D_\tau \sigma^{n+1} \cdot \sigma^{n+1} = \frac{1}{4\tau}\big(|\sigma^{n+1}|^2 - |\sigma^n|^2 + |2\sigma^{n+1} - \sigma^n|^2 - |2\sigma^n - \sigma^{n-1}|^2 + |\sigma^{n+1} - 2\sigma^n + \sigma^{n-1}|^2\big). \tag{2.4}$$

On the condition that $n \ge 2$, an implicit fully-discrete nonconforming FE scheme is established as: Find $U_h^n \in V_h$ such that

$$\begin{aligned}
&\big(D_{\tau\tau}U_h^{n+1}, v_h\big) + \big(D_\tau \nabla_h U_h^{n+1}, \nabla_h v_h\big)_h \\
&+ \Bigg(\phi\Big(\frac{1}{4}\Big(\big\|U_h^{n+1}\big\|_h^2 + \big\|2U_h^{n+1} - U_h^n\big\|_h^2 + \big\|U_h^n\big\|_h^2 \\
&\qquad + \big\|2U_h^n - U_h^{n-1}\big\|_h^2\Big)\Big)\nabla_h U_h^{n+1}, \nabla_h v_h\Bigg)_h \\
&+ \frac{\alpha}{4}\Big(\Big(\big|U_h^{n+1}\big|^2 + \big|2U_h^{n+1} - U_h^n\big|^2 + \big|U_h^n\big|^2 \\
&\qquad + \big|2U_h^n - U_h^{n-1}\big|^2\Big)U_h^{n+1}, v_h\Big) = 0, \quad \forall\, v_h \in V_h.
\end{aligned} \tag{2.5}$$

On the condition that $n = 1$, we compute U_h^2 by the nonlinearized equation

$$\begin{aligned}
&\left(\frac{U_h^2 - 2U_h^1 + U_h^0}{\tau^2}, v_h\right) + \left(\frac{\nabla_h U_h^2 - \nabla_h U_h^0}{\tau}, \nabla_h v_h\right)_h \\
&+ \left(\phi\left(\frac{\big\|U_h^2\big\|_h^2 + \big\|U_h^0\big\|_h^2}{2}\right)\frac{\nabla_h U_h^2 + \nabla_h U_h^0}{2}, \nabla_h v_h\right)_h \\
&+ \alpha\left(\big(U_h^1\big)^2 \frac{U_h^2 + U_h^0}{2}, v_h\right) = 0, \quad \forall\, v_h \in V_h,
\end{aligned} \tag{2.6}$$

where

$$U_h^0 = I_h\psi_0, \quad U_h^1 = I_h\left(\psi_0 + \psi_1\tau + \frac{1}{2}\psi_{tt}(0)\tau^2\right),$$

and $\psi_{tt}(0)$ can be deduced by

$$\psi_{tt}(0) = \Delta\psi_1 + \phi\big(\|\nabla\psi_0\|_0^2\big)\Delta\psi_0 - \alpha|\psi_0|^2\psi_0.$$

Now, we give the energy stability of (2.5), which leads to the prior estimates as follows.

Theorem 2.1. *Suppose $n \ge 2$, the nonconforming FE scheme of* (2.5) *is energy dissipation in the manner*

$$E^{n+1} \le E^n, \tag{2.7}$$

where

$$E^{n+1} = \big(\bar{\partial}_t U_h^{n+1} - \bar{\partial}_t U_h^n, \bar{\partial}_t U_h^{n+1}\big) + \frac{3}{4}\big\|\bar{\partial}_t U_h^{n+1} - \bar{\partial}_t U_h^n\big\|_0^2$$

$$+\frac{1}{2}\left\|\bar{\partial}_t U_h^{n+1}\right\|_0^2+\frac{1}{4}\Big(\left\|U_h^{n+1}\right\|_h^2+\left\|2U_h^{n+1}-U_h^n\right\|_h^2\Big)$$
$$+\frac{1}{16}\left(\left\|U_h^{n+1}\right\|_h^2+\left\|2U_h^{n+1}-U_h^n\right\|_h^2\right)^2+\frac{\alpha}{16}\left\|\left|U_h^{n+1}\right|^2+\left|2U_h^{n+1}-U_h^n\right|^2\right\|_0^2. \quad (2.8)$$

When $n\geq 1$, $\|U_h^{n+1}\|_h$ is bounded.

Proof. To derive the boundedness of the numerical solutions, which is an important factor to derive Theorems 2.2 and 3.1, we prove the energy dissipation of the scheme (2.5).

Firstly, setting $v_h=D_\tau U_h^{n+1}$ in (2.5), we have

$$\left\|D_\tau U_h^{n+1}\right\|_h^2+\left(D_{\tau\tau}U_h^{n+1},D_\tau U_h^{n+1}\right)$$
$$+\phi\Big(\frac{1}{4}\Big(\left\|U_h^n\right\|_h^2+\left\|U_h^{n+1}\right\|_h^2+\left\|2U_h^n-U_h^{n-1}\right\|_h^2+\left\|2U_h^{n+1}-U_h^n\right\|_h^2\Big)\Big)\left(\nabla_h U_h^{n+1},D_\tau\nabla_h U_h^{n+1}\right)_h$$
$$+\frac{\alpha}{4}\Big(\Big(\left|U_h^n\right|^2+\left|U_h^{n+1}\right|^2+\left|U_h^{n-1}-2U_h^n\right|^2+\left|U_h^n-2U_h^{n+1}\right|^2\Big)U_h^{n+1},D_\tau U_h^{n+1}\Big)=0. \quad (2.9)$$

Through splitting the terms $D_{\tau\tau}U_h^{n+1}$ and $D_\tau U_h^{n+1}$, we have

$$\begin{aligned}
&\left(D_{\tau\tau}U_h^{n+1},D_\tau U_h^{n+1}\right)\\
&=\frac{1}{\tau}\left(2\bar{\partial}_t U_h^{n+1}-3\bar{\partial}_t U_h^n+\bar{\partial}_t U_h^{n-1},\frac{3}{2}\bar{\partial}_t U_h^{n+1}-\frac{1}{2}\bar{\partial}_t U_h^n\right)\\
&=\frac{1}{\tau}\left(2\left(\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n\right)-\left(\bar{\partial}_t U_h^n-\bar{\partial}_t U_h^{n-1}\right),\bar{\partial}_t U_h^{n+1}+\frac{1}{2}\left(\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n\right)\right)\\
&=\frac{2}{\tau}\left(\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n,\bar{\partial}_t U_h^{n+1}\right)-\frac{1}{\tau}\left(\bar{\partial}_t U_h^n-\bar{\partial}_t U_h^{n-1},\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n\right)\\
&\quad-\frac{1}{\tau}\left(\bar{\partial}_t U_h^n-\bar{\partial}_t U_h^{n-1},\bar{\partial}_t U_h^n\right)+\frac{1}{2\tau}\left\|\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n\right\|_0^2\\
&\quad+\frac{1}{2\tau}\left(\left(\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n\right)-\left(\bar{\partial}_t U_h^n-\bar{\partial}_t U_h^{n-1}\right),\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n\right)\\
&\geq\frac{1}{\tau}\left(\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n,\bar{\partial}_t U_h^{n+1}\right)-\frac{1}{\tau}\left(\bar{\partial}_t U_h^n-\bar{\partial}_t U_h^{n-1},\bar{\partial}_t U_h^n\right)\\
&\quad+\frac{1}{2\tau}\left\|\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n\right\|_0^2-\frac{1}{2\tau}\left\|\bar{\partial}_t U_h^n-\bar{\partial}_t U_h^{n-1}\right\|_0^2+\frac{1}{2\tau}\left(\left\|\bar{\partial}_t U_h^{n+1}\right\|_0^2-\left\|\bar{\partial}_t U_h^n\right\|_0^2\right)\\
&\quad+\frac{1}{4\tau}\left(\left\|\bar{\partial}_t U_h^{n+1}-\bar{\partial}_t U_h^n\right\|_0^2-\left\|\bar{\partial}_t U_h^n-\bar{\partial}_t U_h^{n-1}\right\|_0^2\right).
\end{aligned} \quad (2.10)$$

By using the definition of $\phi(s)=1+s$, it is easy to see that

$$\begin{aligned}
&\phi\left(\frac{1}{4}\left(\left\|U_h^{n+1}\right\|_h^2+\left\|2U_h^{n+1}-U_h^n\right\|_h^2+\left\|U_h^n\right\|_h^2+\left\|2U_h^n-U_h^{n-1}\right\|_h^2\right)\right)\left(\nabla_h U_h^{n+1},D_\tau\nabla_h U_h^{n+1}\right)_h\\
&=\left(\nabla_h U_h^{n+1},D_\tau\nabla_h U_h^{n+1}\right)_h\\
&\quad+\frac{1}{4}\left(\left\|U_h^{n+1}\right\|_h^2+\left\|2U_h^{n+1}-U_h^n\right\|_h^2+\left\|U_h^n\right\|_h^2+\left\|2U_h^n-U_h^{n-1}\right\|_h^2\right)\left(\nabla_h U_h^{n+1},D_\tau\nabla_h U_h^{n+1}\right)_h\\
&\geq\frac{1}{4\tau}\left(\left\|U_h^{n+1}\right\|_h^2-\left\|U_h^n\right\|_h^2+\left\|U_h^n-2U_h^{n+1}\right\|_h^2-\left\|U_h^{n-1}-2U_h^n\right\|_h^2\right)\\
&\quad+\frac{1}{16\tau}\left(\left\|U_h^{n+1}\right\|_h^2+\left\|2U_h^{n+1}-U_h^n\right\|_h^2+\left\|U_h^n\right\|_h^2+\left\|2U_h^n-U_h^{n-1}\right\|_h^2\right)\\
&\quad\times\left(\left\|U_h^{n+1}\right\|_h^2+\left\|2U_h^{n+1}-U_h^n\right\|_h^2-\left\|U_h^n\right\|_h^2-\left\|2U_h^n-U_h^{n-1}\right\|_h^2\right)\\
&\geq\frac{1}{4\tau}\left(\left\|U_h^{n+1}\right\|_h^2-\left\|U_h^n\right\|_h^2+\left\|U_h^n-2U_h^{n+1}\right\|_h^2-\left\|U_h^{n-1}-2U_h^n\right\|_h^2\right)
\end{aligned}$$

$$+\frac{1}{16\tau}\left(\left(\|U_h^{n+1}\|_h^2+\|2U_h^{n+1}-U_h^n\|_h^2\right)^2-\left(\|U_h^n\|_h^2+\|2U_h^n-U_h^{n-1}\|_h^2\right)^2\right), \tag{2.11}$$

$$\begin{aligned}
&\frac{\alpha}{4}\left(\left(|U_h^n|^2+|U_h^{n+1}|^2+|U_h^{n-1}-2U_h^n|^2+|U_h^n-2U_h^{n+1}|^2\right)U_h^{n+1},D_\tau U_h^{n+1}\right)\\
\geq{}&\frac{\alpha}{16\tau}\int_\Omega\left(|U_h^n|^2+|U_h^{n+1}|^2+|U_h^{n-1}-2U_h^n|^2+|U_h^n-2U_h^{n+1}|^2\right)\\
&\qquad\times\left(|U_h^n-2U_h^{n+1}|^2-|U_h^{n-1}-2U_h^n|^2+|U_h^{n+1}|^2-|U_h^n|^2\right)dX\\
={}&\frac{\alpha}{16\tau}\left(\left\||U_h^n-2U_h^{n+1}|^2+|U_h^{n+1}|^2\right\|_0^2-\left\||U_h^{n-1}-2U_h^n|^2+|U_h^n|^2\right\|_0^2\right),
\end{aligned} \tag{2.12}$$

respectively. Therefore, we have

$$\begin{aligned}
&\frac{1}{\tau}\left(\bar\partial_tU_h^{n+1}-\bar\partial_tU_h^n,\bar\partial_tU_h^{n+1}\right)-\frac{1}{\tau}\left(\bar\partial_tU_h^n-\bar\partial_tU_h^{n-1},\bar\partial_tU_h^n\right)+\frac{3}{4\tau}\|\bar\partial_tU_h^{n+1}-\bar\partial_tU_h^n\|_0^2\\
&\quad-\frac{3}{4\tau}\|\bar\partial_tU_h^n-\bar\partial_tU_h^{n-1}\|_0^2+\frac{1}{2\tau}\left(\|\bar\partial_tU_h^{n+1}\|_0^2-\|\bar\partial_tU_h^n\|_0^2\right)\\
&\quad+\frac{1}{4\tau}\left(\left(\|U_h^{n+1}\|_h^2+\|U_h^n-2U_h^{n+1}\|_h^2\right)-\left(\|U_h^n\|_h^2+\|U_h^{n-1}-2U_h^n\|_h^2\right)\right)\\
&\quad+\frac{1}{16\tau}\left(\left(\|U_h^{n+1}\|_h^2+\|2U_h^{n+1}-U_h^n\|_h^2\right)^2-\left(\|U_h^n\|_h^2+\|2U_h^n-U_h^{n-1}\|_h^2\right)^2\right)\\
&\quad+\frac{\alpha}{16\tau}\left(\left\||U_h^{n+1}|^2+|2U_h^{n+1}-U_h^n|^2\right\|_0^2-\left\||U_h^n|^2+|2U_h^n-U_h^{n-1}|^2\right\|_0^2\right)\leq 0.
\end{aligned} \tag{2.13}$$

That is

$$E^{n+1}\leq E^n.$$

Secondly, to bound the numerical solutions, we sum (2.13) from 2 to n and we can see that

$$\begin{aligned}
&\frac{3}{4\tau}\|\bar\partial_tU_h^{n+1}-\bar\partial_tU_h^n\|_0^2+\frac{1}{2\tau}\|\bar\partial_tU_h^{n+1}\|_0^2+\frac{1}{4\tau}\left(\|U_h^{n+1}\|_h^2+\|2U_h^{n+1}-U_h^n\|_h^2\right)\\
&\quad+\frac{1}{16\tau}\left(\|U_h^{n+1}\|_h^2+\|2U_h^{n+1}-U_h^n\|_h^2\right)^2+\frac{\alpha}{16\tau}\left\||U_h^{n+1}|^2+|2U_h^{n+1}-U_h^n|^2\right\|_0^2\\
\leq{}&\frac{1}{\tau}\left(\bar\partial_tU_h^{n+1}-\bar\partial_tU_h^n,\bar\partial_tU_h^{n+1}\right)+\frac{1}{\tau}\left(\bar\partial_tU_h^2-\bar\partial_tU_h^1,\bar\partial_tU_h^2\right)\\
&\quad+\frac{3}{4\tau}\|\bar\partial_tU_h^2-\bar\partial_tU_h^1\|_0^2+\frac{1}{2\tau}\|\bar\partial_tU_h^2\|_0^2+\frac{1}{4\tau}\left(\|U_h^2\|_h^2+\|2U_h^2-U_h^1\|_h^2\right)\\
&\quad+\frac{1}{16\tau}\left(\|U_h^2\|_h^2+\|2U_h^2-U_h^1\|_h^2\right)^2+\frac{\alpha}{16\tau}\left\||U_h^2|^2+|2U_h^2-U_h^1|^2\right\|_0^2.
\end{aligned} \tag{2.14}$$

When $n=1$, we need to bound $\|\bar\partial_tU_h^2\|_0$ and $\|U_h^2\|_h$. We choose $v_h=(U_h^2-U_h^0)/\tau$ in (2.6) to get

$$\begin{aligned}
&\frac{1}{\tau}\left(\left\|\frac{U_h^2-U_h^1}{\tau}\right\|_0^2-\left\|\frac{U_h^1-U_h^0}{\tau}\right\|_0^2\right)+\left\|\frac{U_h^2-U_h^0}{\tau}\right\|_h^2+\frac{1}{2\tau}\left(\|U_h^2\|_h^2-\|U_h^0\|_h^2\right)\\
&\quad+\frac{\|U_h^2\|_h^4-\|U_h^0\|_h^4}{4\tau}+\frac{\alpha}{2\tau}\left(\|U_h^2U_h^1\|_0^2-\|U_h^1U_h^0\|_0^2\right)=0,
\end{aligned} \tag{2.15}$$

which implies

$$\|\bar\partial_tU_h^2\|_0^2+\frac{1}{2}\|U_h^2\|_h^2\leq\|\bar\partial_tU_h^1\|_0^2+\frac{1}{2}\|U_h^0\|_h^2+\frac{1}{4}\|U_h^0\|_h^4+\frac{\alpha}{2}\|U_h^1U_h^0\|_0^2. \tag{2.16}$$

Therefore, we have the boundedness of $\|\bar{\partial}_t U_h^2\|_0$ and $\|U_h^2\|_h$. With the results above and (2.14), we can bound $\|U_h^{n+1}\|_h$. The proof is complete. $\square$

Remark 2.1. As we know, utilizing BDF method to establish the energy dissipation scheme for hyperbolic equation. Combined with the definition of $D_{\tau\tau}\psi^{n+1}$, which is introduced to simulate ψ_{tt}^{n+1} in our paper, a new energy dissipation scheme is set up by use of the BDF method. As a result, the boundedness of the numerical solution is deduced. The results in this paper can be extended to some other nonlinear hyperbolic equations, such as nonlinear equations with wave operator, and so on. However, we believe using some other scheme, the above advantages cannot be fully utilized.

Theorem 2.2. *The nonconforming FE scheme of* (2.5) *is uniquely solvable.*

Proof. Define the mapping $G(\gamma) : V_h \to V_h$ as

$$\begin{aligned}\big(G(\gamma),\mu\big) &= \big(2\gamma - 5U_h^n + 4U_h^{n-1} - U_h^{n-2},\mu\big) + \tau\left(\frac{3\nabla_h\gamma - 4\nabla_h U_h^n + \nabla_h U_h^{n-1}}{2},\nabla_h\mu\right)_h \\ &+ \tau^2\left(\phi\left(\frac{1}{4}\left(\|\gamma\|_h^2 + \left\|2\gamma - U_h^n\right\|_h^2 + \left\|U_h^n\right\|_h^2 + \left\|2U_h^n - U_h^{n-1}\right\|_h^2\right)\right)\nabla_h\gamma,\nabla_h\mu\right)_h \\ &+ \tau^2\frac{\alpha}{4}\left(\left(|\gamma|^2 + \left|2\gamma - U_h^n\right|^2 + \left|U_h^n\right|^2 + \left|2U_h^n - U_h^{n-1}\right|^2\right)\gamma,\mu\right),\end{aligned} \tag{2.17}$$

where $\gamma \in V_h$ is equipped with norm $\|\gamma\|_1$, and $\mu \in V_h$.

Setting $\mu = \gamma$ in (2.18), we have

$$\begin{aligned}\big(G(\gamma),\gamma\big) &\geq \big(2\gamma - 5U_h^n + 4U_h^{n-1} - U_h^{n-2},\gamma\big) + \tau\left(\frac{3\nabla_h\gamma - 4\nabla_h U_h^n + \nabla_h U_h^{n-1}}{2},\nabla_h\gamma\right)_h \\ &+ \tau^2\left(\phi\left(\frac{1}{4}\left(\|\gamma\|_h^2 + \left\|2\gamma - U_h^n\right\|_h^2 + \left\|U_h^n\right\|_h^2 + \left\|2U_h^n - U_h^{n-1}\right\|_h^2\right)\right)\nabla_h\gamma,\nabla_h\gamma\right)_h \\ &+ \tau^2\frac{\alpha}{4}\left(\left(|\gamma|^2 + \left|2\gamma - U_h^n\right|^2 + \left|U_h^n\right|^2 + \left|2U_h^n - U_h^{n-1}\right|^2\right)\gamma,\gamma\right) \\ &\geq \frac{3}{2}\|\gamma\|_0^2 - \frac{1}{2}\left\|5U_h^n - 4U_h^{n-1} + U_h^{n-2}\right\|_0^2 + \tau\|\gamma\|_h^2 - \frac{1}{8}\tau\left\|4U_h^n - U_h^{n-1}\right\|_h^2 \\ &\geq \frac{1}{2}\left(3\|\gamma\|_0^2 - \left\|5U_h^n - 4U_h^{n-1} + U_h^{n-2}\right\|_0^2\right) + \tau\left(\|\gamma\|_h^2 - \frac{1}{8}\left\|4U_h^n - U_h^{n-1}\right\|_h^2\right),\end{aligned} \tag{2.18}$$

where, with the definition of $\phi(s) = 1 + s$, it follows that

$$\begin{aligned}&\left(\phi\left(\frac{1}{4}\left(\|\gamma\|_h^2 + \left\|2\gamma - U_h^n\right\|_h^2 + \left\|U_h^n\right\|_h^2 + \left\|2U_h^n - U_h^{n-1}\right\|_h^2\right)\right)\nabla_h\gamma,\nabla_h\gamma\right)_h \\ &= (\nabla_h\gamma,\nabla_h\gamma)_h + \left(\frac{1}{4}\left(\|\gamma\|_h^2 + \left\|2\gamma - U_h^n\right\|_h^2 + \left\|U_h^n\right\|_h^2 + \left\|2U_h^n - U_h^{n-1}\right\|_h^2\right)\nabla_h\gamma,\nabla_h\gamma\right)_h \geq 0.\end{aligned}$$

Based on Lemma 2.1, we have the boundedness of $\|5U_h^n + 4U_h^{n-1} - U_h^{n-2}\|_0$ and $\|4U_h^n - U_h^{n-1}\|_h$. So if we choose

$$\|\gamma\|_0 = p = \frac{1}{\sqrt{3}}\left(\left\|5U_h^n + 4U_h^{n-1} - U_h^{n-2}\right\|_0 + \frac{1}{\sqrt{8}}\left\|4U_h^n - U_h^{n-1}\right\|_h^2\right) + 1$$

such that $(G(\gamma),\gamma) \geq 0$, then by applying the Brouwer fixed-point theorem [31], there exists a $\gamma^\star \in V_h$ such that $G(\gamma^\star) = 0$. Thus, there exists a solution U_h^{n+1} to the scheme (2.5).

Next, the uniqueness should be proven. When $n \geq 2$, we assume U_h^{n+1} and V_h^{n+1} are the two solutions of (2.5). Easily, we have

$$\begin{aligned}
&\big(D_{\tau\tau}(U_h^{n+1}-V_h^{n+1}), v_h\big) + \big(D_\tau \nabla_h (U_h^{n+1}-V_h^{n+1}), \nabla_h v_h\big)_h + \big(\nabla_h U_h^{n+1}-\nabla_h V_h^{n+1}, \nabla_h v_h\big)_h \\
= &-\frac{1}{4}\Big(\Big(\|U_h^{n+1}\|_h^2 + \|2U_h^{n+1} - U_h^n\|_h^2 + \|U_h^n\|_h^2 + \|2U_h^n - U_h^{n-1}\|_h^2\Big) \nabla_h U_h^{n+1} \\
&\quad - \Big(\|V_h^{n+1}\|_h^2 + \|2V_h^{n+1} - V_h^n\|_h^2 + \|V_h^n\|_h^2 + \|2V_h^n - V_h^{n-1}\|_h^2\Big) \nabla_h V_h^{n+1}, \nabla_h v_h\Big)_h \\
&- \frac{\alpha}{4}\Big(\Big(|U_h^{n+1}|^2 + |U_h^n|^2 + |U_h^n - 2U_h^{n+1}|^2 + |U_h^{n-1} - 2U_h^n|^2\Big) U_h^{n+1} \\
&\quad - \Big(|V_h^{n+1}|^2 + |2V_h^{n+1} - V_h^n|^2 + |V_h^n|^2 + |2V_h^n - V_h^{n-1}|^2\Big) V_h^{n+1}, v_h\Big) = 0. \qquad (2.19)
\end{aligned}$$

Setting $v_h = D_\tau(U_h^{n+1} - V_h^{n+1})$ in (2.20), we have

$$\begin{aligned}
&\big(D_{\tau\tau}(U_h^{n+1} - V_h^{n+1}), D_\tau(U_h^{n+1} - V_h^{n+1})\big) + \big\|D_\tau(U_h^{n+1} - V_h^{n+1})\big\|_h^2 \\
&\quad + \big(\nabla_h U_h^{n+1} - \nabla_h V_h^{n+1}, \nabla_h D_\tau(U_h^{n+1} - V_h^{n+1})\big)_h \\
= &-\frac{1}{4}\Big(\Big(\|U_h^{n+1}\|_h^2 + \|2U_h^{n+1} - U_h^n\|_h^2 + \|U_h^n\|_h^2 + \|2U_h^n - U_h^{n-1}\|_h^2\Big) \nabla_h U_h^{n+1} \\
&\quad - \Big(\|V_h^{n+1}\|_h^2 + \|2V_h^{n+1} - V_h^n\|_h^2 + \|V_h^n\|_h^2 \\
&\qquad + \|2V_h^n - V_h^{n-1}\|_h^2\Big)\nabla_h V_h^{n+1}, \nabla_h D_\tau(U_h^{n+1} - V_h^{n+1})\Big)_h \\
&- \frac{\alpha}{4}\Big(\Big(|U_h^{n+1}|^2 + |U_h^n|^2 + |U_h^n - 2U_h^{n+1}|^2 + |U_h^{n-1} - 2U_h^n|^2\Big) U_h^{n+1} \\
&\quad - \Big(|V_h^{n+1}|^2 + |2V_h^{n+1} - V_h^n|^2 + |V_h^n|^2 \\
&\qquad + |2V_h^n - V_h^{n-1}|^2\Big) V_h^{n+1}, D_\tau(U_h^{n+1} - V_h^{n+1})\Big). \qquad (2.20)
\end{aligned}$$

Similar to (2.11) and (2.12), we have

$$\begin{aligned}
&\big(D_{\tau\tau}(U_h^{n+1} - V_h^{n+1}), D_\tau(U_h^{n+1} - V_h^{n+1})\big) \\
\geq &\frac{1}{\tau}\Big(\big(\bar\partial_t(U_h^{n+1} - V_h^{n+1}) - \bar\partial_t(U_h^n - V_h^n), \bar\partial_t(U_h^{n+1} - V_h^{n+1})\big) \\
&\quad - \big(\bar\partial_t(U_h^n - V_h^n) - \bar\partial_t(U_h^{n-1} - V_h^{n-1}), \bar\partial_t(U_h^n - V_h^n)\big) \Big) \\
&\quad + \frac{3}{4\tau}\Big(\|\bar\partial_t(U_h^{n+1} - V_h^{n+1}) - \bar\partial_t(U_h^n - V_h^n)\|_0^2 - \|\bar\partial_t(U_h^n - V_h^n) - \bar\partial_t(U_h^{n-1} - V_h^{n-1})\|_0^2\Big) \\
&\quad + \frac{1}{2\tau}\Big(\|\bar\partial_t(U_h^{n+1} - V_h^{n+1})\|_0^2 - \|\bar\partial_t(U_h^n - V_h^n)\|_0^2\Big), \\
&\big(\nabla_h U_h^{n+1} - \nabla_h V_h^{n+1}, \nabla_h D_\tau(U_h^{n+1} - V_h^{n+1})\big)_h \\
\geq &\frac{1}{4\tau}\Big(\|U_h^{n+1} - V_h^{n+1}\|_h^2 - \|U_h^n - V_h^n\|_h^2 + \|2(U_h^{n+1} - V_h^{n+1}) - (U_h^n - V_h^n)\|_h^2 \\
&\quad - \|2(U_h^n - V_h^n) - (U_h^{n-1} - V_h^{n-1})\|_h^2\Big).
\end{aligned}$$

With the boundedness of $\|U_h^{n+1}\|_h, \|V_h^{n+1}\|_h, \|U_h^n\|_h, \|V_h^n\|_h, \|U_h^{n-1}\|_h, \|V_h^{n-1}\|_h$, we have

$$\begin{aligned}
&-\frac{1}{4}\Big(\Big(\|U_h^{n+1}\|_h^2 + \|2U_h^{n+1} - U_h^n\|_h^2 + \|U_h^n\|_h^2 + \|2U_h^n - U_h^{n-1}\|_h^2\Big) \nabla_h U_h^{n+1} \\
&\quad - \Big(\|V_h^{n+1}\|_h^2 + \|2V_h^{n+1} - V_h^n\|_h^2 + \|V_h^n\|_h^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + \|2V_h^n - V_h^{n-1}\|_h^2\Big)\nabla_h V_h^{n+1}, \nabla_h D_\tau\big(U_h^{n+1} - V_h^{n+1}\big)\Big)_h \\
&= -\frac{1}{4}\Big(\|V_h^{n+1}\|_h^2 + \|2V_h^{n+1} - V_h^n\|_h^2 + \|V_h^n\|_h^2 + \|2V_h^n - V_h^{n-1}\|_h^2\Big) \\
&\quad \times \big(\nabla_h U_h^{n+1} - \nabla_h V_h^{n+1}, \nabla_h D_\tau\big(U_h^{n+1} - V_h^{n+1}\big)\big)_h \\
&\quad - \frac{1}{4}\Big(\Big(\|U_h^{n+1}\|_h^2 + \|2U_h^{n+1} - U_h^n\|_h^2 + \|U_h^n\|_h^2 + \|2U_h^n - U_h^{n-1}\|_h^2\Big) \\
&\qquad - \Big(\|V_h^{n+1}\|_h^2 + \|2V_h^{n+1} - V_h^n\|_h^2 + \|V_h^n\|_h^2 + \|2V_h^n - V_h^{n-1}\|_h^2\Big)\Big) \\
&\quad \times \big(\nabla_h U_h^{n+1}, \nabla_h D_\tau\big(U_h^{n+1} - V_h^{n+1}\big)\big)_h \\
&\le C\|U_h^{n+1} - V_h^{n+1}\|_h^2 + C\|U_h^n - V_h^n\|_h^2 \\
&\quad + C\|U_h^{n-1} - V_h^{n-1}\|_h^2 + \frac{1}{8}\|D_\tau\big(U_h^{n+1} - V_h^{n+1}\big)\|_h^2, \qquad (2.21)
\end{aligned}
$$

where

$$
\begin{aligned}
&\|U_h^{n+1}\|_h^2 - \|V_h^{n+1}\|_h^2 + \|2U_h^{n+1} - U_h^n\|_h^2 - \|2V_h^{n+1} - V_h^n\|_h^2 \\
&\quad + \|U_h^n\|_h^2 - \|V_h^n\|_h^2 + \|2U_h^n - U_h^{n-1}\|_h^2 - \|2V_h^n - V_h^{n-1}\|_h^2 \\
&= \big(\|U_h^{n+1}\|_h + \|V_h^{n+1}\|_h\big)\big(\|U_h^{n+1}\|_h - \|V_h^{n+1}\|_h\big) \\
&\quad + \big(\|2U_h^{n+1} - U_h^n\|_h + \|2V_h^{n+1} - V_h^n\|_h\big)\big(\|2U_h^{n+1} - U_h^n\|_h - \|2V_h^{n+1} - V_h^n\|_h\big) \\
&\quad + \big(\|U_h^n\|_h + \|V_h^n\|_h\big)\big(\|U_h^n\|_h - \|V_h^n\|_h\big) \\
&\quad + \big(\|2U_h^n - U_h^{n-1}\|_h + \|2V_h^n - V_h^{n-1}\|_h\big)\big(\|2U_h^n - U_h^{n-1}\|_h - \|2V_h^n - V_h^{n-1}\|_h\big) \\
&\le C\|U_h^{n+1} - V_h^{n+1}\|_h + C\|U_h^n - V_h^n\|_h + C\|U_h^{n-1} - V_h^{n-1}\|_h. \qquad (2.22)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\Big(\Big(|U_h^{n+1}|^2 + |U_h^n|^2 + |U_h^n - 2U_h^{n+1}|^2 + |U_h^{n-1} - 2U_h^n|^2\Big)U_h^{n+1} \\
&\quad - \Big(|V_h^{n+1}|^2 + |2V_h^{n+1} - V_h^n|^2 + |V_h^n|^2 + |2V_h^n - V_h^{n-1}|^2\Big)V_h^{n+1}, D_\tau\big(U_h^{n+1} - V_h^{n+1}\big)\Big) \\
&= \Big(\Big(|V_h^{n+1}|^2 + |2V_h^{n+1} - V_h^n|^2 + |V_h^n|^2 \\
&\qquad + |2V_h^n - V_h^{n-1}|^2\Big)\big(U_h^{n+1} - V_h^{n+1}\big), D_\tau\big(U_h^{n+1} - V_h^{n+1}\big)\Big) \\
&\quad + \Big(\Big(|U_h^{n+1}|^2 - |V_h^{n+1}|^2 + |U_h^n - 2U_h^{n+1}|^2 - |2V_h^{n+1} - V_h^n|^2 + |U_h^n|^2 - |V_h^n|^2 \\
&\qquad + |U_h^{n-1} - 2U_h^n|^2 - |2V_h^n - V_h^{n-1}|^2\Big)U_h^{n+1}, D_\tau\big(U_h^{n+1} - V_h^{n+1}\big)\Big) \\
&\le C\|V_h^{n-1} - U_h^{n-1}\|_h^2 + C\|V_h^n - U_h^n\|_h^2 + C\|V_h^{n+1} - U_h^{n+1}\|_h^2 \\
&\quad + \frac{1}{8}\|D_\tau\big(V_h^{n+1} - U_h^{n+1}\big)\|_h^2 + C\|D_\tau\big(V_h^{n+1} - U_h^{n+1}\big)\|_0^2, \qquad (2.23)
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\frac{1}{\tau}\Big(\big(\bar\partial_t\big(U_h^{n+1} - V_h^{n+1}\big) - \bar\partial_t\big(U_h^n - V_h^n\big), \bar\partial_t\big(U_h^{n+1} - V_h^{n+1}\big)\big) \\
&\qquad - \big(\bar\partial_t\big(U_h^n - V_h^n\big) - \bar\partial_t\big(U_h^{n-1} - V_h^{n-1}\big), \bar\partial_t\big(U_h^n - V_h^n\big)\big)\Big) \\
&\quad + \frac{3}{4\tau}\Big(\|\bar\partial_t\big(U_h^{n+1} - V_h^{n+1}\big) - \bar\partial_t\big(U_h^n - V_h^n\big)\|_0^2 - \|\bar\partial_t\big(U_h^n - V_h^n\big) - \bar\partial_t\big(U_h^{n-1} - V_h^{n-1}\big)\|_0^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2\tau}\left(\|\bar{\partial}_t U_h^{n+1}\|_0^2-\|\bar{\partial}_t U_h^n\|_0^2\right)+\|D_\tau(U_h^{n+1}-V_h^{n+1})\|_h^2\\
&+\frac{1}{4\tau}\Big(\|U_h^{n+1}-V_h^{n+1}\|_h^2-\|U_h^n-V_h^n\|_h^2\\
&\qquad+\|2(U_h^{n+1}-V_h^{n+1})-(U_h^n-V_h^n)\|_h^2-\|2(U_h^n-V_h^n)-(U_h^{n-1}-V_h^{n-1})\|_h^2\Big)\\
\le{}& C\|V_h^{n+1}-U_h^{n+1}\|_h^2+C\|V_h^n-U_h^n\|_h^2+C\|V_h^{n-1}-U_h^{n-1}\|_h^2\\
&+C\|D_\tau(V_h^{n+1}-U_h^{n+1})\|_0^2+\frac{1}{4}\|D_\tau(V_h^{n+1}-U_h^{n+1})\|_h^2. \qquad (2.24)
\end{aligned}
$$

We sum (2.24) from 2 to n

$$
\begin{aligned}
&\big(\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})-\bar{\partial}_t(U_h^n-V_h^n),\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})\big)\\
&\quad+\frac{3}{4}\|\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})-\bar{\partial}_t(U_h^n-V_h^n)\|_0^2+\frac{1}{2}\|\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})\|_0^2\\
&\quad+\frac{1}{4}\left(\|U_h^{n+1}-V_h^{n+1}\|_h^2+\|2(U_h^{n+1}-V_h^{n+1})-(U_h^n-V_h^n)\|_h^2\right)\\
\le{}&\big(\bar{\partial}_t(U_h^2-V_h^2)-\bar{\partial}_t(U_h^1-V_h^1),\bar{\partial}_t(U_h^2-V_h^2)\big)+\frac{3}{4}\|\bar{\partial}_t(U_h^2-V_h^2)-\bar{\partial}_t(U_h^1-V_h^1)\|_0^2\\
&\quad+\frac{1}{2}\|\bar{\partial}_t(U_h^2-V_h^2)\|_0^2+\frac{1}{4}\|U_h^2-V_h^2\|_h^2+\|2(U_h^2-V_h^2)-(U_h^1-V_h^1)\|_h^2\\
&\quad+C\tau\sum_{i=2}^n\|U_h^i-V_h^i\|_h^2+C\tau\sum_{i=2}^n\|\bar{\partial}_t(U_h^i-V_h^i)\|_0^2\\
\le{}& C\tau\sum_{i=2}^n\|U_h^i-V_h^i\|_h^2+C\tau\sum_{i=2}^n\|\bar{\partial}_t(U_h^i-V_h^i)\|_0^2. \qquad (2.25)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\|\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})\|_0^2+\frac{3}{4}\|\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})-\bar{\partial}_t(U_h^n-V_h^n)\|_0^2+\frac{1}{4}\|U_h^{n+1}-V_h^{n+1}\|_h^2\\
\le{}&-\big(\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})-\bar{\partial}_t(U_h^n-V_h^n),\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})\big)\\
&+C\tau\sum_{i=2}^n\|U_h^i-V_h^i\|_h^2+C\tau\sum_{i=2}^n\|\bar{\partial}_t(U_h^i-V_h^i)\|_0^2\\
\le{}&\frac{3}{4}\|\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})-\bar{\partial}_t(U_h^n-V_h^n)\|_0^2+\frac{1}{3}\|\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})\|_0^2\\
&+C\tau\sum_{i=2}^n\|U_h^i-V_h^i\|_h^2+C\tau\sum_{i=2}^n\|\bar{\partial}_t(U_h^i-V_h^i)\|_0^2, \qquad (2.26)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{1}{6}\|\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})\|_0^2+\frac{1}{4}\|U_h^{n+1}-V_h^{n+1}\|_h^2\\
\le{}& C\tau\sum_{i=2}^n\|U_h^i-V_h^i\|_h^2+C\tau\sum_{i=2}^n\|\bar{\partial}_t(U_h^i-V_h^i)\|_0^2. \qquad (2.27)
\end{aligned}
$$

Thus, by Gronwall inequality, we see that

$$
\|\bar{\partial}_t(U_h^{n+1}-V_h^{n+1})\|_0^2+\|U_h^{n+1}-V_h^{n+1}\|_h^2\le 0, \qquad (2.28)
$$

which implies $U_h^{n+1}=V_h^{n+1}$. The proof is complete. □

3. Superconvergent Result for the Nonconforming FEM

In this section, we estimate superconvergent result by some new technique. To simplify the procedure, we denote

$$\psi^n - U_h^n = \psi^n - I_h\psi^n + I_h\psi^n - U_h^n := \eta^n + \xi^n.$$

Theorem 3.1. *Suppose that* ψ *and* U_h^{n+1} *are the solutions of* (1.1) *and* (2.5)-(2.6)*, respectively, for* $n = 1, \ldots, N-1$*. Then, we have*

$$\|\xi^{n+1}\|_h \leq C(h^2 + \tau^2). \tag{3.1}$$

Proof. The error equation can be deduced by (1.1) and (2.5)

$$\begin{aligned}
&\big(D_{\tau\tau}\xi^{n+1}, v_h\big) + \big(\nabla_h D_\tau \xi^{n+1}, \nabla_h v_h\big)_h \\
= &-\big(D_{\tau\tau}\eta^{n+1}, v_h\big) - \big(\nabla_h D_\tau \eta^{n+1}, \nabla_h v_h\big)_h \\
&+ \Bigg(\phi\Big(\frac{1}{4}\left(\|\psi^{n+1}\|_h^2 + \|2\psi^{n+1} - \psi^n\|_h^2 + \|\psi^n\|_h^2 + \|2\psi^n - \psi^{n-1}\|_h^2\right)\Big)\nabla_h\psi^{n+1} \\
&\quad - \phi\Big(\frac{1}{4}\Big(\big\|U_h^{n+1}\big\|_h^2 + \big\|2U_h^{n+1} - U_h^n\big\|_h^2 + \big\|U_h^n\big\|_h^2 + \big\|2U_h^n - U_h^{n-1}\big\|_h^2\Big)\Big)\nabla_h U_h^{n+1}, \nabla_h v_h\Bigg)_h \\
&+ \frac{\alpha}{4}\Big(\left(|\psi^{n+1}|^2 + |2\psi^{n+1} - \psi^n|^2 + |\psi^n|^2 + |2\psi^n - \psi^{n-1}|^2\right)\psi^{n+1} \\
&\quad - \Big(\big|U_h^{n+1}\big|^2 + \big|2U_h^{n+1} - U_h^n\big|^2 + \big|U_h^n\big|^2 + \big|2U_h^n - U_h^{n-1}\big|^2\Big)U_h^{n+1}, v_h\Big) \\
&+ \big\langle \nabla\psi_t^{n+1}, v_h\big\rangle_h + \big\langle \phi\big(\|\nabla\psi\|_0^2\big)\nabla\psi, v_h\big\rangle_h + \big(R_1^n, v_h\big) \\
&+ \big(\nabla_h R_2^n, \nabla_h v_h\big)_h + \big(R_3^n, v_h\big) + \big(R_4^n, v_h\big),
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
R_1^n &= D_{\tau\tau}\psi^{n+1} - \psi_{tt}^{n+1}, \\
R_2^n &= D_\tau\psi^{n+1} - \psi_t^{n+1}, \\
R_3^n &= \phi\left(\frac{1}{4}\left(\|\psi^{n+1}\|_1^2 + \|2\psi^{n+1} - \psi^n\|_1^2 + \|\psi^n\|_1^2 + \|2\psi^n - \psi^{n-1}\|_1^2\right)\right)\nabla\psi^{n+1} - \phi\big(\|\psi^{n+1}\|_1^2\big)\nabla\psi^{n+1}, \\
R_4^n &= \frac{1}{4}\left(|\psi^{n+1}|^2 + |2\psi^{n+1} - \psi^n|^2 + |\psi^n|^2 + |2\psi^n - \psi^{n-1}|^2\right)\psi^{n+1} - |\psi^{n+1}|^2\psi^{n+1}.
\end{aligned}$$

Setting $v_h = D_\tau\xi^{n+1}$ in (3.2), we have

$$\begin{aligned}
&\big(D_{\tau\tau}\xi^{n+1}, D_\tau\xi^{n+1}\big) + \big(\nabla_h D_\tau\xi^{n+1}, \nabla_h D_\tau\xi^{n+1}\big)_h \\
= &-\big(D_{\tau\tau}\eta^{n+1}, D_\tau\xi^{n+1}\big) - \big(\nabla_h D_\tau\eta^{n+1}, \nabla_h D_\tau\xi^{n+1}\big)_h \\
&+ \Bigg(\phi\Big(\frac{1}{4}\left(\|\psi^{n+1}\|_h^2 + \|2\psi^{n+1} - \psi^n\|_h^2 + \|\psi^n\|_h^2 + \|2\psi^n - \psi^{n-1}\|_h^2\right)\Big)\nabla_h\psi^{n+1} \\
&\quad - \phi\Big(\frac{1}{4}\Big(\big\|U_h^{n+1}\big\|_h^2 + \big\|2U_h^{n+1} - U_h^n\big\|_h^2 + \big\|U_h^n\big\|_h^2 \\
&\qquad + \big\|2U_h^n - U_h^{n-1}\big\|_h^2\Big)\Big)\nabla_h U_h^{n+1}, \nabla_h D_\tau\xi^{n+1}\Bigg)_h \\
&+ \frac{\alpha}{4}\Big(\left(|\psi^{n+1}|^2 + |2\psi^{n+1} - \psi^n|^2 + |\psi^n|^2 + |2\psi^n - \psi^{n-1}|^2\right)\psi^{n+1}
\end{aligned}$$

$$
\begin{aligned}
&-\Big(\big(|U_h^{n+1}|^2+|2U_h^{n+1}-U_h^n|^2+|U_h^n|^2+|2U_h^n-U_h^{n-1}|^2\big)U_h^{n+1},D_\tau\xi^{n+1}\Big)\\
&+\big\langle\nabla\psi_t^{n+1},D_\tau\xi^{n+1}\big\rangle_h+\big\langle\phi(\|\nabla\psi\|_0^2)\nabla\psi,D_\tau\xi^{n+1}\big\rangle_h+\big(R_1^n,D_\tau\xi^{n+1}\big)\\
&+\big(\nabla_hR_2^n,\nabla_hD_\tau\xi^{n+1}\big)_h+\big(R_3^n,D_\tau\xi^{n+1}\big)+\big(R_4^n,D_\tau\xi^{n+1}\big). \qquad (3.3)
\end{aligned}
$$

On the one hand, with the help of (2.2) and (2.3), it follows that

$$
\begin{aligned}
&-\big(D_{\tau\tau}\eta^{n+1},D_\tau\xi^{n+1}\big)\le Ch^2\|\psi_{tt}^{n+1}\|_2\|D_\tau\xi^{n+1}\|_0\le Ch^4+C\|D_\tau\xi^{n+1}\|_0^2,\\
&-\big(\nabla_hD_\tau\eta^{n+1},\nabla_hD_\tau\xi^{n+1}\big)_h=0,\\
&\big\langle\nabla\psi_t^{n+1},D_\tau\xi^{n+1}\big\rangle_h+\big\langle\phi(\|\nabla\psi\|_0^2)\nabla\psi,D_\tau\xi^{n+1}\big\rangle_h\le Ch^4+\frac{1}{8}\|D_\tau\xi^{n+1}\|_h^2, \qquad (3.4)\\
&\big(R_1^n,D_\tau\xi^{n+1}\big)+\big(\nabla_hR_2^n,\nabla_hD_\tau\xi^{n+1}\big)_h+\big(R_3^n,D_\tau\xi^{n+1}\big)+\big(R_4^n,D_\tau\xi^{n+1}\big)\\
&\le C\tau^4+C\|D_\tau\xi^{n+1}\|_0^2+\frac{1}{2}\|D_\tau\xi^{n+1}\|_h^2. \qquad (3.5)
\end{aligned}
$$

On the other hand, splitting the third term on the right-hand side of (3.3), we derive

$$
\begin{aligned}
&\bigg(\phi\Big(\frac{1}{4}(\|\psi^{n+1}\|_h^2+\|2\psi^{n+1}-\psi^n\|_h^2+\|\psi^n\|_h^2+\|2\psi^n-\psi^{n-1}\|_h^2)\Big)\nabla_h\psi^{n+1}\\
&\quad-\phi\Big(\frac{1}{4}\Big(\|U_h^{n+1}\|_h^2+\|2U_h^{n+1}-U_h^n\|_h^2+\|U_h^n\|_h^2\\
&\qquad+\|2U_h^n-U_h^{n-1}\|_h^2\Big)\Big)\nabla_hU_h^{n+1},\nabla_hD_\tau\xi^{n+1}\bigg)_h\\
&=\big(\nabla_h\xi^{n+1},\nabla_hD_\tau\xi^{n+1}\big)_h+\big(\nabla_h\eta^{n+1},\nabla_hD_\tau\xi^{n+1}\big)_h\\
&\quad+\frac{1}{4}\Big((\|\psi^{n+1}\|_h^2+\|2\psi^{n+1}-\psi^n\|_h^2+\|\psi^n\|_h^2+\|2\psi^n-\psi^{n-1}\|_h^2)\nabla_h\psi^{n+1}\\
&\qquad-\Big(\|U_h^{n+1}\|_h^2+\|2U_h^{n+1}-U_h^n\|_h^2+\|U_h^n\|_h^2\\
&\qquad+\|2U_h^n-U_h^{n-1}\|_h^2\Big)\nabla_hU_h^{n+1},\nabla_hD_\tau\xi^{n+1}\Big)_h\\
&\le C\|\xi^{n+1}\|_h^2+\frac{1}{8}\|D_\tau\xi^{n+1}\|_h^2\\
&\quad+\frac{1}{4}\Big(\|\psi^{n+1}\|_h^2+\|2\psi^{n+1}-\psi^n\|_h^2+\|\psi^n\|_h^2+\|2\psi^n-\psi^{n-1}\|_h^2\\
&\qquad-\|U_h^{n+1}\|_h^2-\|2U_h^{n+1}-U_h^n\|_h^2-\|U_h^n\|_h^2-\|2U_h^n-U_h^{n-1}\|_h^2\Big)\\
&\quad\times\big(\nabla_hU_h^{n+1},\nabla_hD_\tau\xi^{n+1}\big)_h. \qquad (3.6)
\end{aligned}
$$

To maintain the second order of spatial error estimate, we rewrite the nonlinear term and use the property (2.2) of the nonconforming element EQ_1^{rot} to have

$$
\begin{aligned}
&(\|\psi^{n+1}\|_h^2+\|2\psi^{n+1}-\psi^n\|_h^2+\|\psi^n\|_h^2+\|2\psi^n-\psi^{n-1}\|_h^2)\\
&\quad-\Big(\|U_h^{n+1}\|_h^2+\|2U_h^{n+1}-U_h^n\|_h^2+\|U_h^n\|_h^2+\|2U_h^n-U_h^{n-1}\|_h^2\Big)\\
&=\sum_K\int_K\Big(\big(\nabla_h\psi^{n+1}\big)^2-\big(\nabla_hU_h^{n+1}\big)^2\Big)\,dX\\
&\quad+\sum_K\int_K\Big(\big(2\nabla_h\psi^{n+1}-\nabla_h\psi^n\big)^2-\big(2\nabla_hU_h^{n+1}-\nabla_hU_h^n\big)^2\Big)\,dX
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_K\int_K \Big(\big(\nabla_h\psi^n\big)^2-\big(\nabla_h U_h^n\big)^2\Big)\,dX\\
&+\sum_K\int_K \Big(\big(2\nabla_h\psi^n-\nabla_h\psi^{n-1}\big)^2-\big(2\nabla_h U_h^n-\nabla_h U_h^{n-1}\big)^2\Big)\,dX\\
=&\sum_K\int_K \nabla_h\psi^{n+1}\big(\nabla_h\xi^{n+1}+\nabla_h\eta^{n+1}\big)dX\\
&+\sum_K\int_K \nabla_h U_h^{n+1}\big(\nabla_h\xi^{n+1}+\nabla_h\eta^{n+1}\big)dX\\
&+\sum_K\int_K \big(2\nabla_h\psi^{n+1}+2\nabla_h U_h^{n+1}-\nabla_h\psi^n-\nabla_h U_h^n\big)\\
&\qquad\times\big(2\nabla_h\xi^{n+1}+2\nabla_h\eta^{n+1}-\nabla_h\xi^n-\nabla_h\eta^n\big)dX\\
&+\sum_K\int_K \nabla_h\psi^n\big(\nabla_h\xi^{n+1}+\nabla_h\eta^n\big)dX\\
&+\sum_K\int_K \nabla_h U_h^n\big(\nabla_h\xi^{n+1}+\nabla_h\eta^n\big)dX\\
&+\sum_K\int_K \big(2\nabla_h\psi^n+2\nabla_h U_h^n-\nabla_h\psi^{n-1}-\nabla_h U_h^{n-1}\big)\\
&\qquad\times\big(2\nabla_h\xi^n+2\nabla_h\eta^n-\nabla_h\xi^{n-1}-\nabla_h\eta^{n-1}\big)dX\\
\le\,& Ch^2+C\|\xi^{n+1}\|_h^2+C\|\xi^n\|_h^2+C\|\xi^{n-1}\|_h^2, \qquad (3.7)
\end{aligned}
$$

where

$$
\begin{aligned}
&\sum_K\int_K \nabla_h\psi^{n+1}\nabla_h\eta^{n+1}dX\\
=&\sum_K\int_K \nabla_h\eta^{n+1}\nabla_h\eta^{n+1}dX+\sum_K\int_K \nabla_h\xi^{n+1}\nabla_h\eta^{n+1}dX\\
&\quad+\sum_K\int_K \nabla_h U_h^{n+1}\nabla_h\eta^{n+1}dX\\
\le\,& Ch^2+C\|\xi^{n+1}\|_h^2, \qquad (3.8)
\end{aligned}
$$

$$
\sum_K\int_K \nabla_h U_h^{n+1}\nabla_h\eta^{n+1}dX=0. \qquad (3.9)
$$

Therefore,

$$
\begin{aligned}
&\bigg(\phi\Big(\frac14\big(\|\psi^{n+1}\|_h^2+\|2\psi^{n+1}-\psi^n\|_h^2+\|\psi^n\|_h^2+\|2\psi^n-\psi^{n-1}\|_h^2\big)\Big)\nabla_h\psi^{n+1}\\
&\quad-\phi\Big(\frac14\Big(\big\|U_h^{n+1}\big\|_h^2+\big\|2U_h^{n+1}-U_h^n\big\|_h^2+\big\|U_h^n\big\|_h^2+\big\|2U_h^n-U_h^{n-1}\big\|_h^2\Big)\Big)\nabla_h U_h^{n+1},\nabla_h D_\tau\xi^{n+1}\bigg)_h\\
\le\,& Ch^4+C\|\xi^{n+1}\|_h^2+C\|\xi^n\|_h^2+C\|\xi^{n-1}\|_h^2+\frac18\big\|D_\tau\xi^{n+1}\big\|_h^2. \qquad (3.10)
\end{aligned}
$$

Similarly, we have

$$
\frac{\alpha}{4}\Big(\big(|\psi^{n+1}|^2+|2\psi^{n+1}-\psi^n|^2+|\psi^n|^2+|2\psi^n-\psi^{n-1}|^2\big)\,\psi^{n+1}
$$

$$
\begin{aligned}
&\quad - \Big(\big|U_h^{n+1}\big|^2 + \big|2U_h^{n+1} - U_h^n\big|^2 + \big|U_h^n\big|^2 + \big|2U_h^n - U_h^{n-1}\big|^2 \Big) U_h^{n+1}, D_\tau \xi^{n+1} \Big) \\
&\le Ch^4 + C\|\xi^{n+1}\|_h^2 + C\|\xi^n\|_h^2 + \frac{1}{8}\big\|D_\tau \xi^{n+1}\big\|_h^2. \qquad (3.11)
\end{aligned}
$$

By the similar procedure of (2.10), it follows that

$$
\begin{aligned}
&\big(D_{\tau\tau}\xi^{n+1}, D_\tau \xi^{n+1}\big) + \big(\nabla_h D_\tau \xi^{n+1}, \nabla_h D_\tau \xi^{n+1}\big)_h \\
\ge\ & \frac{1}{\tau}\big(\bar\partial_t \xi^{n+1} - \bar\partial_t \xi^n, \bar\partial_t \xi^{n+1}\big) - \frac{1}{\tau}\big(\bar\partial_t \xi^n - \bar\partial_t \xi^{n-1}, \bar\partial_t \xi^n\big) \\
&+ \frac{1}{2\tau}\big\|\bar\partial_t \xi^{n+1} - \bar\partial_t \xi^n\big\|_0^2 - \frac{1}{2\tau}\big\|\bar\partial_t \xi^n - \bar\partial_t \xi^{n-1}\big\|_0^2 \\
&+ \frac{1}{2\tau}\Big(\big\|\bar\partial_t \xi^{n+1}\big\|_0^2 - \big\|\bar\partial_t \xi^n\big\|_0^2\Big) \\
&+ \frac{1}{4\tau}\Big(\big\|\bar\partial_t \xi^{n+1} - \bar\partial_t \xi^n\big\|_0^2 - \big\|\bar\partial_t \xi^n - \bar\partial_t \xi^{n-1}\big\|_0^2\Big) + \big\|D_\tau \xi^{n+1}\big\|_h^2. \qquad (3.12)
\end{aligned}
$$

Allocating all the results above implies

$$
\begin{aligned}
&\frac{1}{\tau}\big(\bar\partial_t \xi^{n+1} - \bar\partial_t \xi^n, \bar\partial_t \xi^{n+1}\big) - \frac{1}{\tau}\big(\bar\partial_t \xi^n - \bar\partial_t \xi^{n-1}, \bar\partial_t \xi^n\big) \\
&+ \frac{3}{4\tau}\big\|\bar\partial_t \xi^{n+1} - \bar\partial_t \xi^n\big\|_0^2 - \frac{3}{4\tau}\big\|\bar\partial_t \xi^n - \bar\partial_t \xi^{n-1}\big\|_0^2 \\
&+ \frac{1}{2\tau}\Big(\big\|\bar\partial_t \xi^{n+1}\big\|_0^2 - \big\|\bar\partial_t \xi^n\big\|_0^2\Big) \\
&+ \frac{1}{4\tau}\big(\|\xi^{n+1}\|_h^2 - \|\xi^n\|_h^2 + \|2\xi^{n+1} - \xi^n\|_h^2 - \|2\xi^n - \xi^{n-1}\|_h^2\big) \\
\le\ & Ch^4 + C\tau^4 + C\|\xi^{n+1}\|_h^2 + C\|\xi^n\|_h^2. \qquad (3.13)
\end{aligned}
$$

We sum (3.13) from 2 to n

$$
\begin{aligned}
&\frac{1}{2}\big\|\bar\partial_t \xi^{n+1}\big\|_0^2 + \frac{3}{4}\big\|\bar\partial_t \xi^{n+1} - \bar\partial_t \xi^n\big\|_0^2 + \frac{1}{4}\|\xi^{n+1}\|_h^2 \\
\le\ & Ch^4 + C\tau^4 + C\tau\sum_{i=2}^{N}\|\nabla \xi^i\|_0^2 - \big(\bar\partial_t \xi^{n+1} - \bar\partial_t \xi^n, \bar\partial_t \xi^{n+1}\big) \\
&+ \big(\bar\partial_t \xi^2 - \bar\partial_t \xi^1, \bar\partial_t \xi^2\big) + \frac{3}{4}\big\|\bar\partial_t \xi^2 - \bar\partial_t \xi^1\big\|_0^2 \\
&+ \frac{1}{2}\big\|\bar\partial_t \xi^2\big\|_0^2 + \frac{1}{4}\|\xi^2\|_h^2 + \frac{1}{4\tau}\|2\xi^2 - \xi^1\|_h^2. \qquad (3.14)
\end{aligned}
$$

To derive the estimates of $\|\bar\partial_t \xi^2\|_0$ and $\|\xi^2\|_h$, we have the error equation with the help of (1.1) and (2.6),

$$
\begin{aligned}
&\Big(\frac{\xi^2 - 2\xi^1}{\tau^2}, v_h\Big) + \Big(\frac{\nabla_h \xi^2}{\tau}, \nabla_h v_h\Big)_h \\
&+ \Bigg(\phi\Big(\frac{\|\psi^2\|_h^2 + \|\psi^0\|_h^2}{2}\Big)\frac{\nabla_h \psi^2 + \nabla_h \psi^0}{2} \\
&\qquad - \phi\Big(\frac{\big\|U_h^2\big\|_h^2 + \big\|U_h^0\big\|_h^2}{2}\Big)\frac{\nabla_h U_h^2 + \nabla_h U_h^0}{2}, \nabla_h v_h\Bigg)_h \\
&+ \alpha\Big((\psi^1)^2\frac{\psi^2 + \psi^0}{2} - \big(U_h^1\big)^2\frac{U_h^2 + U_h^0}{2}, v_h\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\Big(\frac{\eta^2-2\eta^1+\eta^0}{\tau^2}, v_h\Big) - \Big(\frac{\nabla_h\eta^2-\nabla_h\eta^0}{\tau}, \nabla_h v_h\Big)_h \\
&\quad + \big\langle \nabla\psi_t^1, v_h\big\rangle_h + \big\langle \phi\big(\|\nabla\psi^1\|_0^2\big)\nabla\psi^1, v_h\big\rangle_h + (S_1, v_h) \\
&\quad + (\nabla_h S_2, \nabla_h v_h)_h + (S_3, v_h) + (S_4, v_h),
\end{aligned}
\tag{3.15}
$$

where

$$
\begin{aligned}
S_1 &= \frac{\psi^2-2\psi^1+\psi^0}{\tau^2} - \psi_{tt}^1, \\
S_2 &= \frac{\psi^2-\psi^0}{\tau} - \psi_t^1, \\
S_3 &= \phi\Big(\frac{\|\psi^2\|_1^2+\|\psi^0\|_1^2}{2}\Big)\frac{\nabla\psi^2+\nabla\psi^0}{2} - \phi\big(\|\psi^1\|_1^2\big)\nabla\psi^1, \\
S_4 &= |\psi^1|^2\frac{\psi^2+\psi^0}{2} - |\psi^1|^2\psi^1.
\end{aligned}
$$

To bound $\|\bar\partial_t\xi^2\|_0$ and $|\xi^2\|_h$, we set $v_h = \xi^2/\tau$ in (3.15). Then we have

$$
\begin{aligned}
&\frac{1}{\tau}\bigg(\Big\|\frac{\xi^2-\xi^1}{\tau}\Big\|_0^2 - \Big\|\frac{\xi^1-\xi^0}{\tau}\Big\|_0^2\bigg) + \Big\|\frac{\xi^2}{\tau}\Big\|_h^2 \\
&= -\bigg(\phi\Big(\frac{\|\psi^2\|_h^2+\|\psi^0\|_h^2}{2}\Big)\frac{\nabla_h\psi^2+\nabla_h\psi^0}{2} \\
&\qquad - \phi\Big(\frac{\big\|U_h^2\big\|_h^2+\big\|U_h^0\big\|_h^2}{2}\Big)\frac{\nabla_h U_h^2+\nabla_h U_h^0}{2}, \frac{\nabla_h\xi^2}{\tau}\bigg)_h \\
&\quad - \alpha\bigg((\psi^1)^2\frac{\psi^2+\psi^0}{2} - \big(U_h^1\big)^2\frac{U_h^2+U_h^0}{2}, \frac{\xi^2}{\tau}\bigg) \\
&\quad - \Big(\frac{\eta^2-2\eta^1+\eta^0}{\tau^2}, \frac{\xi^2}{\tau}\Big) - \Big(\frac{\nabla_h\eta^2-\nabla_h\eta^0}{\tau}, \frac{\nabla_h\xi^2}{\tau}\Big)_h \\
&\quad + \Big\langle \nabla\psi_t^1, \frac{\xi^2}{\tau}\Big\rangle_h + \Big\langle \phi\big(\|\nabla\psi^1\|_0^2\big)\nabla\psi^1, \frac{\xi^2}{\tau}\Big\rangle_h \\
&\quad + \Big(S_1, \frac{\xi^2}{\tau}\Big) + \Big(\nabla_h S_2, \frac{\nabla_h\xi^2}{\tau}\Big)_h + \Big(S_3, \frac{\xi^2}{\tau}\Big) + \Big(S_4, \frac{\xi^2}{\tau}\Big).
\end{aligned}
\tag{3.16}
$$

It is not hard to see that

$$
\begin{aligned}
&- \Big(\frac{\eta^2-2\eta^1+\eta^0}{\tau^2}, \frac{\xi^2}{\tau}\Big) - \Big(\frac{\nabla_h\eta^2-\nabla_h\eta^0}{\tau}, \frac{\nabla_h\xi^2}{\tau}\Big)_h \\
&\quad + \Big\langle \nabla\psi_t^1, \frac{\xi^2}{\tau}\Big\rangle_h + \Big\langle \phi\big(\|\nabla\psi^1\|_0^2\big)\nabla\psi^1, \frac{\xi^2}{\tau}\Big\rangle_h + \Big(S_1, \frac{\xi^2}{\tau}\Big) \\
&\quad + \Big(\nabla_h S_2, \frac{\nabla_h\xi^2}{\tau}\Big)_h + \Big(S_3, \frac{\xi^2}{\tau}\Big) + \Big(S_4, \frac{\xi^2}{\tau}\Big) \\
&\le Ch^4 + C\tau^4 + \frac{1}{8}\Big\|\frac{\xi^2}{\tau}\Big\|_h^2.
\end{aligned}
\tag{3.17}
$$

Similar to the idea in (3.8) and (3.11), we have

$$
- \alpha\bigg((\psi^1)^2\frac{\psi^2+\psi^0}{2} - \big(U_h^1\big)^2\frac{U_h^2+U_h^0}{2}, \frac{\xi^2}{\tau}\bigg)
$$

$$
\begin{aligned}
&= -\alpha\bigg((U_h^1)^2\Big(\frac{\xi^2}{2}+\frac{\eta^2+\eta^0}{2}\Big),\frac{\xi^2}{\tau}\bigg) - \alpha\bigg(\frac{\psi^2+\psi^0}{2}(\psi^1+U_h^1)(\xi^1+\eta^1),\frac{\xi^2}{\tau}\bigg)\\
&\le Ch^4 + C\tau^4 + \frac{1}{8}\|\frac{\xi^2}{\tau}\|_h^2, \qquad (3.18)\\
&\quad -\bigg(\phi\Big(\frac{\|\psi^2\|_h^2+\|\psi^0\|_h^2}{2}\Big)\frac{\nabla_h\psi^2+\nabla_h\psi^0}{2} - \phi\Big(\frac{\|U_h^2\|_h^2+\|U_h^0\|_h^2}{2}\Big)\frac{\nabla_h U_h^2+\nabla_h U_h^0}{2},\frac{\nabla_h\xi^2}{\tau}\bigg)_h\\
&= -\phi\Big(\frac{\|U_h^2\|_h^2+\|U_h^0\|_h^2}{2}\Big)\Big(\frac{\nabla_h\xi^2}{2}+\frac{\nabla_h\eta^2+\nabla_h\eta^0}{2},\frac{\nabla_h\xi^2}{\tau}\Big)_h\\
&\quad -\bigg(\phi\Big(\frac{\|\psi^2\|_h^2+\|\psi^0\|_h^2}{2}\Big) - \phi\Big(\frac{\|U_h^2\|_h^2+\|U_h^0\|_h^2}{2}\Big)\bigg)\Big(\frac{\nabla_h\psi^2+\nabla_h\psi^0}{2},\frac{\nabla_h\xi^2}{\tau}\Big)_h\\
&\le Ch^4 + \frac{1}{8}\|\frac{\xi^2}{\tau}\|_h^2 - \sum_K\int_K(\nabla_h\psi^2+\nabla_h U_h^2)(\nabla_h\xi^2+\nabla_h\eta^2)dX + \sum_K\int_K(\nabla_h\psi^0+\nabla_h U_h^0)\\
&\quad \times\nabla_h\eta^0\Big(\frac{\nabla_h\psi^2+\nabla_h\psi^0}{2},\frac{\nabla_h\xi^2}{\tau}\Big)_h\\
&\le Ch^4 + \frac{1}{8}\Big\|\frac{\xi^2}{\tau}\Big\|_h^2\\
&\quad -\bigg(\sum_K\int_K\big(\nabla_h\eta^2(\nabla_h\xi^2+\nabla_h\eta^2)+(\nabla_h I_h\psi^2+\nabla_h U_h^2)\nabla_h\xi^2\big)\,dX + \sum_K\int_K\nabla_h\eta^0\nabla_h\eta^0 dX\bigg)\\
&\quad \times\Big(\frac{\nabla_h\psi^2+\nabla_h\psi^0}{2},\frac{\nabla_h\xi^2}{\tau}\Big)_h\\
&\le Ch^4 + C\|\xi^2\|_h^2 + \frac{1}{4}\Big\|\frac{\xi^2}{\tau}\Big\|_h^2. \qquad (3.19)
\end{aligned}
$$

Therefore,

$$
\frac{1}{\tau}\bigg(\Big\|\frac{\xi^2-\xi^1}{\tau}\Big\|_0^2 - \Big\|\frac{\xi^1-\xi^0}{\tau}\Big\|_0^2\bigg) + \Big\|\frac{\xi^2}{\tau}\Big\|_h^2 \le Ch^4 + C\tau^4 + C\|\xi^2\|_h^2 + \frac{1}{2}\Big\|\frac{\xi^2}{\tau}\Big\|_h^2, \tag{3.20}
$$

which leads to

$$
\|\bar{\partial}_t\xi^2\|_0 + \|\xi^2\|_h \le Ch^2 + C\tau^2. \tag{3.21}
$$

Inserting (3.21) into (3.14), we have

$$
\begin{aligned}
&\frac{1}{2}\|\bar{\partial}_t\xi^{n+1}\|_0^2 + \frac{3}{4}\|\bar{\partial}_t\xi^{n+1}-\bar{\partial}_t\xi^n\|_0^2 + \frac{1}{4}\|\xi^{n+1}\|_h^2\\
&\le Ch^4 + C\tau^4 + C\tau\sum_{i=2}^N\|\nabla\xi^i\|_0^2 - (\bar{\partial}_t\xi^{n+1}-\bar{\partial}_t\xi^n,\bar{\partial}_t\xi^{n+1})\\
&\le Ch^4 + C\tau^4 + C\tau\sum_{i=2}^N\|\xi^i\|_h^2 + \frac{1}{2}\|\bar{\partial}_t\xi^{n+1}\|_0^2 + \frac{1}{2}\|\bar{\partial}_t\xi^{n+1}-\bar{\partial}_t\xi^n\|_0^2. \qquad (3.22)
\end{aligned}
$$

Then we have

$$
\|\xi^{n+1}\|_h \le Ch^2 + C\tau^2. \tag{3.23}
$$

The proof is complete. □

Remark 3.1. On the one hand, the properties (2.1)-(2.2) of EQ_1^{rot} are of crucial importance to the superconvergent characteristics. In fact, similar results of Theorem 2.2 can be continued to the other nonconforming elements, for instance, Q_1^{rot} element, the constrained nonconforming Q_1^{rot} element and so on, which also have the properties of (2.1)-(2.2). On the other hand, the technique particularly used within this section to conduct a analysis of the superconvergence properties is the utilization of a specialized splitting method as outlined in Eqs. (3.7)-(3.9). Combining with the properties of EQ_1^{rot}, by employing this unique splitting approach, one can keep the order in spatial direction and gain profound insights into how EQ_1^{rot} achieves superconvergence.

Utilizing interpolated postprocessing operator I_{2h}^2 in [24], the global superconvergence is gained immediately.

Theorem 3.2. *Suppose ψ and U_h^{n+1} are the solutions of* (1.1) *and* (2.5)-(2.6)*, respectively, for $n = 1, \dots, N-1$. On the conditions of Theorem* 3.1*, we derive*

$$\left\|\psi^{n+1} - I_{2h}^2 U_h^{n+1}\right\|_h = \mathcal{O}(h^2 + \tau^2). \tag{3.24}$$

Remark 3.2. In fact, the current result can be extended to some element on quadrilateral grids.

On the one hand, for the conforming elements, for example the bilinear Q_{11} element in [34], the authors indicate the superconvergent results (may be $\mathcal{O}(h^2)$ or $\mathcal{O}(h^{3/2})$) for several types of quadrilateral meshes.

(1) On the deformed rectangular mesh. For such mesh, we have the result that

$$\big(\nabla(u - I_h u), \nabla v_h\big) = \mathcal{O}(h^2)\|u\|_3\|v_h\|_1, \quad v_h \in V_{h0}.$$

(2) On the piecewise deformed rectangular mesh. For such mesh, we have the result that

$$\big(\nabla(u - I_h u), \nabla v_h\big) = \mathcal{O}\big(h^{\frac{3}{2}}\big)\|u\|_3\|v_h\|_1, \quad v_h \in V_h.$$

(3) On the majority deformed rectangular mesh. For such mesh, we have the result that

$$\big(\nabla(u - I_h u), \nabla v_h\big) = \mathcal{O}\big(h^{\frac{3}{2}}\big)(\|u\|_3 + \|u\|_{2,\infty})\|v_h\|_1, \quad v_h \in V_h.$$

In the above estimations, we note that

$$\begin{aligned} V_{h0} &= \big\{v \in H^1(\Omega);\ v|_K \in Q_{11}(K),\ \forall K \in \Gamma_h,\ v|_{\partial\Omega} = 0\big\}, \\ V_h &= \big\{v \in H^1(\Omega);\ v|_K \in Q_{11}(K),\ \forall K \in \Gamma_h\big\}, \end{aligned}$$

and I_h the associated interpolation operator over V_h. Thus, the results in our paper also valid for Q_{11} element on the deformed rectangular mesh.

On the other hand, for the nonconforming elements, there are also some literatures considering the quadrilateral meshes. For example, the quasi-Wilson element proposed in [27] is convergent for narrow quadrilateral meshes. And we can also apply this element to our problem to derive the convergent result. As to the EQ_1^{rot} element, the authors in [26] studied anisotropic nonparallel meshes and derived the optimal error estimates with order $\mathcal{O}(h)$ for second order elliptic problems, which can also extend to nonlinear Kirchhoff-type equation with damping in our paper. It is particularly important to emphasize that, the authors in [11] introduced the constrained rotated nonconforming finite element and derived the superconvergence at the center points of each element on a general quadrilateral mesh for the partition satisfying $1+\alpha$ $(1 \geq \alpha > 0)$ condition. We will use the idea of [11] to study the similar properties in our future research.

4. Numerical Examples with Nonconforming FEM

Considering nonlinear Kirchhoff-type equation with damping

$$\begin{cases} \psi_{tt} = \Delta\psi_t + \phi\big(\|\nabla\psi\|_0^2\big)\Delta\psi - \alpha|\psi|^2\psi + h(X,t), & (X,t) \in \Gamma \times (0,T], \\ \psi(X,t) = 0, & (X,t) \in \partial\Gamma \times (0,T], \\ \psi(X,0) = \psi_0(X), \quad \psi_t(X,0) = \psi_1(X), & X \in \bar{\Gamma}, \end{cases} \tag{4.1}$$

where $\Gamma = (0,1) \times (0,1)$, we compute by EQ_1^{rot} with a uniform rectangular partition of $w+1$ nodes in each direction.

Example 4.1. Suppose that $h(X,t)$ is the function that is chosen by the help of the exact solution $\psi = e^t(1-x)x(1-y)y$.

We test the numerical results at $t = 0.4, 0.8, 1.0$ (see the results in Tables 4.1-4.3). When $h \to 0$, we have that $\|\psi^n - U_h^n\|_1$ is convergent at $\mathcal{O}(h)$ and $\|U_h^n - \psi^n\|_0, \|U_h^n - I_h\psi^n\|_1$, $\|\psi^n - I_{2h}^2 U_h^n\|_1$ are convergent at $\mathcal{O}(h^2)$. When $h = 1/80$ and $\tau = h, 2h, 4h, 10h$, results in Table 4.4 shows that the numerical scheme is stable for large time steps.

Table 4.1: Results of numerical Example 4.1 with the time $t = 0.4$.

$w \times w$	$\|U_h^n - \psi^n\|_0$	Order	$\|U_h^n - \psi^n\|_1$	Order	$\|I_h\psi^n - U_h^n\|_1$	Order	$\|\psi^n - I_{2h}^2 U_h^n\|_1$	Order
4×4	3.2204×10^{-3}	—	5.5923×10^{-2}	—	1.0592×10^{-2}	—	1.2714×10^{-1}	—
8×8	7.5597×10^{-4}	2.0908	2.7837×10^{-2}	1.0064	2.9870×10^{-3}	1.8262	3.3604×10^{-2}	1.9197
16×16	1.8284×10^{-4}	2.0477	1.3904×10^{-2}	1.0015	7.8422×10^{-4}	1.9294	8.4864×10^{-3}	1.9854
32×32	4.3729×10^{-5}	2.0639	6.9502×10^{-3}	1.0004	2.0746×10^{-4}	1.9185	2.1266×10^{-3}	1.9966

Table 4.2: Results of numerical Example 4.1 with the time $t = 0.8$.

$w \times w$	$\|U_h^n - \psi^n\|_0$	Order	$\|U_h^n - \psi^n\|_1$	Order	$\|I_h\psi^n - U_h^n\|_1$	Order	$\|\psi^n - I_{2h}^2 U_h^n\|_1$	Order
4×4	4.6017×10^{-3}	—	8.3408×10^{-2}	—	1.6924×10^{-2}	—	1.9035×10^{-1}	—
8×8	1.1365×10^{-3}	2.0176	4.1528×10^{-2}	1.0061	4.4091×10^{-3}	1.9405	5.0139×10^{-2}	1.9247
16×16	2.8321×10^{-4}	2.0046	2.0742×10^{-2}	1.0015	1.1133×10^{-3}	1.9857	1.2659×10^{-2}	1.9857
32×32	7.0721×10^{-5}	2.0017	1.0369×10^{-2}	1.0004	2.7915×10^{-4}	1.9957	3.1721×10^{-3}	1.9967

Table 4.3: Results of numerical Example 4.1 with the time $t = 1.0$.

$w \times w$	$\|U_h^n - \psi^n\|_0$	Order	$\|U_h^n - \psi^n\|_1$	Order	$\|I_h\psi^n - U_h^n\|_1$	Order	$\|\psi^n - I_{2h}^2 U_h^n\|_1$	Order
4×4	5.6002×10^{-3}	—	1.0187×10^{-1}	—	2.0785×10^{-2}	—	2.3256×10^{-1}	—
8×8	1.3891×10^{-3}	2.0113	5.0507×10^{-2}	1.0061	5.3797×10^{-3}	1.9499	6.1239×10^{-2}	1.9251
16×16	3.4647×10^{-4}	2.0034	2.5335×10^{-2}	1.0015	1.3568×10^{-3}	1.9873	1.5462×10^{-2}	1.9857
32×32	8.6576×10^{-5}	2.0007	1.2664×10^{-2}	1.0004	3.3990×10^{-4}	1.9971	3.8743×10^{-3}	1.9967

Table 4.4: Results $\|\psi^n - U_h^n\|_1$ of numerical Example 4.1 with $h = 1/80, \tau = lh$.

t	$l = 1$	$l = 2$	$l = 4$	$l = 10$
0.4	0.005560098308689	0.005560147979484	0.005712361232327	0.019582152800077
0.8	0.008294562025776	0.008294563089814	0.008295189680630	0.017525058873552
1.0	0.010131002687870	0.010131002093787	0.010131024088336	0.010131005225386

Example 4.2. $h(X, t)$ is chosen corresponding to the exact solution $\psi = e^t \sin \pi y \sin \pi x$.

Similar to Example 4.1, the results in Tables 4.5-4.8 are in good agreement with our theoretical analysis.

Table 4.5: Results of numerical Example 4.2 with the time $t = 0.4$.

$w \times w$	$\|U_h^n - \psi^n\|_0$	Order	$\|U_h^n - \psi^n\|_1$	Order	$\|I_h\psi^n - U_h^n\|_1$	Order	$\|\psi^n - I_{2h}^2 U_h^n\|_1$	Order
4×4	3.7526×10^{-2}	—	7.4635×10^{-1}	—	1.6898×10^{-1}	—	1.6888	—
8×8	1.0284×10^{-2}	1.8675	3.7505×10^{-1}	0.9928	3.9834×10^{-2}	2.0848	4.4747×10^{-1}	1.9162
16×16	2.6520×10^{-3}	1.9553	1.8776×10^{-1}	0.9982	9.7087×10^{-3}	2.0366	1.1360×10^{-1}	1.9778
32×32	6.6761×10^{-4}	1.9900	9.3911×10^{-2}	0.9996	2.4138×10^{-3}	2.0080	2.8513×10^{-2}	1.9943

Table 4.6: Results of numerical Example 4.2 with the time $t = 0.8$.

$w \times w$	$\|U_h^n - \psi^n\|_0$	Order	$\|U_h^n - \psi^n\|_1$	Order	$\|I_h\psi^n - U_h^n\|_1$	Order	$\|\psi^n - I_{2h}^2 U_h^n\|_1$	Order
4×4	5.6593×10^{-2}	—	1.1134	—	2.4917×10^{-1}	—	2.5180	—
8×8	1.4199×10^{-2}	1.9948	5.5946×10^{-1}	0.9929	6.4691×10^{-2}	1.9455	6.6820×10^{-1}	1.9139
16×16	3.5591×10^{-3}	1.9962	2.8010×10^{-1}	0.9981	1.6298×10^{-2}	1.9889	1.6953×10^{-1}	1.9787
32×32	8.8773×10^{-4}	2.0033	1.4010×10^{-1}	0.9995	4.0942×10^{-3}	1.9930	4.2540×10^{-2}	1.9947

Table 4.7: Results of numerical Example 4.2 with the time $t = 1.0$.

$w \times w$	$\|U_h^n - \psi^n\|_0$	Order	$\|U_h^n - \psi^n\|_1$	Order	$\|I_h\psi^n - U_h^n\|_1$	Order	$\|\psi^n - I_{2h}^2 U_h^n\|_1$	Order
4×4	6.9094×10^{-2}	—	1.3600	—	3.0447×10^{-1}	—	3.0755	—
8×8	1.7352×10^{-2}	1.9935	6.8332×10^{-1}	0.9929	7.8971×10^{-2}	1.9499	8.1614×10^{-1}	1.9140
16×16	4.3424×10^{-3}	1.9985	3.4211×10^{-1}	0.9981	1.9927×10^{-2}	1.9873	2.0707×10^{-1}	1.9787
32×32	1.0864×10^{-3}	1.9990	1.7112×10^{-1}	0.9995	4.9912×10^{-3}	1.9971	5.1959×10^{-2}	1.9947

Table 4.8: Results $\|\psi^n - U_h^n\|_1$ of numerical Example 4.2 with $h = 1/80, \tau = lh$.

t	$l = 1$	$l = 2$	$l = 4$	$l = 10$
0.4	0.075143580053450	0.075192709882750	0.075628561344724	0.167594793362243
0.8	0.112082601651442	0.112082535692757	0.112082720001293	0.270716665523632
1.0	0.136897969851081	0.136897960491304	0.136897917993048	0.136898210542478

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