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REGULARIZATION METHOD WITH TWO DIFFERENTIAL OPERATORS FOR SIMULTANEOUS INVERSION OF SOURCE TERM AND INITIAL VALUE IN A TIME-FRACTIONAL BLACK-SCHOLES EQUATION*

Shuang Yu
School of Arts and Science, Guangzhou Maritime University, Guangzhou 510700, China;
Guangdong Province Key Laboratory of Computational Science, School of Computer Science and Engineering, Sun Yat-sen University, Guangzhou 510006, China
Email: yushuang_math@163.com
Hongqi Yang[1)]
Guangdong Province Key Laboratory of Computational Science, School of Computer Science and Engineering, Sun Yat-sen University, Guangzhou 510006, China
Email: mcsyhq@mail.sysu.edu.cn

Abstract

This paper is devoted to identifying the source term and initial value simultaneously in a time-fractional Black-Scholes equation, which is an ill-posed problem. The inverse problem is transformed into a system of operator equations, and under certain source conditions, conditional stability is established. We propose a regularization method with two differential operators to solve the problem, error estimates by rules of a priori and a posteriori regularization parameter selection are derived, respectively. Numerical experiments are presented to validate the effectiveness of the proposed regularization method.

Mathematics subject classification: 35R30, 35R11, 65M32, 47A52.
Key words: Regularization method with two differential operators, Simultaneous inversion of source term and initial value, Ill-posed problem, Time-fractional Black-Scholes equation.

1. Introduction

Options, as a significant financial derivative, play a positive role in the stability and development of modern financial trading markets. Option pricing is crucial in the study of options because market volatility and supply-demand dynamics are reflected in prices. The accuracy and reasonableness of option pricing are of great significance to both buyers and sellers in the financial market. In 1973, Black and Scholes [9], and Merton [27] made outstanding contributions to the theory of option pricing by proposing a model

$$\frac{\partial C}{\partial \zeta}+\frac{\sigma^2}{2}s^2\frac{\partial^2 C(s,\zeta)}{\partial s^2}+rs\frac{\partial C(s,\zeta)}{\partial s}-rC(s,\zeta)=0, \quad (s,\zeta)\in[0,\infty)\times[0,T],$$

where $C=C(S,\zeta)$ represents the function of the underlying asset price S and time ζ, σ is the volatility of the underlying asset price, T is the option's expiration date, and r is the risk-free interest rate. The establishment of this model has become a significant milestone in the theory of option pricing. Subsequently, in 1997, the Nobel Prize in Economic Sciences was

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1) Corresponding author

awarded to American economists Merton and Scholes, in recognition of their "new method for pricing derivatives". Many researchers have discussed numerical solution methods for the Black-Scholes equation [4, 7, 10, 13, 18, 25, 34]. Through observation and research of stock markets, it was found that the most fundamental characteristic and basic state of capital markets are stochastic fluctuations. This indicates a deviation between traditional pricing theories and actual stock movements, such as the inability to address issues like significant jumps in stock prices over short time intervals [8].

In recent decades, fractional derivatives have emerged as powerful tools for describing the memory and hereditary properties of materials. This has led to the widespread application [6, 14, 15, 26, 33, 35, 37–40] of models based on fractional derivatives across various fields of engineering and science. Wyss [36] initially derived the fractional Black-Scholes equation by replacing the first-order derivative with a fractional derivative, denoted as α $(0 < \alpha \leq 1)$, and provided analytical solutions for European options. Cartea and del-Castillo-Negrete [11] proposed that specific Levy processes satisfy fractional partial differential equations, leading to models for pricing options with fractional jump diffusion and fractional barrier options. Jumarie [21, 22] utilized the fractional Taylor formula to mitigate the impact of non-zero initial values of functions, thereby deriving the fractional Black-Scholes equation in time domain and the time-space fractional Black-Scholes equation.

Assuming $C(s, \zeta)$ represents the price of a European option, if the price dynamics of options in the financial market follow a fractional transmission system, then C should satisfy the following time-fractional Black-Scholes equation [2]:

$$\frac{\partial^\alpha C(s,\zeta)}{\partial \zeta^\alpha} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 C(s,\zeta)}{\partial s^2} + rs\frac{\partial C(s,\zeta)}{\partial s} - rC(s,\zeta) = 0, \quad (s,\zeta) \in (a,b) \times [0,T],$$

where $0 < \alpha < 1$. Currently, many researchers have discussed numerical solution methods for the time-fractional Black-Scholes equation [1,2,12,23,24,28,31]. An *et al.* [2] proposed a space-time spectral method using Jacobi polynomials for temporal discretization and Fourier-like basis functions for spatial discretization. Chen *et al.* [12] found an explicit closed-form analytical solution for double-barrier options, which had been taken to price the single barrier options and European path-independent options under the same framework as a special case of the current solution. Roul [31] presented a high-order numerical approach utilizing a uniform mesh for efficiently solving the time-fractional Black-Scholes equation governing European options, employing a collocation method with quintic B-spline basis functions for space discretization and a backward Euler method for time-stepping. Ahmad *et al.* [1] proposed a local meshless collocation method by using hybrid Gaussian-cubic radial basis functions with polynomials. Kazmi [24] transformed the equation into an integro-differential form, employing numerical integration for time and central difference formulas for space discretization. Time discretization is performed by linear interpolation with a temporally $\tau^{2-\alpha}$ order accuracy, and the Chebyshev collocation is based on the orthogonal polynomials used for spatial discretization [28]. Kaur and Natesan [23] discretized the fractional time derivative using the classical L_1-scheme and spatial derivatives using the cubic spline method. A few studies have tackled the inverse problem of the time-fractional Black-Scholes equation [3,19,20]. An *et al.* [3] aimed to estimate the parameters by using the real option prices of the S&P 500 index options. Jiang and Xu [20] recovered the implied volatility via additional data, while Iqbal and Wei [19] inverted the implied volatility coefficients using regularization method.

In this paper, we present a regularization method with two differential operators in the context of simultaneous inversion of source term and initial value problem for a time-fractional

Black-Scholes equation, so as to tackle the problem such that source term and initial value belonging to two different spaces. In this context, we consider the following time-fractional Black-Scholes model:

$$\frac{\partial^\alpha C(s,\tau)}{\partial \tau^\alpha}+\frac{\sigma^2}{2}s^2\frac{\partial^2 C(s,\tau)}{\partial s^2}+rs\frac{\partial C(s,\tau)}{\partial s}-rC(s,\tau)=\theta(s), \quad (s,\tau)\in(1,b)\times(0,T) \tag{1.1}$$

subject to the homogeneous boundary conditions

$$C(1,\tau)=C(b,\tau)=0,$$

and the terminal condition

$$C(s,0)=\rho(s), \tag{1.2}$$

where the fractional derivative is defined as

$$\frac{\partial^\alpha C(s,\tau)}{\partial \tau^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_\tau^T \frac{u_\eta(s,\eta)}{(\eta-\tau)^\alpha}d\eta, \quad 0<\alpha<1.$$

Using the transformation $\xi=\ln s, t=T-\tau$ and $u(\xi,t)=C(e^\xi,T-\tau)$, then

$$-\frac{\partial^\alpha C(s,\tau)}{\partial \tau^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial_w u(\xi,w)}{(t-w)^\alpha}dw={}_0^C\mathrm{D}_t^\alpha\, u(\xi,t).$$

Here, ${}_0^C\mathrm{D}_t^\alpha\, u(\xi,t)$ is the Caputo derivative. Thus, the model (1.1) becomes

$$\begin{cases} {}_0^C\mathrm{D}_t^\alpha\, u(\xi,t)=M_1\frac{\partial^2 u(\xi,t)}{\partial \xi^2}+M_2\frac{\partial u(\xi,t)}{\partial \xi}-M_3u(\xi,t)+\theta(e^\xi), & (\xi,t)\in(0,\ln b)\times(0,T),\\ u(0,t)=u(\ln b,t)=0, & t\in(0,T),\\ u(\xi,0)=\rho(e^\xi), & \xi\in(0,\ln b), \end{cases} \tag{1.3}$$

where $M_1=\sigma^2/2, M_2=r-M_1$ and $M_3=r$. By changing

$$u(\xi,t)=u(x,t)e^{-\frac{M_2}{2\sqrt{M_1}}x}, \quad x=\frac{\xi}{\sqrt{M_1}},$$

the system (1.3) is transformed into

$$\begin{cases} {}_0^C\mathrm{D}_t^\alpha\, u(x,t)=-m^2u(x,t)+\frac{\partial^2 u(x,t)}{\partial x^2}+f(x), & (x,t)\in(0,W)\times(0,T),\\ u(0,t)=u(W,t)=0, & t\in(0,T),\\ u(x,0)=\varphi(x), & x\in(0,W), \end{cases} \tag{1.4}$$

where

$$W=\frac{\ln b}{\sqrt{M_1}}, \quad m^2=\frac{M_2^2}{4M_1}+M_3, \quad f(x)=\theta\big(e^{\sqrt{M_1}x}\big)e^{\frac{M_2}{2\sqrt{M_1}}x}, \quad \varphi(x)=\rho\big(\sqrt{M_1}x\big)e^{\frac{M_2}{2\sqrt{M_1}}x}.$$

If source term f and initial value φ are given, then problem (1.4) is the so-called direct problem, which aims to find the distribution of u. Here, we consider the following time-fractional Black-Scholes inverse problem.

(BSIP): Given data $g_i(x):=u(x,T_i),\ i=1,2$ with $0<T_1<T_2<T$, simultaneously find source term f and initial value φ in (1.4).

In practical applications, we have the observed data $g_i^\delta(x) \approx u(x,T_i),\ i=1,2$ instead of the exact data $g_i(x) := u(x,T_i),\ i=1,2$. And the observed data generally contain random noise and satisfy

$$\|g_i - g_i^\delta\| \le \delta, \quad i = 1,2, \tag{1.5}$$

where $\|\cdot\|$ is the L^2-norm and δ is the noise level.

The paper is organized into five sections. In Section 2, we provide some preliminaries needed in the following discussion. In Section 3, we give an analysis of the ill-posedness of the problem **(BSIP)** and derive conditional stability results under an a priori condition. In Section 4, we reformulate the linear inverse problem **(BSIP)** as a system of operator equations, then propose a regularization method with two differential operators to solve the problem. We obtain a priori and a posteriori error estimates of the regularization solutions by using an a priori and an a posteriori parameter selection rule respectively. In Section 5, we present numerical experiments to illustrate the validity and effectiveness of the proposed regularization method.

2. Preliminaries

In this section, we present the definition of the two-parameter Mittag-Leffler function with its properties, which plays a crucial role in fractional calculus.

Definition 2.1. *The Mittag-Leffler function $E_{a,b}(z)$ is defined by*

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}, \quad z \in \mathbb{C},$$

where $a > 0$ and $b \in \mathbb{R}$ are arbitrary constants.

Lemma 2.1 ([30]). *For $0 < \alpha < 1$ and $\eta > 0$, we have $0 < E_{\alpha,1}(-\eta) < 1$. Moreover, $E_{\alpha,1}(-\eta)$ is completely monotonic, that is*

$$(-1)^n \frac{d^n}{d\eta^n} E_{\alpha,1}(-\eta) \ge 0.$$

Lemma 2.2 ([17]). *The Mittag-Leffler function can be equivalent expressed as*

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}.$$

Lemma 2.3. *Let*

$$w(t) = \frac{E_{\alpha,1}\big(-tT_1^\alpha\big)}{E_{\alpha,1}\big(-tT_2^\alpha\big)}, \quad t > 0, \quad T_2 > T_1 > 0,$$

then $w(t)$ is a monotone increasing function.

Proof.

$$w'(t) = \frac{T_2^\alpha E'_{\alpha,1}\big(-T_2^\alpha t\big)E_{\alpha,1}\big(-T_1^\alpha t\big) - T_1^\alpha E'_{\alpha,1}\big(-T_1^\alpha t\big)E_{\alpha,1}\big(-T_2^\alpha t\big)}{E_{\alpha,1}^2\big(-T_2^\alpha t\big)}.$$

By Lemma 2.1 we have $E'_{\alpha,1}(-T_2^\alpha t) > E'_{\alpha,1}(-T_1^\alpha t)$, then we conclude $w'(t) > 0$, that is to say, $w(t)$ is a monotone increasing function. This completes the proof of the lemma. □

Lemma 2.4 ([29]). *Let $\alpha \in (0,1)$. The Mittag-Leffler function has the following asymptotic property:*

$$E_{\alpha,1}(z) = -\frac{1}{z\Gamma(1-\alpha)} + \mathcal{O}\left(\frac{1}{z^2}\right) \quad (\Re z < 0, \quad \Re z \ \to \ -\infty).$$

Lemma 2.5 ([29]). *Assume that $0 < \alpha_0 < \alpha_1 < 1$ and $z \in \boldsymbol{R}$, then there exist constants $C_{1,-}$ and $C_{1,+}$ depending only on α_0, α_1 such that*

$$C_{1,-}\frac{1}{1-z} \le E_{\alpha,1}(z) \le C_{1,+}\frac{1}{1-z}, \quad z \le 0,$$

uniformly hold for all $\alpha \in [\alpha_0, \alpha_1]$.

3. Conditional Stability for the Simultaneous Inverse Problem

In this section, we discuss the ill-posedness of the problem **(BSIP)** and provide a conditional stability theorem under the assumption of an a priori condition.

Let λ_n be the eigenvalues of the elliptic operator $-\partial^2 u/\partial x^2 = -L$ satisfying

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty,$$

and its corresponding eigenfunction is $X_n(x) \in H^2(0,W) \cap H_0^1(0,W)$, that is,

$$-LX_n(x) = \lambda_n X_n(x), \quad 0 < x < W$$

with $X_n(0) = X_n(W) = 0$. Here, all $X_n(x)$, $n = 0,1,2,\ldots$, form an orthonormal basis in $L^2(0,W)$ space.

Since problem (1.4) is a linear problem, based on the superposition principle, solution $u(x,t)$ of problem (1.4) can be written as the sum of the solutions $u_1(x,t)$ and $u_2(x,t)$ of the following two subproblems:

$$\begin{cases} {}_0^C\mathrm{D}_t^\alpha\, u_1(x,t) = -m^2 u_1(x,t) + \dfrac{\partial^2 u_1(x,t)}{\partial x^2} + f(x), & (x,t) \in (0,W)\times(0,T), \\ u_1(0,t) = u_1(W,t) = 0, & t \in (0,T), \\ u_1(x,0) = 0, & x \in (0,W), \end{cases} \tag{3.1}$$

$$\begin{cases} {}_0^C\mathrm{D}_t^\alpha\, u_2(x,t) = -m^2 u_2(x,t) + \dfrac{\partial^2 u_2(x,t)}{\partial x^2}, & (x,t) \in (0,W)\times(0,T), \\ u_2(0,t) = u_2(W,t) = 0, & t \in (0,T), \\ u_2(x,0) = \varphi(x), & x \in (0,W). \end{cases} \tag{3.2}$$

Using the separation of variables method, we can get the solutions of problem (3.1) and problem (3.2) as follows:

$$u_1(x,t) = \sum_{n=1}^{+\infty} \frac{1 - E_{\alpha,1}\big(-(m^2+\lambda_n)t^\alpha\big)}{m^2+\lambda_n} f_n X_n(x),$$

$$u_2(x,t) = \sum_{n=1}^{+\infty} E_{\alpha,1}\big(-(m^2+\lambda_n)t^\alpha\big)\varphi_n X_n(x),$$

respectively, where $\varphi_n = (\varphi, X_n)$ and $f_n = (f, X_n)$. By the terminal conditions $g_i(x) = u(x, T_i)$, $i = 1, 2$, and $u(x,t) = u_1(x,t) + u_2(x,t)$, we obtain

$$g_{1,n} = f_n \frac{1 - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)}{m^2+\lambda_n} + E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)\varphi_n, \tag{3.3}$$

$$g_{2,n} = f_n \frac{1 - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big)}{m^2+\lambda_n} + E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big)\varphi_n, \tag{3.4}$$

where $g_{i,n} = (g_i, X_n),\ i = 1, 2$. From the properties of eigenvalues λ_n and Lemma 2.1, we have $f_n = 0$ and $\varphi_n = 0$ if $g_{i,n} = 0$. This yields the uniqueness of simultaneous inversion of source term and initial value problem. Obviously, the exact solutions of source term and initial value of problem (**BSIP**) are

$$f(x) = \sum_{n=1}^{+\infty} \big(m^2+\lambda_n\big) \frac{E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big)g_{1,n} - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)g_{2,n}}{E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big) - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)} X_n(x), \tag{3.5}$$

$$\varphi(x) = \sum_{n=1}^{+\infty} \frac{\big(1 - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)\big)g_{2,n} - \big(1 - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big)\big)g_{1,n}}{E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big) - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)} X_n(x). \tag{3.6}$$

Remark 3.1 (The Ill-posedness of BSIP). The source term f and initial values ϕ can be equivalently rewritten as

$$f(x) = \sum_{n=1}^{+\infty} a_{1,n}g_{1,n} + a_{2,n}g_{2,n}, \quad \varphi(x) = \sum_{n=1}^{+\infty} a_{3,n}g_{1,n} + a_{4,n}g_{2,n}$$

with

$$a_{1,n} = \frac{\big(m^2+\lambda_n\big)E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big)}{E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big) - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)},$$
$$a_{2,n} = \frac{-\big(m^2+\lambda_n\big)E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)}{E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big) - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)},$$

and by Lemma 2.2, we have that

$$a_{3,n} = \frac{-\big(m^2+\lambda_n\big)T_2^\alpha E_{\alpha,\alpha+1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big)}{E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big) - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)},$$
$$a_{4,n} = \frac{\big(m^2+\lambda_n\big)T_1^\alpha E_{\alpha,\alpha+1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)}{E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big) - E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)}.$$

The instability is apparent: small perturbations of the data g_1 and g_2 can, thanks to the factors

$$a_{i,n} \ \to \ \infty \quad \text{as} \quad n \ \to \ \infty, \quad i = 1, 2, 3, 4,$$

introduce very large changes in the solutions f and φ. Therefore, the inverse problem **(BSIP)** is linear ill-posed.

Additionally, regarding the degree of ill-posedness for the ill-posed problem **(BSIP)**, we observe that both the inverse source problem and the inverse initial value problem exhibit characteristics of being "mildly ill-posed". Specifically, the two subproblems (3.1) and (3.2) correspond to the inverse source problem and the inverse initial value problem for the time-fractional B-S equation, respectively. The inverse source term problem using data y_1 represented by $u_1(x,t)$ at time t_1, i.e., $y_1 = u_1(x,t_1)$, yields the exact solution of the source term as

$$f(x) = \sum_{n=1}^{\infty} \frac{(m^2+\lambda_n)y_{1,n}}{1-E_{\alpha,1}\big(-(m^2+\lambda_n)t_1^{\alpha}\big)} X_n.$$

According to Lemma 2.1, term $y_{1,n}/(1-E_{\alpha,1}(-(m^2+\lambda_n)t_1^{\alpha}))$ is bounded, thus confirming that the inverse source problem is "mildly ill-posed". Similarly, using data $y_2 = u_2(x,t_2)$ to inverse the initial value gives

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{y_{2,n}}{E_{\alpha,1}\big(-(m^2+\lambda_n)t_2^{\alpha}\big)} X_n.$$

By the asymptotic property of the Mittag-Leffler function, as shown in Lemma 2.4, we conclude that the inverse initial value problem is also "mildly ill-posed".

Now, for $p_1, p_2 > 0$, we define two sets

$$\mathscr{D}\big((-L)^{p_i}\big) = \left\{\psi \in L^2(0,W) : \left(\sum_{n=1}^{\infty} \lambda_n^{2p_i} |(\psi, X_n)|^2\right)^{\frac{1}{2}} < \infty\right\}, \quad i=1,2. \tag{3.7}$$

Obviously, $\mathscr{D}((-L)^{p_i})$ is Hilbert spaces with the following norm:

$$\|\psi\|_{\mathscr{D}((-L)^{p_i})} = \left(\sum_{n=1}^{\infty} \lambda_n^{2p_i} |(\psi, X_n)|^2\right)^{\frac{1}{2}}, \quad i=1,2.$$

Next, we can state the following theorem.

Theorem 3.1. *Let $f \in \mathscr{D}((-L)^{p_1})$ and $\varphi \in \mathscr{D}((-L)^{p_2})$. Assume there exists a positive constant E such that*

$$\max\{\|f\|_{\mathscr{D}((-L)^{p_1})}, \|\varphi\|_{\mathscr{D}((-L)^{p_2})}\} \le E. \tag{3.8}$$

Then, we have

$$(a) \quad \|f\| \le C_2 E^{\frac{1}{p_1+1}} \left(\|g_1\| + \left(\frac{T_2}{T_1}\right)^{\alpha} \|g_2\|\right)^{\frac{p_1}{p_1+1}}, \tag{3.9}$$

$$(b) \quad \|\varphi\| \le C_3 E^{\frac{1}{p_2+1}} \left(\|g_2\| + \frac{1}{1-E_{\alpha,1}\big(-(m^2+\lambda_1)T_1^{\alpha}\big)}\|g_1\|\right)^{\frac{p_2}{p_2+1}}, \tag{3.10}$$

where C_2 and C_3 are constants depending on $m, \alpha, T_1, T_2, \lambda_1, p_1, p_2$.

Proof. From (3.5) and Hölder inequality, there is

$$\begin{aligned}\|f\|^2 &= \sum_{n=1}^{+\infty} \left((m^2+\lambda_n)\frac{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^{\alpha}\big)g_{1,n} - E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^{\alpha}\big)g_{2,n}}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^{\alpha}\big) - E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^{\alpha}\big)}\right)^2 \\ &\le \left(\sum_{n=1}^{+\infty}\left(\left(\frac{m^2}{\lambda_n}+1\right)\frac{\lambda_n E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^{\alpha}\big)}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^{\alpha}\big) - E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^{\alpha}\big)}\right)^{2(p_1+1)}\right.\end{aligned}$$

$$
\times \left(g_{1,n} - \frac{E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)} g_{2,n}\right)^2\Bigg)^{\frac{1}{p_1+1}}
$$

$$
\times \left(\sum_{n=1}^{+\infty}\left(g_{1,n} - \frac{E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)} g_{2,n}\right)^2\right)^{\frac{p_1}{p_1+1}}
$$

$$
\leq \left(\sum_{n=1}^{+\infty}\left(\frac{m^2/\lambda_1+1}{1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)/\big(E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)\big)}\right)^{2p_1}\lambda_n^{2p_1}f_n^2\right)^{\frac{1}{p_1+1}}
$$

$$
\times \left(\left(\sum_{n=1}^{+\infty}(g_{1,n})^2\right)^{\frac12} + \left(\sum_{n=1}^{+\infty}\left(\frac{E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)} g_{2,n}\right)^2\right)^{\frac12}\right)^{\frac{2p_1}{p_1+1}}.
$$

From Lemmas 2.1 and 2.3-2.4, we get

$$
\frac{E_{\alpha,1}\big(-(m^2+\lambda_1)T_1^\alpha\big)}{E_{\alpha,1}\big(-(m^2+\lambda_1)T_2^\alpha\big)} \leq \frac{E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)},
$$

$$
\frac{E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)} \leq \lim_{n\to\infty}\frac{E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)} = \left(\frac{T_2}{T_1}\right)^\alpha.
$$

Denote

$$
C_2 := \left(\frac{m^2+\lambda_1}{\lambda_1\big(E_{\alpha,1}\big(-(m^2+\lambda_1)T_1^\alpha\big)/\big(E_{\alpha,1}\big(-(m^2+\lambda_1)T_2^\alpha\big)\big)-1\big)}\right)^{\frac{p_1}{p_1+1}},
$$

by the assumption of the a priori condition (3.8), we have

$$
\|f\|^2 \leq C_2^2\|f\|_{\mathscr{D}((-L)^{p_1})}^{\frac{2}{p_1+1}}\left(\|g_1\| + \left(\frac{T_2}{T_1}\right)^\alpha\|g_2\|\right)^{\frac{2p_1}{p_1+1}}.
$$

Similarly, we get

$$
\|\varphi\|^2 = \sum_{n=1}^{+\infty}\left(\frac{\big(1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)\big)g_{2,n} - \big(1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)\big)g_{1,n}}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big) - E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)}\right)^2
$$

$$
\leq \Bigg(\sum_{n=1}^{+\infty}\left(\frac{1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)}{E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big) - E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)}\right)^{2(p_2+1)}
$$

$$
\times \left(g_{2,n} - \frac{1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)}{1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)} g_{1,n}\right)^2\Bigg)^{\frac{1}{p_2+1}}
$$

$$
\times \left(\sum_{n=1}^{+\infty}\left(g_{2,n} - \frac{1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)}{1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)} g_{1,n}\right)^2\right)^{\frac{p_2}{p_2+1}}
$$

$$
\leq \left(\sum_{n=1}^{+\infty}\left(\frac{1/\big(E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)\big)}{1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)/\big(E_{\alpha,1}\big(-(m^2+\lambda_n)T_2^\alpha\big)\big)}\right)^{2p_2}\varphi_n^2\right)^{\frac{1}{p_2+1}}
$$

$$
\times \left(\left(\sum_{n=1}^{+\infty}(g_{2,n})^2\right)^{\frac12} + \left(\sum_{n=1}^{+\infty}\left(\frac{1}{1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_1^\alpha\big)} g_{1,n}\right)^2\right)^{\frac12}\right)^{\frac{2p_2}{p_2+1}}.
$$

In addition, from Lemma 2.5 and there exists a constant $C > 0$ such that

$$\left(\sum_{n=1}^{+\infty}\left(\frac{1/\big(E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big)\big)}{1-\big(E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_1^\alpha\big)\big)/\big(E_{\alpha,1}\big(-\big(m^2+\lambda_n\big)T_2^\alpha\big)\big)}\right)^{2p_2}\varphi_n^2\right)^{\frac{1}{p_2+1}}$$
$$\leq\left(\sum_{n=1}^{+\infty}\left(\frac{\big(C\lambda_1+m^2+\lambda_1\big)T_2^\alpha}{C_{1,-\lambda_1}\big(\big(E_{\alpha,1}\big(-\big(m^2+\lambda_1\big)T_1^\alpha\big)\big)/\big(E_{\alpha,1}\big(-\big(m^2+\lambda_1\big)T_2^\alpha\big)\big)-1\big)}\right)^{2p_2}\lambda_n^{2p_2}\varphi_n^2\right)^{\frac{1}{p_2+1}}.$$

Denote

$$C_3:=\left(\frac{\big(C\lambda_1+m^2+\lambda_1\big)T_2^\alpha}{C_{1,-\lambda_1}\big(\big(E_{\alpha,1}\big(-\big(m^2+\lambda_1\big)T_1^\alpha\big)\big)/\big(E_{\alpha,1}\big(-\big(m^2+\lambda_1\big)T_2^\alpha\big)\big)-1\big)}\right)^{\frac{p_2}{p_2+1}},$$

we have the estimate

$$\|\varphi\|^2\leq C_3^2\|\varphi\|_{\mathscr{D}((-L)^{p_1})}^{\frac{2}{p_2+1}}\left(\|g_2\|+\frac{1}{1-E_{\alpha,1}\big(-\big(m^2+\lambda_1\big)T_1^\alpha\big)}\|g_1\|\right)^{\frac{2p_2}{p_2+1}}.$$

The proof is complete. □

Remark 3.2. It follows from the Theorem 3.1 that the selection of time points T_1 and T_2 influences the conditional stability of the inverse problem **(BSIP)**, specifically concerning the term T_2/T_1 and the ratio $(E_{\alpha,1}(-(m^2+\lambda_1)T_1^\alpha))/(E_{\alpha,1}(-(m^2+\lambda_1)T_2^\alpha))$. Furthermore, our additional experiments reveal that when T_1 and T_2 are chosen to be very close, it will affect the results of regularization.

4. Regularization Method with Two Differential Operators

In this section, we reformulate the linear inverse problem **(BSIP)** as a system of operator equations. Through the analysis in the previous section, it is evident that the problem is ill-posed. Therefore, we propose a regularization method with two differential operators to solve it. In Theorem 4.1 we derive the normal equations of the regularization solution. Then, under a priori and a posteriori parameter selection rules, we obtain its corresponding error estimates.

For any given source term f and initial value function φ for $i = 1, 2$, we define the following operators as:

$$\begin{aligned} K_i:(f,\varphi)&\mapsto u(x,T_i),\\ K_{1,i}:f&\mapsto u_1(x,T_i),\\ K_{2,i}:\varphi&\mapsto u_2(x,T_i). \end{aligned}$$

From the above notations, the problem (**BSIP**) can be expressed as

$$\begin{cases} K_1(f,\varphi)=g_1,\\ K_2(f,\varphi)=g_2, \end{cases}\quad\text{or equivalently}\quad\begin{cases} K_{1,1}f+K_{2,1}\varphi=g_1,\\ K_{1,2}f+K_{2,2}\varphi=g_2. \end{cases}$$

Denote

$$K=\begin{pmatrix} K_{1,1} & K_{2,1}\\ K_{1,2} & K_{2,2} \end{pmatrix},$$

then the problem (**BSIP**) can be rewritten in matrix form

$$K\begin{pmatrix} f \\ \varphi \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{4.1}$$

where $K : L^2(0,W) \times L^2(0,W) \to L^2(0,W) \times L^2(0,W)$ is a linear compact operator and $L^2(0,W) \times L^2(0,W)$ is a Hilbert space with the following inner product.

Definition 4.1. *For any $(f_1, \varphi_1), (f_2, \varphi_2) \in L^2(0,W) \times L^2(0,W)$, then $\langle \cdot, \cdot \rangle$ defined by*

$$\left\langle \begin{pmatrix} f_1 \\ \varphi_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix} \right\rangle = (f_1, f_2) + (\varphi_1, \varphi_2)$$

is an inner product on $L^2(0,W) \times L^2(0,W)$.

One can easily to see that $L^2(0,W) \times L^2(0,W)$ is a Hilbert space. As a result, the operator equations (4.1) is ill-posed, with

$$K^* = \begin{pmatrix} K^*_{1,1} & K^*_{1,2} \\ K^*_{2,1} & K^*_{2,2} \end{pmatrix}$$

is the adjoint operator of K as shown in Remark 3.2.

Remark 4.1. In fact, from Definition 4.1, we have

$$\left\langle K\begin{pmatrix} f_1 \\ \varphi_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix} \right\rangle = (K_{1,1} f_1 + K_{2,1}\varphi_1, f_2) + (K_{1,2} f_1 + K_{2,2}\varphi_1, \varphi_2).$$

On the other hand, let $K^*_{i,j},\ i,j = 1,2$ be the self-adjoint operator of $K_{i,j}$, then

$$\begin{aligned} &(K_{1,1} f_1 + K_{2,1}\varphi_1, f_2) + (K_{1,2} f_1 + K_{2,2}\varphi_1, \varphi_2) \\ =& (f_1, K^*_{1,1} f_2 + K^*_{1,2}\varphi_2) + (\varphi_1, K^*_{2,1} f_2 + K^*_{2,2}\varphi_2) \\ =& \left\langle \begin{pmatrix} f_1 \\ \varphi_1 \end{pmatrix}, \begin{pmatrix} K^*_{1,1} f_2 + K^*_{1,2}\varphi_2 \\ K^*_{2,1} f_2 + K^*_{2,2}\varphi_2 \end{pmatrix} \right\rangle. \end{aligned}$$

So, the adjoint operator

$$K^* = \begin{pmatrix} K^*_{1,1} & K^*_{1,2} \\ K^*_{2,1} & K^*_{2,2} \end{pmatrix}$$

of K is a linear compact operator from $L^2(0,W) \times L^2(0,W)$ to $L^2(0,W) \times L^2(0,W)$ satisfying

$$\left\langle K\begin{pmatrix} f_1 \\ \varphi_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} f_1 \\ \varphi_1 \end{pmatrix}, K^*\begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix} \right\rangle.$$

In order to reconstruct source term f and initial value φ stably from the observed data g_i^δ, $i = 1,2$, we propose a regularization method with two differential operators to solve the problem, that is, to solve the following optimization problem:

$$\min J_\gamma(f,\varphi), \quad J_\gamma(f,\varphi) := \left\|K_1(f,\varphi) - g_1^\delta\right\|^2 + \left\|K_2(f,\varphi) - g_2^\delta\right\|^2 + \gamma\|D_1 f\|^2 + \gamma\|D_2\varphi\|^2, \tag{4.2}$$

where $\gamma > 0$ is regularization parameter, D_1 and D_2 are derivative operators defined as

$$D_1 f = f^{(m_1)}, \quad D_2\varphi = \varphi^{(m_2)}, \quad m_1, m_2 \in \mathbb{N}_+ := \{1,2,3,\ldots\} \tag{4.3}$$

with

$$\mathscr{D}(D_i) = \{x \in L^2[0,W] \,|\, x \text{ absolutely continuous, } x^{(m_i)} \in L^2[0,W]\}, \quad i = 1,2.$$

We know that the adjoint operator $D_i^*,\ i = 1,2$ is also a differential operator in $L^2[0,W]$, which is determined by

$$\begin{aligned}\mathscr{D}(D_i^*) = \{x \in L^2[0,W] \,|\, & x \text{ absolutely continuous, } x^{(m_i)} \in L^2[0,W], \\ & x^{(l)}(0) = x^{(l)}(W),\ l = 1,2,\dots,m_i - 1\},\end{aligned}$$

and $D_i^* x = (-1)^{m_i} x^{(m_i)}$. Also, the product operator $D_i^* D_i$ given by $D_i^* D_i x = (-1)^{m_i} x^{(2m_i)}$ with domain

$$\begin{aligned}\mathscr{D}(D_i^* D_i) &= \{x \in \mathscr{D}(D_i) \,|\, D_i x \in \mathscr{D}(D_i^*)\} \\ &= \big\{x \in \mathscr{D}(D_i) \,|\, x^{(m_i)} \text{ absolutely continuous, } x^{(2m_i)} \in L^2[0,W], \\ &\qquad x^{(l)}(0) = x^{(l)}(W),\ l = m_i, m_i+1, \dots, 2m_i - 1\big\}.\end{aligned}$$

Theorem 4.1. *The functional $J_\gamma(\cdot)$ define by (4.2) has a minimum at $(f_\gamma^\delta, \varphi_\gamma^\delta)$ in $\mathscr{D}(D_1) \times \mathscr{D}(D_2)$, and $(f_\gamma^\delta, \varphi_\gamma^\delta) \in \mathscr{D}(D_1^* D_1) \times \mathscr{D}(D_2^* D_2)$ satisfying the following normal equation:*

$$\left(K^*K + \begin{pmatrix} \gamma D_1^* D_1 & 0 \\ 0 & \gamma D_2^* D_2 \end{pmatrix}\right) \begin{pmatrix} f_\gamma^\delta \\ \varphi_\gamma^\delta \end{pmatrix} = K^* \begin{pmatrix} g_1^\delta \\ g_2^\delta \end{pmatrix}, \tag{4.4}$$

where $K^* : L^2(0,W) \times L^2(0,W) \to L^2(0,W) \times L^2(0,W)$ is the adjoint operator of K.

Proof. Taking any elements $\omega = (\omega_1, \omega_2)$ and $x = (f, \varphi) \in \mathscr{D}(D_1) \times \mathscr{D}(D_2)$ to form the quadratic polynomials

$$\phi(t) = J_\gamma(x + t\omega).$$

If $J_\gamma(\cdot)$ achieves a minimum at $(f_\gamma^\delta\ \varphi_\gamma^\delta)$, then ϕ has a minimum value at $t = 0$, and hence,

$$\begin{aligned}0 = \phi'(0) = &\big(K_{1,1} f_\gamma^\delta + K_{2,1}\varphi_\gamma^\delta - g_1^\delta, K_{1,1}\omega_1\big) + \big(K_{1,2} f_\gamma^\delta + K_{2,2}\varphi_\gamma^\delta - g_2^\delta, K_{1,2}\omega_1\big) \\ &+ \gamma\big(D_1 f_\gamma^\delta, D_1\omega_1\big) + \big(K_{1,1} f_\gamma^\delta + K_{2,1}\varphi_\gamma^\delta - g_1^\delta, K_{2,1}\omega_2\big) \\ &+ \big(K_{1,2} f_\gamma^\delta + K_{2,2}\varphi_\gamma^\delta - g_2^\delta, K_{2,2}\omega_2\big) + \gamma\big(D_2\varphi_\gamma^\delta, D_2\omega_2\big),\end{aligned}$$

or

$$\begin{aligned}0 = &\big(K_{1,1}^*\big(K_{1,1} f_\gamma^\delta + K_{2,1}\varphi_\gamma^\delta - g_1^\delta\big) + K_{1,2}^*\big(K_{1,2} f_\gamma^\delta + K_{2,2}\varphi_\gamma^\delta - g_2^\delta\big) + \gamma D_1^* D_1 f_\gamma^\delta, \omega_1\big) \\ &+ \big(K_{2,1}^*\big(K_{1,1} f_\gamma^\delta + K_{2,1}\varphi_\gamma^\delta - g_1^\delta\big) + K_{2,2}^*\big(K_{1,2} f_\gamma^\delta + K_{2,2}\varphi_\gamma^\delta - g_2^\delta\big) + \gamma D_2^* D_2 \varphi_\gamma^\delta, \omega_2\big).\end{aligned}$$

Since the above equality holds for all $\omega_1 \in \mathscr{D}(D_1)$ and $\omega_2 \in \mathscr{D}(D_2)$, we conclude that $D_1 f_\gamma^\delta \in \mathscr{D}(D_1^*)$ and $D_2\varphi_\gamma^\delta \in \mathscr{D}(D_2^*)$, so that $(f_\gamma^\delta\ \varphi_\gamma^\delta) \in \mathscr{D}(D_1^* D_1) \times \mathscr{D}(D_2^* D_2)$. Therefore, we get

$$\begin{pmatrix} K_{1,1}^* K_{1,1} + K_{1,2}^* K_{1,2} + \gamma D_1^* D_1 & K_{1,1}^* K_{2,1} + K_{1,2}^* K_{2,2} \\ K_{2,1}^* K_{1,1} + K_{2,2}^* K_{1,2} & K_{2,1}^* K_{2,1} + K_{2,2}^* K_{2,2} + \gamma D_2^* D_2 \end{pmatrix} \begin{pmatrix} f_\gamma^\delta \\ \varphi_\gamma^\delta \end{pmatrix} = \begin{pmatrix} K_{1,1}^* g_1^\delta + K_{1,2}^* g_2^\delta \\ K_{2,1}^* g_1^\delta + K_{2,2}^* g_2^\delta \end{pmatrix}.$$

The above equation is also equivalent to the following equation:

$$\left(K^*K + \begin{pmatrix} \gamma D_1^* D_1 & 0 \\ 0 & \gamma D_2^* D_2 \end{pmatrix}\right) \begin{pmatrix} f_\gamma^\delta \\ \varphi_\gamma^\delta \end{pmatrix} = K^* \begin{pmatrix} g_1^\delta \\ g_2^\delta \end{pmatrix}.$$

So the theorem is proved. □

In Theorem 4.1, we obtain the normal equation system of the regularization functional. Consequently, the regularization solutions f_γ and φ_γ would satisfy the following equation:

$$\left(K^*K+\begin{pmatrix}\gamma D_1^*D_1 & 0\\ 0 & \gamma D_2^*D_2\end{pmatrix}\right)\begin{pmatrix}f_\gamma\\ \varphi_\gamma\end{pmatrix}=K^*\begin{pmatrix}g_1\\ g_2\end{pmatrix}.$$

To simplify notations, let

$$l_{1,n}^i=\frac{1-E_{\alpha,1}\big(-(m^2+\lambda_n)T_i^\alpha\big)}{m^2+\lambda_n},\quad l_{2,n}^i=E_{\alpha,1}\big(-(m^2+\lambda_n)T_i^\alpha\big),\quad i=1,2. \tag{4.5}$$

Using singular value decomposition [5] to solve the normal equation (4.4), we get the regularization solutions

$$f_\gamma^\delta(x)=\sum_{n=1}^{+\infty}\frac{\big(\gamma\lambda_n^{m_2}l_{1,n}^1+Ml_{2,n}^2\big)g_{1,n}^\delta+\big(\gamma\lambda_n^{m_2}l_{1,n}^2-Ml_{2,n}^1\big)g_{2,n}^\delta}{M^2+\gamma\lambda_n^{m_1}\big(\big(l_{2,n}^1\big)^2+\big(l_{2,n}^2\big)^2\big)+\gamma\lambda_n^{m_2}\big(\big(l_{1,n}^1\big)^2+(l_{1,n}^2)^2\big)+\gamma^2\lambda_n^{m_1+m_2}}X_n(x), \tag{4.6}$$

$$\varphi_\gamma^\delta(x)=\sum_{n=1}^{+\infty}\frac{\big(\gamma\lambda_n^{m_1}l_{2,n}^1-Ml_{1,n}^2\big)g_{1,n}^\delta+\big(\gamma\lambda_n^{m_1}l_{2,n}^2+Ml_{1,n}^1\big)g_{2,n}^\delta}{M^2+\gamma\lambda_n^{m_1}\big(\big(l_{2,n}^1\big)^2+\big(l_{2,n}^2\big)^2\big)+\gamma\lambda_n^{m_2}\big(\big(l_{1,n}^1\big)^2+\big(l_{1,n}^2\big)^2\big)+\gamma^2\lambda_n^{m_1+m_2}}X_n(x), \tag{4.7}$$

where $M=l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1$. Next, we present two lemmas, which will be needed in the following discussion.

Lemma 4.1. *Let $l_{1,n}^i$ and $l_{2,n}^i$ are defined as (4.5), then there exists an positive integer N and a positive constant C_4, for all $n>N$, we have that*

$$\frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{\big(l_{1,n}^2\big)^2+\big(l_{2,n}^2\big)^2}\ge\frac{C_4}{\lambda_n^2},\quad \frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{\big(l_{1,n}^1\big)^2+\big(l_{2,n}^1\big)^2}\ge\frac{C_4}{\lambda_n^2},\quad \frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{\big(l_{1,n}^1\big)^2+\big(l_{1,n}^2\big)^2}\ge\frac{C_4}{\lambda_n^2},$$

$$\frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{\big(l_{2,n}^1\big)^2+\big(l_{2,n}^2\big)^2}\ge\frac{C_4}{\lambda_n^2},\quad \frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{l_{1,n}^1l_{1,n}^2+l_{2,n}^1l_{2,n}^2}\ge\frac{C_4}{\lambda_n^2},\quad \frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{l_{1,n}^1l_{2,n}^1+l_{1,n}^2l_{2,n}^2}\ge\frac{C_4}{\lambda_n^2},$$

where C_4 depends on parameters T_1,T_2 and α.

Proof. From Lemma 2.4, we get

$$\lim_{n\to\infty}\lambda_n^2\frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{\big(l_{1,n}^2\big)^2+\big(l_{2,n}^2\big)^2}=\frac{\big(T_2^\alpha-T_1^\alpha\big)^2}{T_1^{2\alpha}\big(1+\Gamma^2(1-\alpha)T_2^{2\alpha}\big)}, \tag{4.8}$$

$$\lim_{n\to\infty}\lambda_n^2\frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{\big(l_{1,n}^1\big)^2+\big(l_{2,n}^1\big)^2}=\frac{\big(T_2^\alpha-T_1^\alpha\big)^2}{T_2^{2\alpha}\big(1+\Gamma^2(1-\alpha)T_1^{2\alpha}\big)}, \tag{4.9}$$

$$\lim_{n\to\infty}\lambda_n^2\frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{\big(l_{1,n}^1\big)^2+\big(l_{1,n}^2\big)^2}=\frac{1}{\Gamma^2(1-\alpha)}\left(\frac{1}{T_2^\alpha}-\frac{1}{T_1^\alpha}\right)^2,$$

$$\lim_{n\to\infty}\lambda_n^2\frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{\big(l_{2,n}^1\big)^2+\big(l_{2,n}^2\big)^2}=\frac{\big(T_1^\alpha-T_2^\alpha\big)^2}{T_1^{2\alpha}+T_2^{2\alpha}},$$

$$\lim_{n\to\infty}\lambda_n^2\frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{l_{1,n}^1l_{1,n}^2+l_{2,n}^1l_{2,n}^2}=\frac{\big(T_1^\alpha-T_2^\alpha\big)^2}{T_1^\alpha T_2^\alpha\big(1+\Gamma^2(1-\alpha)T_1^\alpha T_2^\alpha\big)}, \tag{4.10}$$

$$\lim_{n\to\infty}\lambda_n^2\frac{\big(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\big)^2}{l_{1,n}^1l_{2,n}^1+l_{1,n}^2l_{2,n}^2}=\frac{\big(T_1^\alpha-T_2^\alpha\big)^2}{\Gamma(1-\alpha)T_1^\alpha T_2^\alpha\big(T_1^\alpha+T_2^\alpha\big)}.$$

Therefore, take the minimum value of the above limits to be C, then for any $C_4 \in (0, C)$, there exists an integer $N > 0$ such that the conclusion holds when $n > N$. The lemma is proved. □

Lemma 4.2. *Let $l_{1,n}^i$ and $l_{2,n}^i$ are defined as* (4.5)*, then there exists an positive integer N' and a positive constant C_5, for all $n > N'$, the following inequalities hold:*

$$\frac{\left(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\right)^2}{\lambda_n^{m_1}\left(l_{2,n}^2\right)^2 + \lambda_n^{m_2}\left(l_{1,n}^2\right)^2} \geq \frac{C_5}{\lambda_n^{\max\{m_1,m_2\}+2}},$$
$$\frac{\left(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\right)^2}{\lambda_n^{m_1}\left(l_{2,n}^1\right)^2 + \lambda_n^{m_2}\left(l_{1,n}^1\right)^2} \geq \frac{C_5}{\lambda_n^{\max\{m_1,m_2\}+2}},$$
$$\frac{\left(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\right)^2}{\lambda_n^{m_1} l_{2,n}^1 l_{2,n}^2 + \lambda_n^{m_2} l_{1,n}^1 l_{1,n}^2} \geq \frac{C_5}{\lambda_n^{\max\{m_1,m_2\}+2}},$$

where C_5 depends on parameters T_1, T_2 and α.

Proof. Based on Eq. (4.8) in Lemma 4.1. If $m_1 < m_2$, i.e. $\max\{m_1, m_2\} = m_2$, then

$$\lim_{n\to\infty} \lambda_n^{2+m_2} \frac{\left(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\right)^2}{\lambda_n^{m_1}\left(l_{2,n}^2\right)^2 + \lambda_n^{m_2}\left(l_{1,n}^2\right)^2} = \frac{\left(T_2^\alpha - T_1^\alpha\right)^2}{T_1^{2\alpha} T_2^{2\alpha} \Gamma^2(1-\alpha)}.$$

Conversely, if $\max\{m_1, m_2\} = m_1$, then

$$\lim_{n\to\infty} \lambda_n^{2+m_1} \frac{\left(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\right)^2}{\lambda_n^{m_1}\left(l_{2,n}^2\right)^2 + \lambda_n^{m_2}\left(l_{1,n}^2\right)^2} = \frac{\left(T_2^\alpha - T_1^\alpha\right)^2}{T_1^{2\alpha}}.$$

Similarly, based on (4.9) and (4.10), the limits

$$\lim_{n\to\infty} \lambda_n^{\max\{m_1,m_2\}+2} \frac{\left(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\right)^2}{\lambda_n^{m_1}\left(l_{2,n}^1\right)^2 + \lambda_n^{m_2}\left(l_{1,n}^1\right)^2},$$
$$\lim_{n\to\infty} \lambda_n^{\max\{m_1,m_2\}+2} \frac{\left(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\right)^2}{\lambda_n^{m_1} l_{2,n}^1 l_{2,n}^2 + \lambda_n^{m_2} l_{1,n}^1 l_{1,n}^2}$$

exist. We can still select the minimum value of the aforementioned limits as C. Therefore, for any $C_5 \in (0, C)$, there exists $N' > 0$ such that the conclusion holds when $n > N'$. The lemma is proved. □

Assumption 4.1. For $i = 1, 2$, suppose p_i defined in (3.7) and m_i defined in (4.3) satisfying the following constraint condition:

$$p_i \geq \max\{m_1, m_2\}. \tag{4.11}$$

Theorem 4.2. *Suppose the noise assumption* (1.5)*, a priori condition* (3.8) *and the constraint condition* (4.11) *hold, then we have the following error estimates.*

(a) If $0 < p_i - \max\{m_1, m_2\} < 2$, choosing γ as

$$\gamma = \begin{cases} \delta^{\frac{2(m_1+2)}{2\min\{p_1,p_2\}+3m_1-2m_2+2}}, & m_1 < m_2, \\ \delta^{\frac{2(m_2+2)}{2\min\{p_1,p_2\}+3m_2-2m_1+2}}, & m_1 > m_2, \end{cases}$$

then

$$\left\|f_\gamma^\delta - f\right\| + \left\|\varphi_\gamma^\delta - \varphi\right\| \le \begin{cases} C\delta^{\frac{2(\min\{p_1,p_2\}+m_1-m_2)}{2\min\{p_1,p_2\}+3m_1-2m_2+2}}, & m_1 < m_2, \\ C\delta^{\frac{2(\min\{p_1,p_2\}+m_2-m_1)}{2\min\{p_1,p_2\}+3m_2-2m_1+2}}, & m_1 > m_2, \end{cases}$$

(b) If $p_i - \max\{m_1, m_2\} \ge 2$*, and choosing* $\gamma = \delta^{2/3}$*, then we have*

$$\left\|f_\gamma^\delta - f\right\| + \left\|\varphi_\gamma^\delta - \varphi\right\| \le C\delta^{\frac{2}{3}},$$

where C *depends on* $p_i, m_i, T_i\ (i = 1, 2), \alpha, m, \lambda_1$.

Proof. By triangular inequality, we have

$$\left\|f_\gamma^\delta - f\right\| \le \left\|f_\gamma^\delta - f_\gamma\right\| + \|f_\gamma - f\|, \tag{4.12}$$
$$\left\|\varphi_\gamma^\delta - \varphi\right\| \le \left\|\varphi_\gamma^\delta - \varphi_\gamma\right\| + \|\varphi_\gamma - \varphi\|. \tag{4.13}$$

We first estimate $\|f_\gamma^\delta - f\|$. For the first term on the right-hand side of the inequality (4.12), there exists constants $C_6 > 0$ and $C_7 > 0$ such that

$$\begin{aligned}
&\left\|f_\gamma^\delta - f_\gamma\right\| \\
\le& \left\| \sum_{n=1}^{+\infty} \frac{\left(\gamma\lambda_n^{m_2} l_{1,n}^1 + M l_{2,n}^2\right)\left(g_{1,n}^\delta - g_{1,n}\right)}{M^2 + \gamma\lambda_n^{m_1}\left(\left(l_{2,n}^1\right)^2 + (l_{2,n}^2)^2\right) + \gamma\lambda_n^{m_2}\left(\left(l_{1,n}^1\right)^2 + \left(l_{1,n}^2\right)^2\right) + \gamma^2\lambda_n^{m_1+m_2}} X_n \right\| \\
&+ \left\| \sum_{n=1}^{+\infty} \frac{\left(\gamma\lambda_n^{m_2} l_{1,n}^2 - M l_{2,n}^1\right)\left(g_{2,n}^\delta - g_{2,n}\right)}{M^2 + \gamma\lambda_n^{m_1}\left(\left(l_{2,n}^1\right)^2 + \left(l_{2,n}^2\right)^2\right) + \gamma\lambda_n^{m_2}\left(\left(l_{1,n}^1\right)^2 + \left(l_{1,n}^2\right)^2\right) + \gamma^2\lambda_n^{m_1+m_2}} X_n \right\| \\
\le& \left\| \sum_{n=1}^{+\infty} \left(\frac{\gamma\lambda_n^{m_2} l_{1,n}^1}{\gamma\lambda_n^{m_2}\left(l_{1,n}^1\right)^2 + \gamma^2\lambda_n^{m_1+m_2}} + \frac{M l_{2,n}^2}{\gamma\lambda_n^{m_1}\left(l_{2,n}^2\right)^2 + M^2} \right) \left(g_{1,n}^\delta - g_{1,n}\right) X_n \right\| \\
&+ \left\| \sum_{n=1}^{+\infty} \left(\frac{\gamma\lambda_n^{m_2} l_{1,n}^2}{\gamma\lambda_n^{m_2}\left(l_{1,n}^2\right)^2 + \gamma^2\lambda_n^{m_1+m_2}} + \frac{M l_{2,n}^1}{\gamma\lambda_n^{m_1}\left(l_{2,n}^1\right)^2 + M^2} \right) \left(g_{2,n}^\delta - g_{2,n}\right) X_n \right\| \\
\le& \left(\sup_n \left(\frac{\lambda_n}{C_6 + \gamma\lambda_n^{m_1+2}} \right) + \sup_n \left(\frac{1}{2\sqrt{\gamma}\lambda_n^{m_1/2}} \right) \right) \left\| \sum_{n=1}^{\infty} \left(g_{1,n}^\delta - g_{1,n}\right) X_n \right\| \\
&+ \left(\sup_n \left(\frac{\lambda_n}{C_7 + \gamma\lambda_n^{m_1+2}} \right) + \sup_n \left(\frac{1}{2\sqrt{\gamma}\lambda_n^{m_1/2}} \right) \right) \left\| \sum_{n=1}^{\infty} \left(g_{2,n}^\delta - g_{2,n}\right) X_n \right\|,
\end{aligned}$$

where $M = l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1$. By straightforward calculation, we obtain the inequality

$$\begin{aligned}
&\sup_n \left(\frac{\lambda_n}{C_i + \gamma\lambda_n^{m_1+2}} \right) \le \frac{(1+m_1)^{(1+m_1)/(2+m_1)}}{(2+m_1) C_i^{(1+m_1)/(2+m_1)}} \gamma^{-\frac{1}{m_1+2}}, \quad i = 6, 7, \\
&\sup_n \left(\frac{1}{2\sqrt{\gamma}\lambda_n^{m_1/2}} \right) \le \frac{1}{2\lambda_1^{m_1/2}} \gamma^{-\frac{1}{2}}.
\end{aligned}$$

Therefore, there exists positive constant C_8 and by the noise assumption (1.5) yields

$$\left\|f_\gamma^\delta - f_\gamma\right\| \le C_8 \frac{\delta}{\sqrt{\gamma}}. \tag{4.14}$$

Similarly,

$$
\begin{aligned}
&\left\|\varphi_\gamma^\delta-\varphi_\gamma\right\| \\
&=\left\|\sum_{n=1}^{+\infty}\frac{\left(\gamma\lambda_n^{m_1}l_{2,n}^1-Ml_{1,n}^2\right)\left(g_{1,n}^\delta-g_{1,n}\right)+\left(\gamma\lambda_n^{m_1}l_{2,n}^2+Ml_{1,n}^1\right)\left(g_{2,n}^\delta-g_{2,n}\right)}{M^2+\gamma\lambda_n^{m_1}\left(\left(l_{2,n}^1\right)^2+\left(l_{2,n}^2\right)^2\right)+\gamma\lambda_n^{m_2}\left(\left(l_{1,n}^1\right)^2+\left(l_{1,n}^2\right)^2\right)+\gamma^2\lambda_n^{m_1+m_2}}\right\| \\
&\le\left\|\sum_{n=1}^{+\infty}\left(\frac{\gamma\lambda_n^{m_1}l_{2,n}^1}{\gamma\lambda_n^{m_1}\left(l_{2,n}^1\right)^2+\gamma^2\lambda_n^{m_1+m_2}}+\frac{Ml_{1,n}^2}{\gamma\lambda_n^{m_2}\left(l_{1,n}^2\right)^2+M^2}\right)\left(g_{1,n}^\delta-g_{1,n}\right)\right\| \\
&\quad+\left\|\sum_{n=1}^{+\infty}\left(\frac{\gamma\lambda_n^{m_1}l_{2,n}^2}{\gamma\lambda_n^{m_1}\left(l_{2,n}^2\right)^2+\gamma^2\lambda_n^{m_1+m_2}}+\frac{Ml_{1,n}^1}{\gamma\lambda_n^{m_2}\left(l_{1,n}^1\right)^2+M^2}\right)\left(g_{2,n}^\delta-g_{2,n}\right)\right\|,
\end{aligned}
$$

where $M=l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1$. Then there exists a constant $C_9>0$ such that

$$
\left\|\varphi_\gamma^\delta-\varphi_\gamma\right\|\le C_9\frac{\delta}{\sqrt{\gamma}}. \tag{4.15}
$$

We now estimate the term $\|f_\gamma-f\|$. From Lemma 4.1, we get

$$
\frac{\left(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\right)^2}{l_{1,n}^1l_{2,n}^1+l_{1,n}^2l_{2,n}^2}\ge\frac{C_4}{\lambda_n^2},\quad\frac{\left(l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1\right)^2}{\left(l_{2,n}^1\right)^2+\left(l_{2,n}^2\right)^2}\ge\frac{C_4}{\lambda_n^2},
$$

it follows that

$$
\begin{aligned}
\|f_\gamma-f\|&=\left\|\sum_{n=1}^{+\infty}\frac{\gamma\lambda_n^{m_2}\left(l_{1,n}^1l_{2,n}^1+l_{1,n}^2l_{2,n}^2\right)\varphi_n-\left(\gamma^2\lambda_n^{m_1+m_2}+\gamma\lambda_n^{m_1}\left(\left(l_{2,n}^1\right)^2+\left(l_{2,n}^2\right)^2\right)\right)f_n}{M^2+\gamma\lambda_n^{m_1}\left(\left(l_{2,n}^1\right)^2+\left(l_{2,n}^2\right)^2\right)+\gamma\lambda_n^{m_2}\left(\left(l_{1,n}^1\right)^2+\left(l_{1,n}^2\right)^2\right)+\gamma^2\lambda_n^{m_1+m_2}}X_n\right\| \\
&\le\left\|\sum_{n=1}^{+\infty}\frac{\gamma\lambda_n^{m_2}\varphi_n}{\gamma\lambda_n^{(m_1+m_2)/2}+M^2/\left(l_{1,n}^1l_{2,n}^1+l_{1,n}^2l_{2,n}^2\right)}X_n\right\| \\
&\quad+\left\|\sum_{n=1}^{+\infty}\frac{\gamma\lambda_n^{m_1}f_n}{\gamma\lambda_n^{m_1}+\left(\gamma\lambda_n^{m_2}\left(\left(l_{1,n}^1\right)^2+\left(l_{1,n}^2\right)^2\right)+M^2\right)/\left(\gamma\lambda_n^{m_2}+\left(l_{2,n}^1\right)^2+\left(l_{2,n}^2\right)^2\right)}X_n\right\| \\
&\le\left\|\sum_{n=1}^{+\infty}\frac{\gamma\lambda_n^{m_2}\varphi_n}{\gamma\lambda_n^{(m_1+m_2)/2}+M^2/\left(l_{1,n}^1l_{2,n}^1+l_{1,n}^2l_{2,n}^2\right)}X_n\right\| \\
&\quad+\left\|\sum_{n=1}^{+\infty}\frac{\gamma\lambda_n^{m_1}f_n}{\gamma\lambda_n^{m_1}+M^2/\left(\left(l_{2,n}^1\right)^2+\left(l_{2,n}^2\right)^2\right)}X_n\right\| \\
&\le\sup_n\left(\frac{\gamma\lambda_n^{m_2+2-p_2}}{C_4+\gamma\lambda_n^{(m_1+m_2+4)/2}}\right)\left\|\sum_{n=1}^{+\infty}\lambda_n^{p_2}\varphi_nX_n\right\| \\
&\quad+\sup_n\left(\frac{\gamma\lambda_n^{m_1+2-p_1}}{C_4+\gamma\lambda_n^{m_1+2}}\right)\left\|\sum_{n=1}^{+\infty}\lambda_n^{p_1}f_nX_n\right\|,
\end{aligned}
$$

where $M=l_{1,n}^1l_{2,n}^2-l_{1,n}^2l_{2,n}^1$. From the a priori condition (3.8) yields

$$
\|f_\gamma-f\|\le\sum_{i=1,2}\sup_n\left(\frac{\gamma\lambda_n^{\max\{m_1,m_2\}+2-p_i}}{C_4+\gamma\lambda_n^{\min\{m_1,m_2\}+2}}\right)E.
$$

For $i=1,2$, we rewrite

$$
\max\{m_1,m_2\}+2-p_i=2-(p_i-\max\{m_1,m_2\}),
$$

where $p_i \geq \max\{m_1, m_2\}$ denotes the constraint condition (4.11). If $0 < p_i - \max\{m_1, m_2\} < 2$, there exists a constant $C_{10} > 0$ such that

$$\sup_n \left(\frac{\gamma \lambda_n^{\max\{m_1,m_2\}+2-p_i}}{C_4 + \gamma \lambda_n^{\min\{m_1,m_2\}+2}} \right) \leq C_{10} \gamma^{\frac{p_i + \min\{m_1,m_2\} - \max\{m_1,m_2\}}{\min\{m_1,m_2\}+2}},$$

which also implies that

$$\|f_\gamma - f\| \leq C_{10} E \gamma^{\frac{\min\{p_1,p_2\} + \min\{m_1,m_2\} - \max\{m_1,m_2\}}{\min\{m_1,m_2\}+2}}.$$

If $p_i - \max\{m_1, m_2\} \geq 2$, we get

$$\sup_n \left(\frac{\gamma \lambda_n^{\max\{m_1,m_2\}+2-p_i}}{C_4 + \gamma \lambda_n^{\min\{m_1,m_2\}+2}} \right) \leq \frac{1}{C_4 \lambda_1^{p_i - \max\{m_1,m_2\}-2}} \gamma,$$

and hence that

$$\|f_\gamma - f\| \leq \frac{2}{C_4 \lambda_1^{\min\{p_1,p_2\} - \max\{m_1,m_2\}-2}} E\gamma.$$

Therefore, the above two cases are summarized as

$$\|f_\gamma - f\| \leq \begin{cases} C_{10} E \gamma^{\frac{\min\{p_1,p_2\} + \min\{m_1,m_2\} - \max\{m_1,m_2\}}{\min\{m_1,m_2\}+2}}, & 0 < p_i - \max\{m_1,m_2\} < 2, \\ 2\left(C_4 \lambda_1^{\min\{p_1,p_2\} - \max\{m_1,m_2\}-2}\right)^{-1} E\gamma, & p_i - \max\{m_1,m_2\} \geq 2. \end{cases} \tag{4.16}$$

On the other hand, from Lemma 4.1 and a priori condition (3.8) yields

$$\begin{aligned}
\|\varphi_\gamma - \varphi\| &= \left\| \sum_{n=1}^{+\infty} \frac{\left(\gamma^2 \lambda_n^{m_1+m_2} + \gamma \lambda_n^{m_2}\left(\left(l_{1,n}^1\right)^2 + \left(l_{1,n}^2\right)^2\right)\right)\varphi_n - \gamma \lambda_n^{m_1}\left(l_{1,n}^1 l_{2,n}^1 + l_{1,n}^2 l_{2,n}^2\right) f_n}{M^2 + \gamma \lambda_n^{m_1}\left(\left(l_{2,n}^1\right)^2 + \left(l_{2,n}^2\right)^2\right) + \gamma \lambda_n^{m_2}\left(\left(l_{1,n}^1\right)^2 + \left(l_{1,n}^2\right)^2\right) + \gamma^2 \lambda_n^{m_1+m_2}} X_n \right\| \\
&\leq \left\| \sum_{n=1}^{+\infty} \frac{\gamma \lambda_n^{m_1} f_n}{\gamma \lambda_n^{(m_1+m_2)/2} + M^2 / \left(l_{1,n}^1 l_{2,n}^1 + l_{1,n}^2 l_{2,n}^2\right)} X_n \right\| \\
&\quad + \left\| \sum_{n=1}^{+\infty} \frac{\gamma \lambda_n^{m_2} \varphi_n}{\gamma \lambda_n^{m_2} + \left(\gamma \lambda_n^{m_1}\left(\left(l_{2,n}^1\right)^2 + \left(l_{2,n}^2\right)^2\right) + M^2\right) / \left(\gamma \lambda_n^{m_1} + \left(l_{1,n}^1\right)^2 + \left(l_{1,n}^2\right)^2\right)} X_n \right\| \\
&\leq \left\| \sum_{n=1}^{+\infty} \frac{\gamma \lambda_n^{m_1} f_n}{\gamma \lambda_n^{(m_1+m_2)/2} + M^2 / \left(l_{1,n}^1 l_{2,n}^1 + l_{1,n}^2 l_{2,n}^2\right)} X_n \right\| \\
&\quad + \left\| \sum_{n=1}^{+\infty} \frac{\gamma \lambda_n^{m_2} \varphi_n}{\gamma \lambda_n^{m_2} + M^2 / \left(\left(l_{1,n}^1\right)^2 + \left(l_{1,n}^2\right)^2\right)} X_n \right\| \\
&\leq \sum_{i=1,2} \sup_n \left(\frac{\gamma \lambda_n^{\max\{m_1,m_2\}+2-p_i}}{C_4 + \gamma \lambda_n^{\min\{m_1,m_2\}+2}} \right) E,
\end{aligned}$$

where $M = l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1$. In a similar way, we get that

$$\|\varphi_\gamma - \varphi\| \leq \begin{cases} C_{10} E \gamma^{\frac{\min\{p_1,p_2\} + \min\{m_1,m_2\} - \max\{m_1,m_2\}}{\min\{m_1,m_2\}+2}}, & 0 < p_i - \max\{m_1,m_2\} < 2, \\ 2\left(C_4 \lambda_1^{\min\{p_1,p_2\} - \max\{m_1,m_2\}-2}\right)^{-1} E\gamma, & p_i - \max\{m_1,m_2\} \geq 2 \end{cases} \tag{4.17}$$

for the second term on the right-hand side of inequality (4.13). Now by choose the regularization parameter

$$\gamma = \begin{cases} \delta^{\frac{2(\min\{m_1,m_2\}+2)}{2\min\{p_1,p_2\} + 3\min\{m_1,m_2\} - 2\max\{m_1,m_2\} + 2}}, & 0 < p_i - \max\{m_1,m_2\} < 2, \\ \delta^{\frac{2}{3}}, & p_i - \max\{m_1,m_2\} \geq 2, \end{cases}$$

this together with (4.14) and (4.16) yields that

$$\|f_\gamma^\delta - f\| \le \begin{cases} (C_8 + C_{10}E)\delta^{\frac{2(\min\{p_1,p_2\}+\min\{m_1,m_2\}-\max\{m_1,m_2\})}{2\min\{p_1,p_2\}+3\min\{m_1,m_2\}-2\max\{m_1,m_2\}+2}}, & 0 < p_i - \max\{m_1,m_2\} < 2, \\ \Big(C_8 + 2\big(C_4\lambda_1^{\min\{p_1,p_2\}-\max\{m_1,m_2\}-2}\big)^{-1}E\Big)\delta^{\frac{2}{3}}, & p_i - \max\{m_1,m_2\} \ge 2. \end{cases}$$

Also, by (4.15) and (4.17), we have

$$\|\varphi_\gamma^\delta - \varphi\| \le \begin{cases} (C_9 + C_{10}E)\delta^{\frac{2(\min\{p_1,p_2\}+\min\{m_1,m_2\}-\max\{m_1,m_2\})}{2\min\{p_1,p_2\}+3\min\{m_1,m_2\}-2\max\{m_1,m_2\}+2}}, & 0 < p_i - \max\{m_1,m_2\} < 2, \\ \Big(C_9 + 2\big(C_4\lambda_1^{\min\{p_1,p_2\}-\max\{m_1,m_2\}-2}\big)^{-1}E\Big)\delta^{\frac{2}{3}}, & p_i - \max\{m_1,m_2\} \ge 2. \end{cases}$$

The proof is complete. □

Theorem 4.2 provides an a priori strategy for choosing regularization parameter. The regularity of exact solutions f and φ should be known in advance. In many practical applications, the regularity of f and φ remains unknown. As a result, it is necessary to investigate a posteriori strategy for choosing regularization parameter.

Here, we introduce Morozov's discrepancy principle [32], that is, we choose the regularization parameter γ as the solution of the following equation:

$$\big\|K_1\big(f_\gamma^\delta, \varphi_\gamma^\delta\big) - g_1^\delta\big\| + \big\|K_2\big(f_\gamma^\delta, \varphi_\gamma^\delta\big) - g_2^\delta\big\| = \tau\delta, \tag{4.18}$$

where τ is a given constant. The solvability of the discrepancy equation (4.18) is guaranteed by the following lemma if $0 < \tau\delta < \|g_1^\delta\| + \|g_2^\delta\|$ hold.

Lemma 4.3. *Let $g_1^\delta, g_2^\delta \in L^2(0,W)$,*

$$\theta(\gamma) := \big\|K_1\big(f_\gamma^\delta, \varphi_\gamma^\delta\big) - g_1^\delta\big\| + \big\|K_2\big(f_\gamma^\delta, \varphi_\gamma^\delta\big) - g_2^\delta\big\|,$$

then $\theta(\gamma)$ is a continuous function satisfying

$$\lim_{\gamma\to 0}\theta(\gamma) = 0, \quad \lim_{\gamma\to\infty}\theta(\gamma) = \big\|g_1^\delta\big\|^2 + \big\|g_2^\delta\big\|^2.$$

Proof. Since $\theta(\gamma)$ is given by

$$\theta(\gamma) = \left(\sum_{n=1}^{\infty}\left(\frac{-\gamma\big(\lambda_n^{m_1}\big(l_{2,n}^2\big)^2 + \lambda_n^{m_2}\big(l_{1,n}^2\big)^2 + \gamma\lambda_n^{m_1+m_2}\big)g_{1,n}^\delta + \gamma L g_{2,n}^\delta}{M^2 + \gamma\lambda_n^{m_1}\big(\big(l_{2,n}^1\big)^2 + \big(l_{2,n}^2\big)^2\big) + \gamma\lambda_n^{m_2}\big(\big(l_{1,n}^1\big)^2 + \big(l_{1,n}^2\big)^2\big) + \gamma^2\lambda_n^{m_1+m_2}}\right)^2\right)^{\frac{1}{2}}$$
$$+\left(\sum_{n=1}^{\infty}\left(\frac{\gamma L g_{1,n}^\delta - \gamma\big(\lambda_n^{m_1}\big(l_{2,n}^1\big)^2 + \lambda_n^{m_2}\big(l_{1,n}^1\big)^2 + \gamma\lambda_n^{m_1+m_2}\big)g_{2,n}^\delta}{M^2 + \gamma\lambda_n^{m_1}\big(\big(l_{2,n}^1\big)^2 + \big(l_{2,n}^2\big)^2\big) + \gamma\lambda_n^{m_2}\big(\big(l_{1,n}^1\big)^2 + \big(l_{1,n}^2\big)^2\big) + \gamma^2\lambda_n^{m_1+m_2}}\right)^2\right)^{\frac{1}{2}},$$

where $M = l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1, L = \lambda_n^{m_1} l_{2,n}^1 l_{2,n}^2 + \lambda_n^{m_2} l_{1,n}^1 l_{1,n}^2$. So we have $\lim_{\gamma\to 0}\theta(\gamma) = 0$ and $\lim_{\gamma\to\infty}\theta(\gamma) = \|g_1^\delta\|^2 + \|g_2^\delta\|^2$. The lemma is proved. □

Next, we obtain error estimates for the regularization solution under the a posteriori choice of regularization parameter.

Theorem 4.3. *Assume that the observed data g_i^δ, $i = 1, 2$ satisfies* (1.5), *the a priori condition* (3.8) *and the constraint condition* (4.11) *hold, and there exists $\tau > 4$ such that $0 < \tau\delta < \|g_1^\delta\| + \|g_2^\delta\|$, the parameter γ is selected by the Morozov's discrepancy principle* (4.18). *Then,* *(a) If* $0 < p_i - \max\{m_1, m_2\} < 1$*, we get*

$$\|f_\gamma^\delta - f\| + \|\varphi_\gamma^\delta - \varphi\| \le \begin{cases} C\delta^{\frac{2\min\{p_1,p_2\}-m_2}{2(1+\min\{p_1,p_2\})}}, & m_1 < m_2, \\ C\delta^{\frac{2\min\{p_1,p_2\}-m_1}{2(1+\min\{p_1,p_2\})}}, & m_1 > m_2. \end{cases}$$

(b) If $p_i - \max\{m_1, m_2\} \ge 1$*, we have*

$$\|f_\gamma^\delta - f\| + \|\varphi_\gamma^\delta - \varphi\| \le C\delta^{\frac{1}{2}},$$

where C *depends on* p_i, m_i, T_i $(i = 1, 2), \alpha, m, \lambda_1, \tau$*.*

Proof. For $\tau\delta$ defined in (4.18), there is the following inequality estimation:

$$\begin{aligned} \tau\delta = {} & \|K_{1,1}(f_\gamma^\delta) + K_{2,1}(\varphi_\gamma^\delta) - g_1^\delta\| + \|K_{1,2}(f_\gamma^\delta) + K_{2,2}(\varphi_\gamma^\delta) - g_2^\delta\| \qquad (4.19) \\ = {} & \left\| \sum_{n=1}^{+\infty} \frac{-\gamma\big(\lambda_n^{m_1}(l_{2,n}^2)^2 + \lambda_n^{m_2}(l_{1,n}^2)^2 + \gamma\lambda_n^{m_1+m_2}\big)g_{1,n}^\delta + \gamma\big(\lambda_n^{m_1}l_{2,n}^1 l_{2,n}^2 + \lambda_n^{m_2}l_{1,n}^1 l_{1,n}^2\big)g_{2,n}^\delta}{M^2 + \gamma\lambda_n^{m_1}\big((l_{2,n}^1)^2 + (l_{2,n}^2)^2\big) + \gamma\lambda_n^{m_2}\big((l_{1,n}^1)^2 + (l_{1,n}^2)^2\big) + \gamma^2\lambda_n^{m_1+m_2}} X_n \right\| \\ & + \left\| \sum_{n=1}^{+\infty} \frac{\gamma\big(\lambda_n^{m_1}l_{2,n}^1 l_{2,n}^2 + \lambda_n^{m_2}l_{1,n}^1 l_{1,n}^2\big)g_{1,n}^\delta - \gamma\big(\lambda_n^{m_1}(l_{2,n}^1)^2 + \lambda_n^{m_2}(l_{1,n}^1)^2 + \gamma\lambda_n^{m_1+m_2}\big)g_{2,n}^\delta}{M^2 + \gamma\lambda_n^{m_1}\big((l_{2,n}^1)^2 + (l_{2,n}^2)^2\big) + \gamma\lambda_n^{m_2}\big((l_{1,n}^1)^2 + (l_{1,n}^2)^2\big) + \gamma^2\lambda_n^{m_1+m_2}} X_n \right\| \\ \le {} & 4\delta + \left\| \sum_{n=1}^{+\infty} \frac{\gamma g_{1,n} X_n}{\gamma + \big(\gamma\big(\lambda_n^{m_1}(l_{2,n}^1)^2 + \lambda_n^{m_2}(l_{1,n}^1)^2\big) + M^2\big)/\big(\lambda_n^{m_1}(l_{2,n}^2)^2 + \lambda_n^{m_2}(l_{1,n}^2)^2 + \gamma\lambda_n^{m_1+m_2}\big)} \right\| \\ & + \left\| \sum_{n=1}^{+\infty} \frac{\gamma g_{2,n} X_n}{\gamma + M^2/\big(\lambda_n^{m_1}l_{2,n}^1 l_{2,n}^2 + \lambda_n^{m_2}l_{1,n}^1 l_{1,n}^2\big)} \right\| \\ & + \left\| \sum_{n=1}^{+\infty} \frac{\gamma g_{1,n} X_n}{\gamma + M^2/\big(\lambda_n^{m_1}l_{2,n}^1 l_{2,n}^2 + \lambda_n^{m_2}l_{1,n}^1 l_{1,n}^2\big)} \right\| \\ & + \left\| \sum_{n=1}^{+\infty} \frac{\gamma g_{2,n} X_n}{\gamma + \big(\gamma\big(\lambda_n^{m_1}(l_{2,n}^2)^2 + \lambda_n^{m_2}(l_{1,n}^2)^2\big) + M^2\big)/\big(\lambda_n^{m_1}(l_{2,n}^1)^2 + \lambda_n^{m_2}(l_{1,n}^1)^2 + \gamma\lambda_n^{m_1+m_2}\big)} \right\|, \end{aligned}$$

where $M = l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1$. It is easy to derive that the following two inequalities:

$$\frac{\gamma\big(\lambda_n^{m_1}(l_{2,n}^1)^2 + \lambda_n^{m_2}(l_{1,n}^1)^2\big) + \big(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\big)^2}{\lambda_n^{m_1}(l_{2,n}^2)^2 + \lambda_n^{m_2}(l_{1,n}^2)^2 + \gamma\lambda_n^{m_1+m_2}} \ge \frac{\big(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\big)^2}{\lambda_n^{m_1}(l_{2,n}^2)^2 + \lambda_n^{m_2}(l_{1,n}^2)^2},$$

$$\frac{\gamma\big(\lambda_n^{m_1}(l_{2,n}^2)^2 + \lambda_n^{m_2}(l_{1,n}^2)^2\big) + \big(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\big)^2}{\lambda_n^{m_1}(l_{2,n}^1)^2 + \lambda_n^{m_2}(l_{1,n}^1)^2 + \gamma\lambda_n^{m_1+m_2}} \ge \frac{\big(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\big)^2}{\lambda_n^{m_1}(l_{2,n}^1)^2 + \lambda_n^{m_2}(l_{1,n}^1)^2}.$$

Recall, from Lemma 4.2, we have

$$\frac{\big(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\big)^2}{\lambda_n^{m_1}(l_{2,n}^2)^2 + \lambda_n^{m_2}(l_{1,n}^2)^2} \ge \frac{C_5}{\lambda_n^{\max\{m_1,m_2\}+2}},$$

$$\frac{\big(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\big)^2}{\lambda_n^{m_1}l_{2,n}^1 l_{2,n}^2 + \lambda_n^{m_2}l_{1,n}^1 l_{1,n}^2} \ge \frac{C_5}{\lambda_n^{\max\{m_1,m_2\}+2}},$$

$$\frac{\big(l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1\big)^2}{\lambda_n^{m_1}(l_{2,n}^1)^2 + \lambda_n^{m_2}(l_{1,n}^1)^2} \ge \frac{C_5}{\lambda_n^{\max\{m_1,m_2\}+2}}.$$

This together with

$$g_{1,n} = l_{1,n}^1 f_n + l_{2,n}^1 \varphi_n, \quad g_{2,n} = l_{1,n}^2 f_n + l_{2,n}^2 \varphi_n$$

gives

$$\begin{aligned}(\tau-4)\delta \le{}& 2\left(\left\|\sum_{n=1}^{+\infty} \frac{\gamma\lambda_n^{2+\max\{m_1,m_2\}} l_{1,n}^1 f_n X_n}{\gamma\lambda_n^{2+\max\{m_1,m_2\}}+C_5}\right\| + \left\|\sum_{n=1}^{+\infty} \frac{\gamma\lambda_n^{2+\max\{m_1,m_2\}} l_{2,n}^1 \varphi_n X_n}{\gamma\lambda_n^{2+\max\{m_1,m_2\}}+C_5}\right\|\right)\\ &+ 2\left(\left\|\sum_{n=1}^{+\infty} \frac{\gamma\lambda_n^{2+\max\{m_1,m_2\}} l_{1,n}^2 f_n X_n}{\gamma\lambda_n^{2+\max\{m_1,m_2\}}+C_5}\right\| + \left\|\sum_{n=1}^{+\infty} \frac{\gamma\lambda_n^{2+\max\{m_1,m_2\}} l_{2,n}^2 \varphi_n X_n}{\gamma\lambda_n^{2+\max\{m_1,m_2\}}+C_5}\right\|\right).\end{aligned}$$

Thus, by Lemmas 2.1, 2.5 and a priori condition (3.8), we get

$$\begin{aligned}\frac{(\tau-4)\delta}{4} &\le \sup_n \left(\frac{\gamma\lambda_n^{1+\max\{m_1,m_2\}-p_1}}{\gamma\lambda_n^{\max\{m_1,m_2\}+2}+C_5}\right)\left\|\sum_{n=1}^{+\infty}\lambda_n^{p_1} f_n X_n\right\|\\ &\quad+ \sup_n \left(\frac{\gamma\lambda_n^{1+\max\{m_1,m_2\}-p_2}}{\gamma\lambda_n^{\max\{m_1,m_2\}+2}+C_5}\right)\left\|\sum_{n=1}^{+\infty}\lambda_n^{p_2} \varphi_n X_n\right\|\\ &\le \sum_{i=1,2}\sup_n \left(\frac{\gamma\lambda_n^{1+\max\{m_1,m_2\}-p_i}}{\gamma\lambda_n^{\max\{m_1,m_2\}+2}+C_5}\right)E.\end{aligned}$$

Now by the constraint condition (4.11) yields $p_i - \max\{m_1,m_2\} > 0$. If $0 < p_i - \max\{m_1,m_2\} < 1$, there exists a constant $C_{11} > 0$ such that

$$\sup_n \left(\frac{\gamma\lambda_n^{1+\max\{m_1,m_2\}-p_i}}{\gamma\lambda_n^{\max\{m_1,m_2\}+2}+C_5}\right) \le C_{11}\gamma^{\frac{1+p_i}{\max\{m_1,m_2\}+2}}.$$

If $p_i - \max\{m_1,m_2\} \ge 1$, we get

$$\sup_n \left(\frac{\gamma\lambda_n^{1+\max\{m_1,m_2\}-p_i}}{\gamma\lambda_n^{\max\{m_1,m_2\}+2}+C_5}\right) \le \frac{1}{C_5\lambda_1^{p_i-\max\{m_1,m_2\}-1}}\gamma.$$

Therefore, we conclude that

$$\frac{(\tau-4)}{4}\delta \le \begin{cases} C_{11}E\gamma^{\frac{1+\min\{p_1,p_2\}}{\max\{m_1,m_2\}+2}}, & 0 < p_i - \max\{m_1,m_2\} < 1,\\ 2\left(C_5\lambda_1^{\min\{p_1,p_2\}-\max\{m_1,m_2\}-1}\right)^{-1}E\gamma, & p_i - \max\{m_1,m_2\} \ge 1.\end{cases}$$

Denote constants

$$C_{12} = \left(\frac{4C_{11}E}{\tau-4}\right)^{\frac{\max\{m_1,m_2\}+2}{2(1+\min\{p_1,p_2\})}}, \quad C_{13} = \left(\frac{8E}{(\tau-4)C_5\lambda_1^{\min\{p_1,p_2\}-1-\max\{m_1,m_2\}}}\right)^{\frac{1}{2}},$$

the following inequality:

$$\frac{1}{\sqrt{\gamma}} \le \begin{cases} C_{12}\delta^{-\frac{\max\{m_1,m_2\}+2}{2(1+\min\{p_1,p_2\})}}, & 0 < p_i - \max\{m_1,m_2\} < 1,\\ C_{13}\delta^{-\frac{1}{2}}, & p_i - \max\{m_1,m_2\} \ge 1\end{cases}$$

is obviously hold. From (4.14) and (4.15), we obtain

$$\left\|f_\gamma^\delta - f_\gamma\right\| \le \begin{cases} C_8C_{12}\delta^{\frac{2\min\{p_1,p_2\}-\max\{m_1,m_2\}}{2(1+\min\{p_1,p_2\})}}, & 0 < p_i - \max\{m_1,m_2\} < 1,\\ C_8C_{13}\delta^{\frac{1}{2}}, & p_i - \max\{m_1,m_2\} \ge 1.\end{cases} \tag{4.20}$$

$$\left\|\varphi_\gamma^\delta - \varphi_\gamma\right\| \le \begin{cases} C_9C_{12}\delta^{\frac{2\min\{p_1,p_2\}-\max\{m_1,m_2\}}{2(1+\min\{p_1,p_2\})}}, & 0 < p_i - \max\{m_1,m_2\} < 1,\\ C_9C_{13}\delta^{\frac{1}{2}}, & p_i - \max\{m_1,m_2\} \ge 1.\end{cases} \tag{4.21}$$

Furthermore, by a priori condition (3.8), we have

$$
\begin{aligned}
&\|f_\gamma - f\|^2_{\mathscr{D}((-L)^{p_1})}\\
\le &\sum_{n=1}^{+\infty}\left(\frac{\gamma\lambda_n^{m_2}\big(l^1_{1,n}l^1_{2,n}+l^2_{1,n}l^2_{2,n}\big)}{M^2+\gamma\lambda_n^{m_1}\big(\big(l^1_{2,n}\big)^2+\big(l^2_{2,n}\big)^2\big)+\gamma\lambda_n^{m_2}\big(\big(l^1_{1,n}\big)^2+\big(l^2_{1,n}\big)^2\big)+\gamma^2\lambda_n^{m_1+m_2}}\right)^2\lambda_n^{2p_2}|\varphi_n|^2\\
&+\sum_{n=1}^{+\infty}\left(\frac{\gamma^2\lambda_n^{m_1+m_2}+\gamma\lambda_n^{m_1}\big(\big(l^1_{2,n}\big)^2+\big(l^2_{2,n}\big)^2\big)}{M^2+\gamma\lambda_n^{m_1}\big(\big(l^1_{2,n}\big)^2+\big(l^2_{2,n}\big)^2\big)+\gamma\lambda_n^{m_2}\big(\big(l^1_{1,n}\big)^2+\big(l^2_{1,n}\big)^2\big)+\gamma^2\lambda_n^{m_1+m_2}}\right)^2\lambda_n^{2p_1}|f_n|^2\\
\le &\sum_{n=1}^{+\infty}\lambda_n^{2p_2}|\varphi_n|^2+\sum_{n=1}^{+\infty}\lambda_n^{2p_1}|f_n|^2\le 2E^2, \qquad (4.22)
\end{aligned}
$$

where $M = l^1_{1,n}l^2_{2,n} - l^2_{1,n}l^1_{2,n}$. Similarly, we have the same result on $\|\varphi_\gamma - \varphi\|^2_{\mathscr{D}((-L)^{p_2})}$. On the other hand, let

$$F_i(n) = \frac{l^i_{1,n}l^i_{2,n}}{l^1_{1,n}l^2_{2,n} - l^2_{1,n}l^1_{2,n}}, \quad i = 1, 2.$$

From the noise assumptions (1.5), and there exists $C_{14} > 0$ such that

$$
\begin{aligned}
&\|K_1(f_\gamma - f, \varphi_\gamma - \varphi)\|\\
\le\ & C_{14}\delta+\left\|\sum_{n=1}^{+\infty}\frac{-\gamma\big(\lambda_n^{m_1}\big(l^2_{2,n}\big)^2+\lambda_n^{m_2}\big(l^2_{1,n}\big)^2+\gamma\lambda_n^{m_1+m_2}\big)g^\delta_{1,n}+\gamma L g^\delta_{2,n}}{M^2+\gamma\lambda_n^{m_1}\big(\big(l^1_{2,n}\big)^2+\big(l^2_{2,n}\big)^2\big)+\gamma\lambda_n^{m_2}\big(\big(l^1_{1,n}\big)^2+\big(l^2_{1,n}\big)^2\big)+\gamma^2\lambda_n^{m_1+m_2}}X_n\right\|\\
&\times\lim_{n\to\infty}\left|F_1(n)\frac{l^2_{2,n}}{l^1_{2,n}}\right|\\
&+\left\|\sum_{n=1}^{+\infty}\frac{\gamma L g^\delta_{1,n}-\gamma\big(\lambda_n^{m_1}\big(l^1_{2,n}\big)^2+\lambda_n^{m_2}\big(l^1_{1,n}\big)^2+\gamma\lambda_n^{m_1+m_2}\big)g^\delta_{2,n}}{M^2+\gamma\lambda_n^{m_1}\big(\big(l^1_{2,n}\big)^2+\big(l^2_{2,n}\big)^2\big)+\gamma\lambda_n^{m_2}\big(\big(l^1_{1,n}\big)^2+\big(l^2_{1,n}\big)^2\big)+\gamma^2\lambda_n^{m_1+m_2}}X_n\right\|\\
&\times\lim_{n\to\infty}|F_1(n)|\\
&+\left\|\sum_{n=1}^{+\infty}\frac{-\gamma\big(\lambda_n^{m_1}\big(l^2_{2,n}\big)^2+\lambda_n^{m_2}\big(l^2_{1,n}\big)^2+\gamma\lambda_n^{m_1+m_2}\big)g^\delta_{1,n}+\gamma L g^\delta_{2,n}}{M^2+\gamma\lambda_n^{m_1}\big(\big(l^1_{2,n}\big)^2+\big(l^2_{2,n}\big)^2\big)+\gamma\lambda_n^{m_2}\big(\big(l^1_{1,n}\big)^2+\big(l^2_{1,n}\big)^2\big)+\gamma^2\lambda_n^{m_1+m_2}}X_n\right\|\\
&\times\lim_{n\to\infty}\left|F_1(n)\frac{l^2_{1,n}}{l^1_{1,n}}\right|\\
&+\left\|\sum_{n=1}^{+\infty}\frac{-\gamma L g^\delta_{1,n}+\gamma\big(\lambda_n^{m_1}(l^1_{2,n})^2+\lambda_n^{m_2}(l^1_{1,n})^2+\gamma\lambda_n^{m_1+m_2}\big)g^\delta_{2,n}}{M^2+\gamma\lambda_n^{m_1}\big(\big(l^1_{2,n}\big)^2+\big(l^2_{2,n}\big)^2\big)+\gamma\lambda_n^{m_2}\big(\big(l^1_{1,n}\big)^2+\big(l^2_{1,n}\big)^2\big)+\gamma^2\lambda_n^{m_1+m_2}}X_n\right\|\\
&\times\lim_{n\to\infty}|F_1(n)|,
\end{aligned}
$$

where $M = l^1_{1,n}l^2_{2,n} - l^2_{1,n}l^1_{2,n}$, $L = \lambda_n^{m_1}l^1_{2,n}l^2_{2,n} + \lambda_n^{m_2}l^1_{1,n}l^2_{1,n}$. And by Lemma 2.4 yields

$$\lim_{n\to\infty}|F_1(n)| = \lim_{n\to\infty}\left|F_1(n)\frac{l^2_{2,n}}{l^1_{2,n}}\right| = \lim_{n\to\infty}\left|F_1(n)\frac{l^1_{1,n}}{l^2_{1,n}}\right| \le \frac{T_1^\alpha T_2^\alpha\Gamma(1-\alpha)}{T_1^\alpha - T_2^\alpha},$$

this shows that

$$\|K_1(f_\gamma - f, \varphi_\gamma - \varphi)\| \le \left(C_{14} + \frac{2\tau T_1^\alpha T_2^\alpha\Gamma(1-\alpha)}{T_1^\alpha - T_2^\alpha}\right)\delta, \qquad (4.23)$$

which is ensured by Morozov's discrepancy principle (4.18). Similarly, for operator K_2, there exists positive constant C_{15} such that

$$
\begin{aligned}
&\|K_2(f_\gamma - f, \varphi_\gamma - \varphi)\| \\
&\le C_{15}\delta + \left\| \sum_{n=1}^{+\infty} \frac{-\gamma\big(\lambda_n^{m_1}(l_{2,n}^2)^2 + \lambda_n^{m_2}(l_{1,n}^2)^2 + \gamma\lambda_n^{m_1+m_2}\big) g_{1,n}^\delta + \gamma L g_{2,n}^\delta}{M^2 + \gamma\lambda_n^{m_1}\big(\big(l_{2,n}^1\big)^2 + \big(l_{2,n}^2\big)^2\big) + \gamma\lambda_n^{m_2}\big(\big(l_{1,n}^1\big)^2 + \big(l_{1,n}^2\big)^2\big) + \gamma^2\lambda_n^{m_1+m_2}} X_n \right\| \\
&\quad \times \lim_{n\to\infty} |F_2(n)| \\
&\quad + \left\| \sum_{n=1}^{+\infty} \frac{\gamma L g_{1,n}^\delta - \gamma\big(\lambda_n^{m_1}\big(l_{2,n}^1\big)^2 + \lambda_n^{m_2}\big(l_{1,n}^1\big)^2 + \gamma\lambda_n^{m_1+m_2}\big) g_{2,n}^\delta}{M^2 + \gamma\lambda_n^{m_1}\big(\big(l_{2,n}^1\big)^2 + \big(l_{2,n}^2\big)^2\big) + \gamma\lambda_n^{m_2}\big(\big(l_{1,n}^1\big)^2 + \big(l_{1,n}^2\big)^2\big) + \gamma^2\lambda_n^{m_1+m_2}} X_n \right\| \\
&\quad \times \lim_{n\to\infty} \left| F_2(n) \frac{l_{1,n}^1}{l_{1,n}^2} \right| \\
&\quad + \left\| \sum_{n=1}^{+\infty} \frac{-\gamma\big(\lambda_n^{m_1}(l_{2,n}^2)^2 + \lambda_n^{m_2}(l_{1,n}^2)^2 + \gamma\lambda_n^{m_1+m_2}\big) g_{1,n}^\delta + \gamma L g_{2,n}^\delta}{M^2 + \gamma\lambda_n^{m_1}\big(\big(l_{2,n}^1\big)^2 + \big(l_{2,n}^2\big)^2\big) + \gamma\lambda_n^{m_2}\big(\big(l_{1,n}^1\big)^2 + \big(l_{1,n}^2\big)^2\big) + \gamma^2\lambda_n^{m_1+m_2}} X_n \right\| \\
&\quad \times \lim_{n\to\infty} |F_2(n)| \\
&\quad + \left\| \sum_{n=1}^{+\infty} \frac{-\gamma L g_{1,n}^\delta + \gamma\big(\lambda_n^{m_1}(l_{2,n}^1)^2 + \lambda_n^{m_2}(l_{1,n}^1)^2 + \gamma\lambda_n^{m_1+m_2}\big) g_{2,n}^\delta}{M^2 + \gamma\lambda_n^{m_1}\big(\big(l_{2,n}^1\big)^2 + \big(l_{2,n}^2\big)^2\big) + \gamma\lambda_n^{m_2}\big(\big(l_{1,n}^1\big)^2 + \big(l_{1,n}^2\big)^2\big) + \gamma^2\lambda_n^{m_1+m_2}} X_n \right\| \\
&\quad \times \lim_{n\to\infty} \left| F_2(n) \frac{l_{2,n}^1}{l_{2,n}^2} \right|,
\end{aligned}
$$

where $M = l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1, L = \lambda_n^{m_1} l_{2,n}^1 l_{2,n}^2 + \lambda_n^{m_2} l_{1,n}^1 l_{1,n}^2$, and hence that

$$
\|K_2(f_\gamma - f, \varphi_\gamma - \varphi)\| \le \left(C_{15} + \frac{2\tau T_1^\alpha T_2^\alpha \Gamma(1-\alpha)}{T_1^\alpha - T_2^\alpha} \right) \delta. \tag{4.24}
$$

Now, we estimate term $\|f_\gamma - f\|$. Here,

$$
\|K_i(f_\gamma - f, \varphi_\gamma - \varphi)\|, \quad i = 1, 2,
$$

is obtained by replacing g_i^δ with g_i in (4.19), which, along with

$$
f_n = \frac{l_{2,n}^2 g_{1,n} - l_{2,n}^1 g_{2,n}}{l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1}, \quad \varphi_n = \frac{l_{1,n}^1 g_{2,n} - l_{1,n}^2 g_{1,n}}{l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1}
$$

yields

$$
\|f_\gamma - f\| = \left\| \sum_{n=1}^{+\infty} \frac{l_{2,n}^2 K_1(f_{\gamma,n} - f_n, \varphi_{\gamma,n} - \varphi_n) + l_{2,n}^1 K_2(f_{\gamma,n} - f_n, \varphi_{\gamma,n} - \varphi_n)}{l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1} X_n \right\|.
$$

Therefore, by the Hölder inequality

$$
\begin{aligned}
&\|f_\gamma - f\|^2 \\
&= \sum_{n=1}^{+\infty} \left(\frac{l_{2,n}^2}{l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1} \left(K_1(f_{\gamma,n} - f_n, \varphi_{\gamma,n} - \varphi_n) - \frac{l_{2,n}^1}{l_{2,n}^2} K_2(f_{\gamma,n} - f_n, \varphi_{\gamma,n} - \varphi_n) \right) \right)^2 \\
&\le \left(\sum_{n=1}^{+\infty} \left(\frac{l_{2,n}^2}{l_{1,n}^1 l_{2,n}^2 - l_{1,n}^2 l_{2,n}^1} \right)^{2(1+p_1)} \right.
\end{aligned}
$$

$$\times\left(K_1(f_{\gamma,n}-f_n,\varphi_{\gamma,n}-\varphi_n)-\frac{l^1_{2,n}}{l^2_{2,n}}K_2(f_{\gamma,n}-f_n,\varphi_{\gamma,n}-\varphi_n)\right)^2\Bigg)^{\frac{1}{1+p_1}}$$
$$\times\left(\sum_{n=1}^{+\infty}\left(K_1(f_{\gamma,n}-f_n,\varphi_{\gamma,n}-\varphi_n)-\frac{l^1_{2,n}}{l^2_{2,n}}K_2(f_{\gamma,n}-f_n,\varphi_{\gamma,n}-\varphi_n)\right)^2\right)^{\frac{p_1}{1+p_1}}$$
$$\le\left(\sum_{n=1}^{+\infty}\left(\frac{l^2_{2,n}}{l^1_{1,n}l^2_{2,n}-l^2_{1,n}l^1_{2,n}}\right)^{2p_1}(f_{\gamma,n}-f_n)^2\right)^{\frac{1}{1+p_1}}$$
$$\times\Bigg(\left(\sum_{n=1}^{+\infty}\big(K_1(f_{\gamma,n}-f_n,\varphi_{\gamma,n}-\varphi_n)\big)^2\right)^{\frac12}$$
$$+\left(\sum_{n=1}^{+\infty}\left(\frac{l^1_{2,n}}{l^2_{2,n}}\big(K_2(f_{\gamma,n}-f_n,\varphi_{\gamma,n}-\varphi_n)\big)\right)^2\right)^{\frac12}\Bigg)^{\frac{2p_1}{1+p_1}}.$$

Hence, similar to the proof of Theorem 3.1, we obtain

$$\|f_\gamma-f\|\le C_2\|f_\gamma-f\|^{\frac{1}{p_1+1}}_{\mathscr{D}((-L)^{p_1})}\left(\|K_1(f_\gamma-f,\varphi_\gamma-\varphi)\|+\left(\frac{T_2}{T_1}\right)^\alpha\|K_2(f_\gamma-f,\varphi_\gamma-\varphi)\|\right)^{\frac{p_1}{p_1+1}},$$

and based on (4.22)-(4.24), there exists a constant $C_{16}>0$ such that

$$\|f_\gamma-f\|\le C_{16}\delta^{\frac{p_1}{p_1+1}}.$$

In a similar manner, there exist $C_{17}>0$ we obtain

$$\begin{aligned}\|\varphi_\gamma-\varphi\|&\le C_3\|\varphi_\gamma-\varphi\|^{\frac{1}{p_2+1}}_{\mathscr{D}((-L)^{p_2})}\\&\quad\times\left(\|K_2(f_\gamma-f,\varphi_\gamma-\varphi)\|+\frac{1}{1-E_{\alpha,1}\big(-(m^2+\lambda_1)T_1^\alpha\big)}\|K_1(f_\gamma-f,\varphi_\gamma-\varphi)\|\right)^{\frac{p_2}{p_2+1}}\\&\le C_{17}\delta^{\frac{p_2}{p_2+1}}.\end{aligned}$$

And hence, from (4.20) and (4.21), we have conclusions

$$\left\|f^\delta_\gamma-f_\gamma\right\|\le\begin{cases}C_8C_{12}\delta^{\frac{2\min\{p_1,p_2\}-\max\{m_1,m_2\}}{2(1+\min\{p_1,p_2\})}}+C_{16}\delta^{\frac{p_1}{p_1+1}}, & 0<p_i-\max\{m_1,m_2\}<1,\\ C_8C_{13}\delta^{\frac12}+C_{16}\delta^{\frac{p_1}{p_1+1}}, & p_i-\max\{m_1,m_2\}\ge1,\end{cases}$$
$$\left\|\varphi^\delta_\gamma-\varphi\right\|\le\begin{cases}C_9C_{12}\delta^{\frac{2\min\{p_1,p_2\}-\max\{m_1,m_2\}}{2(1+\min\{p_1,p_2\})}}+C_{17}\delta^{\frac{p_2}{p_2+1}}, & 0<p_i-\max\{m_1,m_2\}<1,\\ C_9C_{13}\delta^{\frac12}+C_{17}\delta^{\frac{p_2}{p_2+1}}, & p_i-\max\{m_1,m_2\}\ge1.\end{cases}$$

As a result, there exists a constant C dependent on the constants p_i,m_i,T_i $(i=1,2),\alpha,m,\lambda_1$ and τ such that when $0<p_i-\max\{m_1,m_2\}<1$, we get

$$\left\|f^\delta_\gamma-f_\gamma\right\|\le C\delta^{\frac{2\min\{p_1,p_2\}-\max\{m_1,m_2\}}{2(1+\min\{p_1,p_2\})}},$$

and when $p_i-\max\{m_1,m_2\}\ge1$, we obtain

$$\left\|f^\delta_\gamma-f_\gamma\right\|\le C\delta^{\frac12}.$$

Similarly, we can derive the same estimates for $\|\varphi^\delta_\gamma-\varphi_\gamma\|$. So, the proof is complete. □

5. Numerical Experiments

In this section, we give some examples to demonstrate the effectiveness and flexibility of the proposed method. In the initial-boundary value problem (1.4), we consistently specify the parameters as $m^2 = 4, W = 1, T = 1$, that is, we consider the following system:

$$\begin{cases} {}_0^C\mathrm{D}_t^\alpha\, u(x,t) = -4u(x,t) + \dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x), & (x,t) \in (0,1)\times(0,1), \\ u(0,t) = u(W,t) = 0, & t \in (0,1), \\ u(x,0) = \varphi(x), & x \in (0,1). \end{cases} \tag{5.1}$$

In our numerical examples, we will take $T_1 = 1/3$ and $T_2 = 1/2$, and the time-fractional Black-Scholes inverse problem we considered is the following problem.

(**BSIP**): From data $u(x,T_i) = g_i(x),\ i = 1,2$, simultaneously find source term f and initial value φ in (5.1).

We know that the eigenvalues and its corresponding eigenvectors of the elliptic operator $-L$ are $\lambda_n = (n\pi)^2, X_n = \sin(n\pi x), n = 1,2,\ldots$, respectively, and the observed data is generated by

$$g_i^\delta(x) = g_i(x) + \delta\big(2 * \mathrm{rand}(x) - 1\big)g_i(x), \quad i = 1,2,$$

where $\mathrm{rand}(x)$ is the random function obeying uniform distribution in the interval $[0,1]$. In our numerical simulations, we will show the simulation results of the regularized solutions, as the parameters (such as the noise level δ and Caputo derivative order α) vary in some range, while the regularization parameters γ are always selected by Morozov's discrepancy principle (4.18).

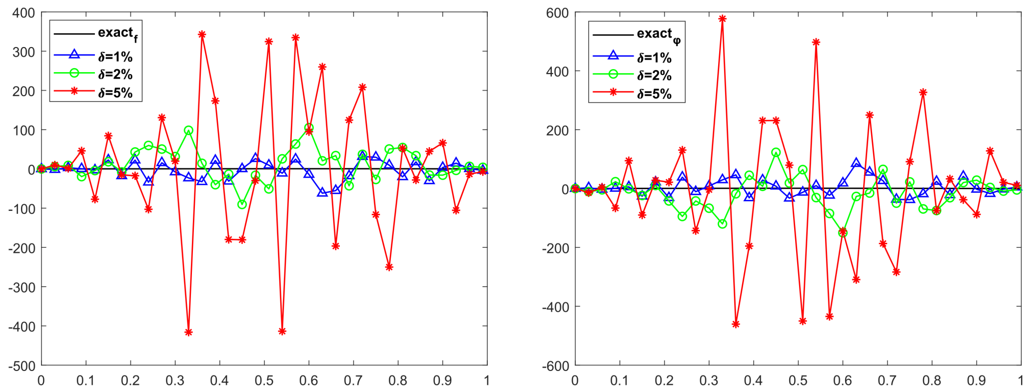
Example 5.1. Consider the model system (5.1) with smooth source term

$$f(x) = x(1-x)\exp(x),$$

and smooth initial value

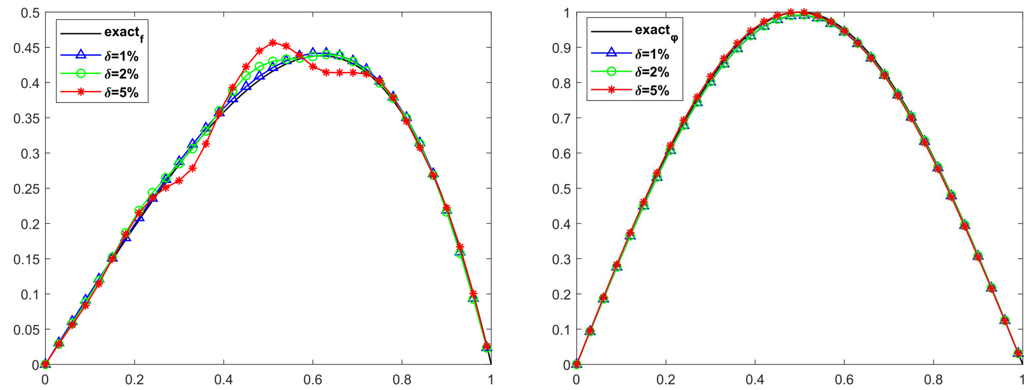
$$\varphi(x) = \sin(\pi x).$$

The simulation results are presented in Figs. 5.1 and 5.2.



(a) The source term $f(x)$ (b) The initial value $\varphi(x)$

Fig. 5.1. Example 5.1. Numerical results without regularization.



(a) The source term $f(x)$ (b) The initial value $\varphi(x)$

Fig. 5.2. Example 5.1. Numerical results with $D_1 f = f'$ and $D_2\varphi = \varphi''$.

Figs. 5.1 and 5.2 show the inversion results at different noise levels $\delta = [1\%, 2\%, 5\%]$ when $\alpha = 0.8$, respectively, without using the regularization method and using the regularization method with two differential operators given by $D_1 f = f'$ and $D_2\varphi = \varphi''$. From the results in Figs. 5.1-5.2, one can see that the simultaneous inversion problem **(BSIP)** is an ill-posed problem, and the regularization method with two differential operators proposed is an effective method for dealing with this problem. Clearly, we observe that as the noise level δ decreases, the regularized solution gets closer and closer to the exact solution, resulting in better numerical inversion results.

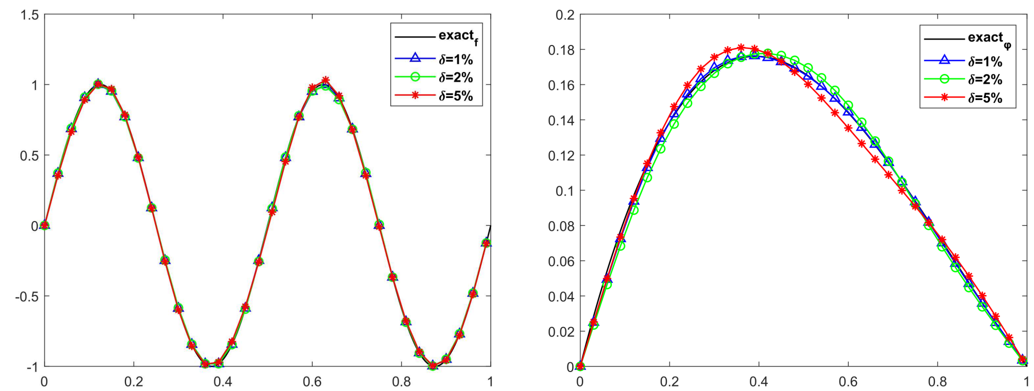
Example 5.2. Consider the model system (5.1) with smooth source term

$$f(x) = \sin(4\pi x),$$

and smooth initial value

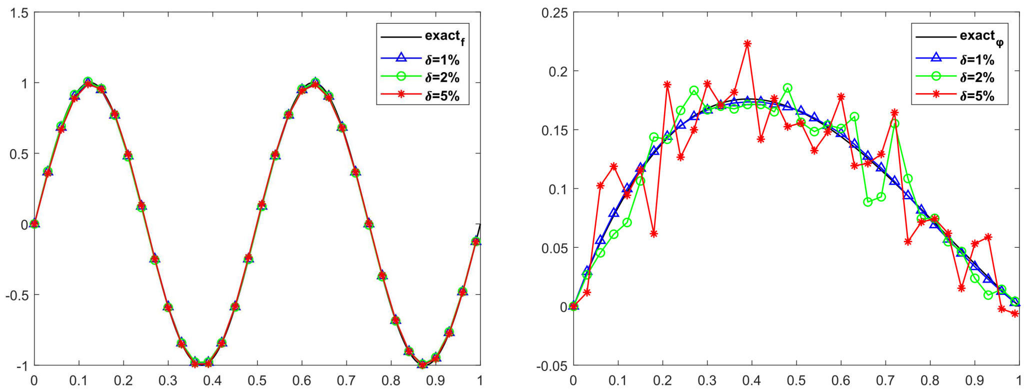
$$\varphi(x) = x(1-x)\left(1 - \frac{2x}{3}\right).$$

The simulation results are presented in Figs. 5.3 and 5.4.



(a) The source term $f(x)$ (b) The initial value $\varphi(x)$

Fig. 5.3. Example 5.2. Numerical results with $D_1 f = f''$ and $D_2\varphi = \varphi^{(3)}$.



(a) The source term $f(x)$ (b) The initial value $\varphi(x)$

Fig. 5.4. Example 5.2. Numerical results with $D_1 f = f^{(3)}$ and $D_2\varphi = \varphi''$.

The numerical inversion results of the simultaneous inversion problem **(BSIP)**, at different noise levels $\delta = [1\%, 2\%, 5\%]$ with $\alpha = 0.8$ and γ being selected by Morozov's discrepancy principle (4.18), are shown in Figs. 5.3 and 5.4. In Fig. 5.3, we choose differential operators as $D_1 f = f''$ and $D_2\varphi = \varphi^{(3)}$, while in Fig. 5.4, we take $D_1 f = f^{(3)}$ and $D_2\varphi = \varphi''$. Both figures show that, the initial value φ seems more sensitive to error variation, and also more sensitive to the selection of differential operator.

Example 5.3. Consider the model system (5.1) with piecewise smooth source term

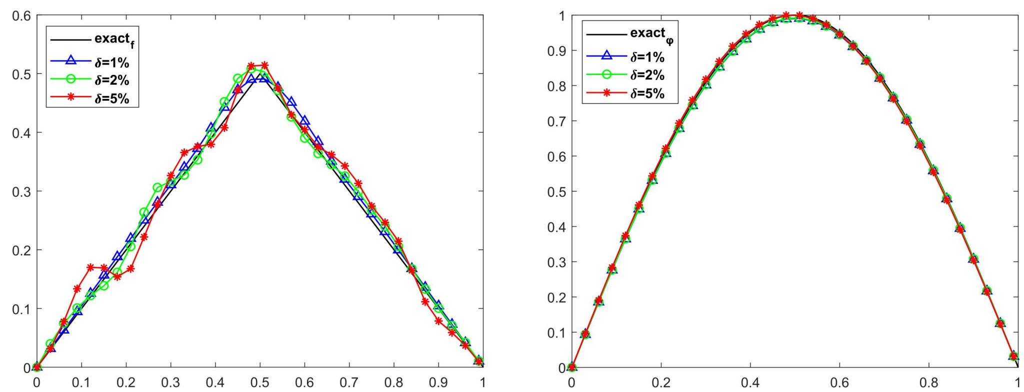
$$f(x) = \begin{cases} x, & x \in [0, 1/2), \\ 1 - x, & x \in [1/2, 1], \end{cases}$$

and smooth initial value

$$\varphi(x) = \sin(\pi x).$$

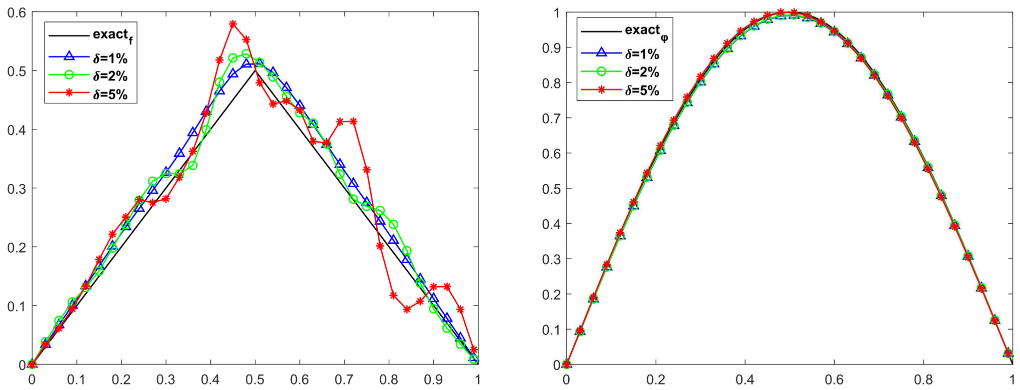
The simulation results are presented in Figs. 5.5 and 5.6.

In Example 5.3, source term f is a piecewise smooth function with a "cusp", and initial value φ is a smooth function. we choose the two differential operators D_1 and D_2 as $D_1 f = f'$



(a) The source term $f(x)$ (b) The initial value $\varphi(x)$

Fig. 5.5. Example 5.3. Numerical results with $\alpha = 0.9$.



(a) The source term $f(x)$ (b) The initial value $\varphi(x)$

Fig. 5.6. Example 5.3. Numerical results with $\alpha = 0.1$.

and $D_2\varphi = \varphi^{(3)}$. In Fig. 5.5, we take the order of fractional derivative $\alpha = 0.9$, and $\alpha = 0.1$ in Fig. 5.6, the simulation results are shown in Figs. 5.5-5.6. The results from Figs. 5.5 and 5.6 indicate that the larger the value of α, the better the fitting effect of the regularization solutions.

6. Conclusion

This paper investigates the inverse problem **(BSIP)** concerning the simultaneous reconstruction of the source term and initial value for the time-fractional Black-Scholes equation, establishing it as a linear ill-posed problem exhibiting a mild degree of ill-posedness. By establishing conditional stability in Theorem 3.1, we reveal that the selection of time points T_1 and T_2 influences stability. In particular, selecting these points too close together can adversely affect the regularized solution. To address the **(BSIP)**, we propose a regularization method with two differential operators. Convergence analyses are provided for both a priori (Theorem 4.2) and a posteriori (Theorem 4.3) selections of the regularization parameter. While theoretical applicability may be constrained by the restrictive condition (4.11), numerical experiments, which employ diverse exact solutions for the source term and initial value – validate the effectiveness of the proposed approach, despite its reliance on the spectral expansion of the differential operator within the eigensystem.

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