The 3-coloring of planar graphs with adjacent triangles

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Abstract. Erdős raised the following problem according to Steinberg's conjecture: Is there an integer such that every planar graph without cycles of length from 4 to k is 3-colorable. By far, the result about the problem was improved to $k \le 7$ by Borodin et al. However, by permitting the existence of adjacent triangles except K_4, for an arbitrary integer $k \ge 5$, there exists a planar graph without cycles of length from 5 to k such that G is not 3-colorable. Let d denote the minimum distance between two diamonds in G, where a diamond is the union of two adjacent triangles. In this paper, we prove that a planar graph G with $d \ge 2$ and without cycles of length from 5 to 18 is 3-colorable. The reader is invited to find the smallest integer k such that a planar graph G with $d \ge 2$ and without cycles of length from 5 to k is 3-colorable.

1 Introduction

The Three Color Problem of planar graphs is to ask whether a planar graph can be colored with three colors such that no two adjacent vertices are assigned the same color. In 1976, Steinberg conjectured that every planar graph without 4- and 5-cycles is 3-colorable. In 2017, Steinberg's conjecture was disproved by constructing a counterexample to the conjecture [6]. However, the question whether every planar graph without cycles of length from 4 to 6 is 3-colorable is still open. Erdős raised

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the following relaxation of Steinberg's conjecture: Is there an integer such that every planar graph without cycles of length from 4 to k is 3-colorable. Abbott and Zhou confirmed that such k exists and $k \leq 11$ [1]. This result was later improved to $k \leq 9$ independently by Borodin [2] and by Sanders and Zhao [8] , and to $k \leq 7$ by Borodin et al [5]. It was proved by Borodin et al [3] and independently by Xu [9] that every planar graph having neither 5- and 7-cycles nor adjacent 3-cycles is 3-colorable which further improved the results above. In 2017, Jin et al proved that plane graphs without 4- and 5-cycles and without ext-triangular 7-cycles are 3-colorable [7].

However, for an arbitrary integer $k \geq 5$, we can construct a planar graph G without K_4 and cycles of length from 5 to k such that G is not 3-colorable as follow. Let D_i be the union of two adjacent triangles $[w_i u_i v_i]$ and $[u_i v_i w_{i+1}]$ which share one public edge for $i=1,2,...,k$. Let G be the graph obtained from the union of D_i's by adding the edge $w_1 w_{k+1}$. Obviously, G has neither K_4 nor cycles of length from 5 to $2k$ and G is not 3-colorable. Hence, it is an interesting thing to research the 3-coloring of planar graphs with adjacent triangles.

A *diamond D* is the union of two adjacent triangles $T_1 = [wuv]$ and $T_2 = [tuv]$ which share one public edge uv. Let d denote the minimum distance between two diamonds in G. In this paper, we prove that a planar graph G with $d \geq 2$ and without cycles of length from 5 to 18 is 3-colorable. The reader is invited to find the smallest integer k such that a planar graph G with $d \geq 2$ and without cycles of length from 5 to k is 3-colorable.

2 The 3-coloring of planar graphs with adjacent triangles

First, we introduce some notations as follow. A graph is *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *plane graph* is a planar graph G together with an embedding of G into the plane. Let G be a plane graph and C be a cycle of G. By $Int(C)$ (or $Ext(C)$), we denote the subgraph of G induced by the vertices lying inside (or outside) C. The cycle C is *separating* if neither $Int(C)$ nor $Ext(C)$ is empty. A plane graph G partitions the plane into a number of arcwise-connected open sets which are called the *faces* of G. Each plane graph has exactly one unbounded face, called the *exterior face* of G. Let a *k-face* be a face of degree k. Let a k^+-face be a face of degree at least k. The distance between two subgraphs H_1 and H_2 in G is the length of the shortest path between H_1 and H_2. By Euler's formula and the idea of discharging charge, we directly prove the following result.