Ann. Appl. Math.
doi: 10.4208/aam.OA-2025-0016

Vol. **41**, No. 3, pp. 295-301
August 2025

A note on hypergraph extensions of Mantel's theorem

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Received 8 July 2025

Abstract. Chao and Yu introduced an entropy method for hypergraph Turán problems, and used it to show that the family of $\lfloor k/2 \rfloor$ k-uniform tents have Turán density $k!/k^k$. Il'kovič and Yan [5] improved this by reducing to a subfamily of $\lceil k/e \rceil$ tents. In this note, enhancing Il'kovič-Yan's result, we give a significantly shorter entropy proof, with optimal bounds within this framework.

AMS subject classifications: 05C35, 05C65

Key words: Turán density, entropy, hypergraph.

1 Introduction

The *Turán number* $\mathrm{ex}(n,\mathcal{F})$ of a family $\mathcal{F}$ of k-uniform hypergraphs (k-*graphs* for short) denotes the maximum number of edges in an n-vertex k-graph not containing any member of $\mathcal{F}$ as its subgraph. Its *Turán density* is given by $\pi(\mathcal{F}) := \lim_{n\to\infty}\mathrm{ex}(n,\mathcal{F})/\binom{n}{k}$. For integer n, let $[n] := \{1,2,\ldots,n\}$. The study of Turán number and Turán density of graphs and hypergraphs is one of the central topics in extremal combinatorics. While the celebrated theorems of Turán [11] and Erdős-Stone-Simonovits [2] completely characterize Turán densities for all graph families, the hypergraph setting remains largely open, with exact densities known only for

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very few cases (see [6]). Significant progress has been made through sustained investigations of hypergraph extensions of Mantel's theorem concerning $\mathrm{ex}(n,K_3)$. These extensions have identified many families of k-graphs with Turán density $k!/k^k$. This particular value has attracted special attention, owing in part to Erdős' famous conjecture regarding whether $k!/k^k$ is a jump for k-graphs (see [4] for details). To proceed, for each $i\in[k]$, we define the $(k-i,i)$-tent $\Delta_{(k-i,i)}$ to be the k-graph with vertex set $[2k-1]$ and edge set

$$\big\{\{1,2,\ldots k\},\{1,\ldots,i,k+1,\ldots,2k-i\},\{i+1,\ldots,k+1,2k-i+1,\ldots,2k-1\}\big\}. \tag{1.1}$$

More generally, for any partition $\lambda=(\lambda_1,\ldots,\lambda_\ell)$ of k with $\lambda_1\geq\ldots\geq\lambda_\ell\geq1$, the λ-tent Δ_λ denotes the k-graph with $(k-1)\ell+1$ vertices and $\ell+1$ edges $e,e_1,\ldots,e_\ell$ defined as follows:

- There exists a vertex v (called *apex*) such that $e_i\cap e_j=\{v\}$ for all $1\leq i<j\leq\ell$;
- The subsets $e\cap e_1,\ldots,e\cap e_\ell$ form a partition of e, where $|e\cap e_i|=\lambda_i$ for each $i\in[\ell]$.

Frankl and Füredi [3] were the first to determine the exact Turán number for a hypergraph, where they showed $\pi(\Delta_{(2,1)})=3!/3^3$. Pikhurko [10] proved $\pi(\Delta_{(3,1)})=4!/4^4$ and the exact Turán number for large n. Generalizing a result of Mubayi [8], Mubayi and Pikhurko [9] determined $\pi(\Delta_{(1,1,\ldots,1)})=k!/k^k$. Recently, Chao and Yu [1] developed an innovative entropy-based approach to hypergraph Turán problems. Applying this method, they [1] proved the following generalization of Mantel's theorem: for every $k\geq2$, the family $\mathcal{F}_k:=\{\Delta_{(k-i,i)}:1\leq i\leq\lfloor k/2\rfloor\}$ satisfies $\pi(\mathcal{F}_k)=k!/k^k$. This implies that for any $1\leq q\leq\lceil k/2\rceil$, $\pi(\Delta_{(q,1,\ldots,1)})=k!/k^k$, generalizing the above results of [3,8,9]. Combining the entropy method with other techniques, Liu [7] determined the exact Turán number of $\mathcal{F}_k$ for large n. For each $1\leq s\leq\lfloor k/2\rfloor$, define

$$\mathcal{F}_k^{\leq s}=\{\Delta_{(k-i,i)}:1\leq i\leq s\}. \tag{1.2}$$

Very recently, Il'kovič and Yan [5] improved both of these results in [1, 7] by determining $\pi(\mathcal{F}_k^{\leq\lceil k/e\rceil})=k!/k^k$ for $k\geq4$ and the exact Turán number of $\mathcal{F}_k^{\leq\lceil k/e\rceil}$ for large n.

In this note, building on the entropy method of Chao and Yu [1], we provide a much short proof of the above Turán density results with a slightly better bound. Our main result is as follows. Throughout this note, for any integer $k\geq2$, define

$$t(k) \text{ to be the largest integer } s \text{ such that } 1/s+1/(s+1)+\cdots+1/k>1. \tag{1.3}$$

Theorem 1.1. *For every integer $k\geq2$, $\pi(\mathcal{F}_k^{\leq t(k)})=k!/k^k$.*

Note that the integer $t=t(k)$ satisfies $1<\sum\limits_{i=t}^{k}\frac{1}{i}\leq\int_{t-1}^{k}\frac{1}{x}\mathrm{d}x=\ln\frac{k}{t-1}$, which implies $t(k)\leq\lceil k/e\rceil$. So Theorem 1.1 generalizes the result of Il'kovič and Yan [5] that $\pi(\mathcal{F}_k^{\leq\lceil k/e\rceil})=k!/k^k$ for all $k\geq4$. Table 1 compares $\lceil k/e\rceil$ and $t(k)$ for $4\leq k\leq19$.