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Decoupled and Unconditionally Energy Stable Numerical Schemes for the Thermally Coupled Incompressible Magnetohydrodynamics Flow

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Abstract. This paper proposes two efficient numerical schemes for the time dependent thermally coupled incompressible Magnetohydrodynamics (TMHD) equations. Firstly, the existence and uniqueness of the weak solutions are established by employing the Galerkin method and extracting the subsequences. Secondly, the Euler semi-implicit scheme is designed for the target problem. The energy dissipation and stability of the numerical scheme are developed, and the optimal error estimates are also presented. Thirdly, the implicit/explicit (IMEX) scheme is utilized to simplify the computational complexity, then the aimed problem splits into three linear elliptic subproblems with the constant coefficients at each time level t_n. Meanwhile, the "zero-energy contribution" (ZEC) approach is used to surmount the restriction $\Delta t \leq C$ caused by the IMEX scheme. The corresponding mathematical findings, including the energy dissipation and stability of the IMEX-ZEC scheme and the convergence of the numerical solutions are obtained via the energy method and the Gronwall lemma. Finally, we conduct some numerical results to confirm the established theoretical findings and show the performances of the considered numerical schemes.

AMS subject classifications: 65M10, 65N30, 76Q10

Key words: Thermally coupled incompressible magnetohydrodynamics equations, unconditional stability, implicit/explicit scheme, ZEC approach, error estimates.

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1 Introduction

In this paper, we focus on the following time-dependent thermally coupled incompressible Magnetohydrodynamics model (see [4,16,21,23])

$$\begin{cases} \mathbf{u}_t - R_e^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + s_1\mathbf{B}\times\nabla\times\mathbf{B} = \lambda\mathbf{j}\theta & \text{in } \Omega_T, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega_T, \\ \mathbf{B}_t + R_m^{-1}\nabla\times\nabla\times\mathbf{B} - s_1\nabla\times(\mathbf{u}\times\mathbf{B}) = 0 & \text{in } \Omega_T, \\ \nabla\cdot\mathbf{B} = 0 & \text{in } \Omega_T, \\ \theta_t - \kappa\Delta\theta + \mathbf{u}\cdot\nabla\theta = 0 & \text{in } \Omega_T, \end{cases} \tag{1.1}$$

with the initial and boundary conditions:

$$\begin{cases} \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{u}|_{\partial\Omega} = 0, & \text{(no-slip condition)}, \\ (\mathbf{B}\cdot\mathbf{n})|_{\partial\Omega} = 0, \quad (\mathbf{n}\times\nabla\times\mathbf{B})|_{\partial\Omega} = 0, & \text{(perfectly wall)}, \\ \theta(x,0) = \theta_0(x), \quad \theta|_{\partial\Omega} = 0, & \text{(The Dirichlet boundary condition)}, \end{cases} \tag{1.2}$$

where the variables $\mathbf{u}$, $\mathbf{B}$, p, θ are the velocity, the magnetic field, the pressure and the temperature. The notation $\Omega_T = \Omega\times(0,T]$ and the domain $\Omega\subset\mathbb{R}^d$, $(d=2,3)$ is a bounded domain with the boundary $\partial\Omega$, The notations T, $\mathbf{j}$, ∇ and $\nabla\times$ are the final time, the vector of the gravitational acceleration, the gradient operator and the curl operator, respectively. The notations R_e, R_m, s_1, κ, λ represent the Reynolds number, the magnetic Reynolds number, the coupling number, the thermal conductivity and the thermal expansion coefficient, respectively.

The incompressible TMHD equations can be used to model the flow of an electrically charged fluid under the influence of an externally electromagnetic field driven by the heat difference (see [16,21,22]), it has been widely used in the industry and engineering, such as the cooling of nuclear reactors of the electrically conducting fluids, the continuous metal casting (see [1,4,5]). From the representation of the problem (1.1), one can see that it consists of an incompressible MHD equations and a nonlinear heat equation. From the perspective of theoretical analysis and numerical testing, the problem (1.1) not only inherits the all numerical difficulties of the incompressible flow, such as the coupling of variables, the divergence-free of velocity and magnetic fields, the incompressibility, but also contains more nonlinear terms. Therefore, it is a challenging task to develop the efficient numerical scheme for the thermally coupled incompressible Magnetohydrodynamics problem.

There were several works for the incompressible TMHD problem in the last decades years [23,26,30,42,43]. The earlier related works can be traced back to [21]. In this work, the authors discussed the existence and regularity results of the solutions of (1.1) by the Galerkin method and the fixed point arguments (the Leray-Schuader principle). Later, Meir considered the Newton iterative scheme for the steady TMHD problem in [22], the well-posdness and convergence of the numerical solutions were given. Yang et al. established the stability and convergence results of the numerical solutions for the TMHD